

# Double Truncated Transmuted Fréchet Distribution: Properties and Applications

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## Abstract

In this paper, we modify the Mahmoud and Mandouh (2013) model by adopting double truncation technique. It is referred to as Double Truncated Transmuted Fréchet (DTTF) distribution. Diverse probabilistic and reliability measures are developed and discussed. The MLEs of parameters are derived and a simulation study is also made. The DTTF distribution is modeled by two real-time datasets and supportive rationalized results provide the evidence that DTTF distribution is a reasonably better fit model than its competing models.

**Keywords:** Fréchet Distribution, Double Truncation, Hazard Function, Moments, MLE, Quadratic Rank Transmutation Map (QRTM), Rényi entropy, Order Statistics.

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## 1. Introduction

In numerous continuous probability distributions, particularly extreme value theory is an important part of statistical literature and one of the special cases (inverse Weibull, inverse Rayleigh, inverse Exponential or Gumbel type-II) is Fréchet distribution. Maurice Fréchet (1878-1973) a French mathematician developed a significant relation with Pareto distribution in 1927 when he discovered a limiting distribution for higher order statistic. The vital role of Fréchet distribution is observed in applied fields, for instance, through accelerated life-testing to engineering, geology, hydrology, horse racing, insurance, meteorology, sea currents, wind speed and many other diverse problems of life. Several generalizations and modifications, as well as progressive expansions over the last two decades, have been studied and a lot more is about to happen.

Nadarajah and Kotz (2003) developed an Exponentiated Fréchet distribution. Krishna *et al.* (2013) developed Fréchet distribution in Marshall-Olkin family of distributions and discovered its application in time series modeling. Mead and Abd-Eltawab (2014) expressed the Fréchet distribution in Kumaraswamy family of distributions and illustrated its application in the breaking stress of carbon fibers and strength of 1.5cm glass fibers datasets. Afify *et al.* (2016a) developed Marshall-Olkin Fréchet distribution in Kumaraswamy family of distributions and discussed its application in survival times of guinea pigs and the strength of 1.5cm glass fibers datasets. Transmuted Marshall Oklin Fréchet distribution introduced in Kumaraswamy family of distributions by Yousof *et al.* (2016). They developed its application in carbon fiber and glass fiber datasets. Mead *et al.* (2017) expressed the Exponentiated Fréchet distribution in the beta distribution and discovered its application in the Myelogeneous Leukemia and carbon fiber datasets. Mansour *et al.* (2018) illustrated the Exponentiated Fréchet distribution in Kumaraswamy family of distributions. They studied its application in the strength of 1.5cm glass

fibers dataset. Mansour *et al.* (2018) this time exponentiated the Marshall-Olkin Fréchet distribution and discovered its application in Myelogeneous Leukemia and flood peaks datasets. Mansour *et al.* (2018) generalized the Fréchet distribution in Odd Lindley family of distributions. They illustrated its application in exceedances of flood peaks and the breaking stress of carbon fibers datasets. Oguntunde *et al.* (2019) developed a compound of the Gompertz and Fréchet distribution. They developed its application in strength of carbon fibers and hailing times datasets.

Quadratic Rank Transmutation Map (QRTM) as a generator first time introduced by Shaw and Buckley (2009) for non-Gaussian distributions by adding a new parameter  $\lambda$  to base distribution  $G(x)$ . CDF of QRTM is followed by

$$F(x) = (1 + \lambda)G(x) - \lambda G^2(x) \quad , \quad for \quad |\lambda| < 1.$$

For  $\lambda = 0$ ,  $F(x) = G(x)$ , where  $G(x)$  is base distribution.

Scientific literature extended by Mahmoud and Mandouh (2013) when they generalized the Fréchet distribution in QRTM family of distributions and developed its application in breaking stress of carbon fiber and simulated datasets. The CDF of TFD is

$$F(x) = (1 + \lambda)e^{-\alpha x^{-\beta}} - \lambda e^{-2\alpha x^{-\beta}} \quad , \quad for \alpha, \beta > 0 \quad and \quad |\lambda| < 1.$$

Afify *et al.* (2015) investigated Marshall Olkin Fréchet distribution in QRTM family of distributions and discussed its application in breaking stress of carbon fibers and strength of 1.5cm glass fibers datasets. Arshad *et al.* (2018) generalized Exponentiated Moment Pareto in QRTM and found its application in four-lifetime datasets. Abayomia and Adeleke (2019) developed the transmuted edition of half normal distribution and discussed its application in purchasing behavior of customers from a wholesale outlet. Bhatti and Ali (2019) developed a range of characterizations of the transmuted edition of Exponentiated Pareto-I distribution.

