Inequalities For Numerical Radius And The Spectral Norm Of Hilbert Space Operators

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Abstract

Let (H,< .,. >) be a complex Hilbert space and B(H) denote the $\mathbb{C}^*$-algebra of all bounded linear operators on H. In this paper we establish inequalities for numerical radius and the spectral norm of Hilbert space operators , from the previous inequalities
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1 Introduction:

Let $B(H)$ denote the $\mathbb{C}^*$-algebra of all bounded linear operators on a complex Hilbert space H with inner product $\langle .,. \rangle$. For $A \in B(H)$. let $w(A), r(A)$ and $\|A\|$denote the numerical radius , the spectral radius and the usual operator norm of A, respectively.It is well known that $w(.)$ defines a norm on $B(H)$,and that for every $A \in B(H)$,

$$r(A) \leq w(A) \leq \|A\|, \tag{1}$$

and that equality holds if A is normal.
The numerical radius w(A) of an operator A on H is defined by

$$w(A) = \max\{| \langle Ax, x \rangle |: x \in H, \|x\| = 1\} \tag{2}$$

It has been recently shown in [4], that if A^2 does not converge to the zero operator in $M_n(\mathbb{C})$,then

$$w(A) \leq \left\|A^2\right\|^{\frac{1}{2}}, \tag{3}$$

Moreover, it has been shown in [3] that if $A, B \in B(H)$, then

$$w(AB + BA) \leq 2\sqrt{2}w(A).w(B) \tag{4}$$

In addition, It has been shown by Kittaneh [2], that if $A \in M_n(\mathbb{C})$, then

$$\frac{1}{4}\|AA^* + A^*A\| \leq w^2(A) \leq \frac{1}{2}\|AA^* + A^*A\|. \tag{5}$$

Also, It has been shown by Faryad-Khosravi [6] , that if $A \in B(H)$ such that $\|A\| \neq 1$or $w(A) \notin (0,1)$, then

$$\|A\| \leq 2^{\frac{3}{4}} w(A). \tag{6}$$

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In this paper we establish inequalities between numerical radius and the spectral norm of Hilbert space operators , from the previous inequalities

2 Main Results

Theorem 2.1: For every $A \in B(H)$,such that A^2 does not converge to the zero operator in $B(H)$,then

$$w^2(A \leq w(A^2) \leq 2^{\frac{9}{4}} w^2(A). \tag{7}$$

Proof: For every$A \in B(H)$,such that A^2 does not converge to the zero operator in $B(H)$ and from the inequality (3),we have

$$w^2(A) \leq \left\|A^2\right\|, \tag{8}$$

Also, from the inequality (6), we obtain

$$\|A^2\| \leq 2^{\frac{3}{4}} w(A^2). \tag{9}$$

And , from the inequality (4), we obtain

$$w(A^2) \leq 2\sqrt{2} w^2(A). \tag{10}$$

So,

$$2^{\frac{3}{4}} w(A^2) \leq 2^{\frac{9}{4}} w^2(A). \tag{11}$$

And hence , from the inequalities (8),(9),(10),and (11) we get the result.

From the proof of the previous theorem ,we can present the following corollary

COROLLARY 2.2: For every $A \in B(H)$,such that A^2 does not converge to the zero operator in $B(H)$,then

$$w^2(A \leq \|A^2\| \leq 2^{\frac{3}{4}} w(A^2). \tag{12}$$

By using the inequalities (11) and (12) and by taking the square roots of the both sides , we can write the inequality (12) as following

COROLLARY 2.3: For every $A \in B(H)$,such that A^2 does not converge to the zero operator in $B(H)$,then

$$w(A) \leq \|A^2\|^{\frac{1}{2}} \leq 2\sqrt{2} w(A). \tag{13}$$

From the inequality (5), we get the following inequality

$$\frac{1}{4}\|AA^* + A^*A\| \leq \|A^2\| \leq 2^{\frac{3}{4}} w(A^2). \tag{14}$$

These estimates yields new bounds for the zeros of monic polynomials by applying the previous inequalities to the Frobenius companion matrices of these polynomials.So,any researcher in mathematics interested in the subject of polynomial zeros, as well as numerical radius inequalities can benefit from these results, presented in this research

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