 

# A New Result On Adomian Decomposition Method For Solving Bratu's Problem

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**Abstract**

This paper investigates the properties of solution to the nonlinear Bratu's problem. Approximate solution of the strongly nonlinear problem is obtained using the rapidly convergent Adomian decomposition method. The result shows that the problem has two solutions, bifurcated and has no solution depending on the value of the Frank-Kameneskii parameter. Of particular interest is the determination of the bifurcation point using Adomian decomposition method.

**Keywords**: nonlinear eigenvalue problem, rational function, Adomian decomposition method, Bratu's problem.

## 1. Introduction

Studies on fuel ignition in thermal combustion theory have been on the increase over the last few years. The reason for the increased study is to ensure the safety of working environment especially when working with combustible fluid in some petro-chemical engineering processes. Combustion problems are generally characterized by strong nonlinearity and singularity, as such in most cases exact solution of combustion problems are very difficult to get. Therefore, researchers working in this area have resolved to approximate solutions by either analytical or numerical method.

Over the years, Bratu's problem has been a benchmark for many numerical and analytical methods in the literature [1-10]. Here, Bratu's problem is treated as an eigenvalue problem and approximate solution is obtained using modified Adomian decomposition method. In this paper attention is focused on the work done in [10] in which Adomian decomposition method was applied to Bratu's problem. The study was a major breakthrough in that Adomian decomposition was applied for the very first time to the strongly nonlinear problem arising from combustion problems. However, the paper was limited to predetermined value of the Frank-Kameneskii parameter and in the context of thermal ignition [11], the important properties associated with the problem cannot be determined. In standard form, the problem is given by

$$\frac{d^2u}{dx^2} + \lambda e^u = 0, \quad 0 < x < 1 \tag{1}$$

With the following boundary conditions

$$u(0) = 0 = u(1) \tag{2}$$

Exact solution of the problem can be written as

$$u(x) = -2\ln\left[\frac{Cosh\left(\left(x-\frac{1}{2}\right)\frac{\theta}{2}\right)}{Cosh\left(\frac{\theta}{4}\right)}\right], \quad (3)$$

With $\theta = \sqrt{2\lambda}Cosh\left(\frac{\theta}{4}\right)$ and $1 = \frac{\sqrt{2\lambda_c}}{4}Sinh\left(\frac{\theta_c}{4}\right)$. (4)

## 2. Method of Solution

Integrating (1) together with the first boundary condition in (2), we get

$$u(x) = \mu x - \int_0^x\int_0^x \lambda e^u dxdx \quad (5)$$

where $\mu = u'(0)$ is a constant to be evaluated using the boundary condition

$$u(1) = 0. \quad (6)$$

The standard Adomian decomposition method [12] assumes a series solution of the form

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \quad (7)$$

Substituting (7) in (1), one obtains

$$\sum_{n=0}^{\infty} u_n(x) = \mu x - \lambda\int_0^x\int_0^x\left(\sum_{n=0}^{\infty} A_n\right)dxdx \quad (8)$$

Here $A_n$ represents the nonlinear term $e^u$. From (8) the zeroth component gives

$$u_0(x) = \mu x \quad (9)$$

So that the recurrence relation for the problem is

$$u_{n+1}(x) = -\lambda\int_0^x\int_0^x A_n dxdx \quad (10)$$

The Adomian polynomial for the nonlinear term is given as

$$\begin{aligned} A_0 &= e^{u_0} \\ A_1 &= u_1 e^{u_0} \\ A_2 &= \left(u_2 + \frac{1}{2}u_1^2\right)e^{u_0} \end{aligned} \quad (11)$$

Using mathematica, the partial sum

$$u(x) = \sum_{n=0}^{k} u_n(x) \quad (12)$$

is given as the approximate solutions. To obtain the eigenvalues, the partial sum (12) is solved subject to (6), so as to obtain an expression for the unknown constant $\mu$ in the form

 

$$u(1) = \sum_{n=0}^{k} u_n(1) \tag{13}$$

Then (13) is Taylor's series expanded about $\mu$ up to the quadratic term and solved, this returns two rational functions for the unknown constant in terms of $\lambda$. The numerical results are given as Tables 1-2.