In distribution theory, truncation is referred to as conditional distribution that provides more constructive and reliable results. One can study the tail behavior of the underlying model by truncating the lower, upper or both points. For more information, Abid (2016) double truncated the Fréchet distribution and simulate the model to recognize the performance of the estimates. Abid and Abdulrazak (2017) truncated the Fréchet, Fréchet Uniform and Fréchet Exponential distributions and discussed their strength-stress models as well. Castillo *et al.* (2018) discussed the half normal distribution by considering zero as a truncation point for  $x > 0$  and fit it to two-lifetime datasets.

The present distribution is initiated on this motivation that it has not been studied earlier and will present more flexible estimates on the skewed datasets. Furthermore, it is applicable to many diverse problems other than income and wealth studies.

Rest of the article is arranged into several sections as follows: CDF, PDF, graphical representation alongside special cases are developed in Section 2. In Section 3, we illustrate the moments and various reliability measures in Section 4. Quantiles function along with several descriptive statistics, Rényi entropy, The Mellin

transformation and order statistics is discussed in Section 5. Estimation of the parameters by the maximum likelihood, the study of simulation and application is developed in Section 6 and lastly, the conclusion is reported in Section 7.

## 2. Double Truncated Transmuted Fréchet Distribution

Here we establish a model by applying the technique of double truncation to Transmuted Fréchet distribution, originally developed by Mahmoud and Mandouh (2013). It is referred to a Double Truncated Transmuted Fréchet (DTTF) distribution. The CDF of DTTF distribution is followed by

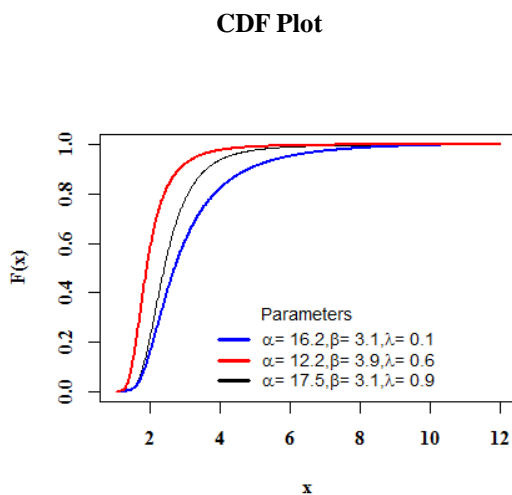
$$F(x) = \frac{[(1 + \lambda)e^{-\alpha x^{-\beta}} - \lambda e^{-2\alpha x^{-\beta}}] - [(1 + \lambda)e^{-\alpha m^{-\beta}} - \lambda e^{-2\alpha m^{-\beta}}]}{[(1 + \lambda)A - \lambda A^2] - [(1 + \lambda)B - \lambda B^2]}, \quad (1)$$

and PDF

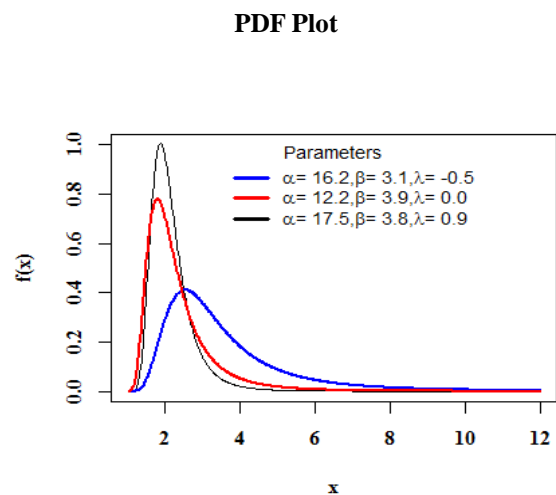
$$f(x) = \frac{[(1 + \lambda)\alpha\beta x^{-\beta-1}e^{-\alpha x^{-\beta}} - 2\alpha\beta\lambda x^{-\beta-1}e^{-2\alpha x^{-\beta}}]}{[(1 + \lambda)A - \lambda A^2] - [(1 + \lambda)B - \lambda B^2]}, \quad (2)$$