Table 1: Convergence of the two solutions with increasing partial sum when $\lambda = 1, k = 5$

| | $u_1$ | $u_2$ |
|---|---|---|
| | 0.549445 | 6.1348 |
| | 0.549444 | 3.32281 |
| | 0.549444 | 4.06463 |
| | 0.549444 | 4.06484 |
| | 0.549444 | 4.06484 |
| | 0.549444 | 4.06484 |

Table 2: Computation showing single solution at with increasing partial sum $\lambda = \lambda_c$

| | $u_1, u_2$ | $\lambda_c$ | Absolute error |
|---|---|---|---|
| | 3.4641, 3.4641 | 2.196152422706 | 1.322154767 |
| | 3.58114, 3.58114 | 2.741603437062 | 0.776703753 |
| | 3.80917, 3.80917 | 3.03355132742305 | 0.484755863 |
| | 4.01023, 4.01023 | 3.2338336873713 | 0.284473503 |
| | 4.17804, 4.17804 | 3.383964733123 | 0.134342457 |
| | 4.31811, 4.31811 | 3.50206042609 | 0.016246764 |

Table 1 show the convergence of the two solutions to the problem whenever $\lambda < \lambda_c$ while Table 2 confirms that the problem has a single solution at the critical point with increasing partial sum. As observed from the table the absolute error reduces with increasing partial sum. All these properties can be found at different stages of the solution shown in Figure 1

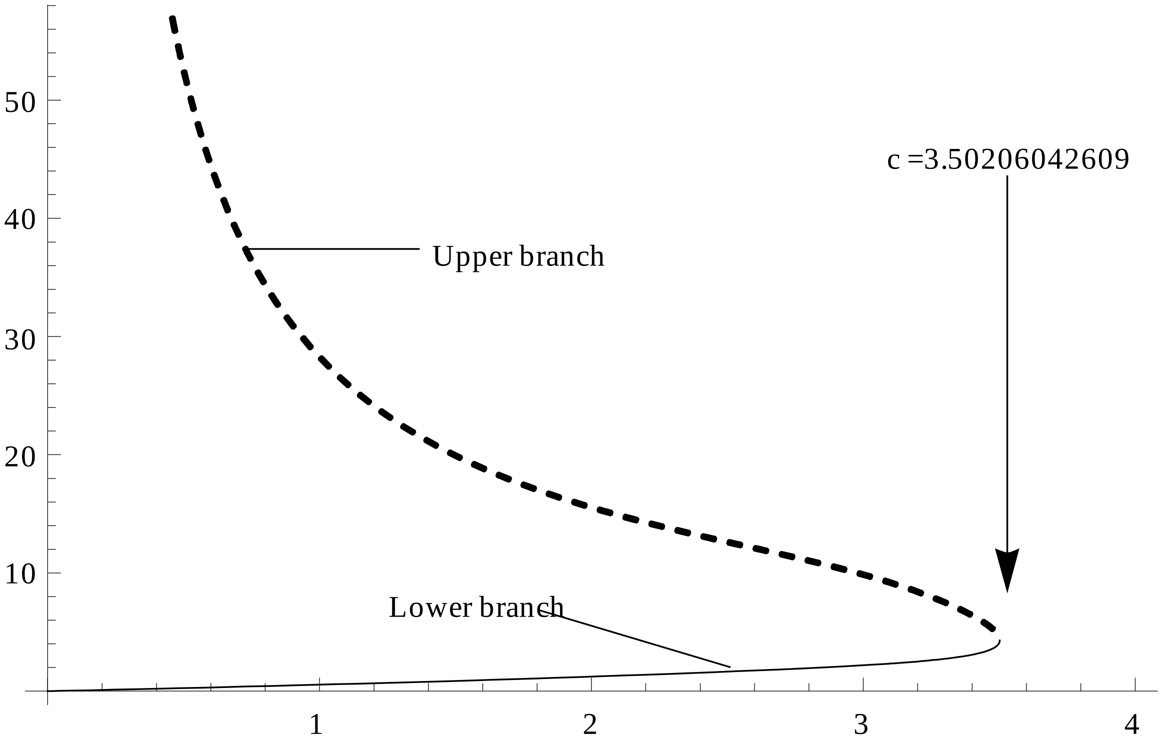


Figure 1: - A slice of the bifurcation of the approximate solution for *k=13*

## 3. Concluding remarks

The aim of this paper is to investigate the properties of solution to the Bratu's problem. Approximate solution to the nonlinear boundary valued problem is obtained using Adomian decomposition method. Although, the results obtained are convergent and well-behaved but for future research on Bratu problem, the combination of Adomian decomposition method together with Pade ´approximants is suggested so as to understand the blow up dynamics.

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