where  $x > 0, \alpha, \beta > 0, |\lambda| < 1, A = e^{-\alpha g^{-\beta}}, B = e^{-\alpha m^{-\beta}}, m$  and  $g$  are lower and upper truncation points.

Fig.1 and Fig.2 illustrate the reasonable shapes of CDF and PDF for selected values of the parameters  $\alpha, \beta$  and  $\lambda$ .



**Fig. 1**



**Fig. 2**

**Table 1**  
**Sub-Models of DTrTF Distribution**

| Models | $\alpha$ | $\beta$ | $\lambda$ | $m$ | $g$   | Author                     |
|--------|----------|---------|-----------|-----|-------|----------------------------|
| DTrF   | $\alpha$ | $\beta$ | 0         | $m$ | G     | Abid (2016)                |
| TF     | $\alpha$ | $\beta$ | $\lambda$ | 0   | $Inf$ | Mahmoud and Mandouh (2013) |
| F      | $\alpha$ | $\beta$ | 0         | 0   | $Inf$ | Fréchet (1924)             |

D= Double, Tr=Truncation, T=Transmutation, F= Frechet

### 3. Properties of DTrTF Distribution

#### 3.1. Theorem:

Suppose the r.v.  $X \sim \text{DTrTF}(x; \alpha, \beta, m, g)$ , subsequently,  $r$ -th raw moment of the Double Truncated Transmuted Fréchet distribution is given by

$$\mu'_r = -\frac{\alpha^{\frac{r}{\beta}}(1+\lambda)}{C} \left[ \Gamma\left(1 - \frac{r}{\beta}, \alpha m^{-\beta}\right) - \Gamma\left(1 - \frac{r}{\beta}, \alpha g^{-\beta}\right) \right] + \frac{2\lambda\alpha^{\frac{r}{\beta}}}{C} \left[ \Gamma\left(1 - \frac{r}{\beta}, 2\alpha m^{-\beta}\right) - \Gamma\left(1 - \frac{r}{\beta}, 2\alpha g^{-\beta}\right) \right],$$

where  $x > 0, \alpha, \beta > 0, |\lambda| < 1, C = [(1 + \lambda)A - \lambda A^2] - [(1 + \lambda)B - \lambda B^2], A = e^{-\alpha g^{-\beta}}$  and  $B = e^{-\alpha m^{-\beta}}, m$  and  $g$  are lower and upper truncation points.

By definition

$$\mu'_r = \int x^r f(x) dx$$

by equation (2),  $r$ -th moment of DTrTF distribution is written as

$$\mu'_r = \int_m^g x^r \frac{[(1 + \lambda)\alpha\beta x^{-\beta-1} e^{-\alpha x^{-\beta}} - 2\alpha\beta\lambda x^{-\beta-1} e^{-2\alpha x^{-\beta}}]}{[(1 + \lambda)A - \lambda A^2] - [(1 + \lambda)B - \lambda B^2]} dx, \quad (3)$$

say  $C = [(1 + \lambda)A - \lambda A^2] - [(1 + \lambda)B - \lambda B^2]$ , equation (3) in simplified form

$$\mu'_r = \frac{\alpha\beta(1+\lambda)}{C} \int_m^g x^{r-\beta-1} e^{-\alpha x^{-\beta}} dx - \frac{2\alpha\beta\lambda}{C} \int_m^g x^{r-\beta-1} e^{-2\alpha x^{-\beta}} dx,$$

$$\mu'_r = \frac{\alpha\beta(1+\lambda)}{C} I_1 - \frac{2\alpha\beta\lambda}{C} I_2, \quad (4)$$

for  $I_1$ : suppose  $\alpha x^{-\beta} = y \Rightarrow x = \left(\frac{y}{\alpha}\right)^{-\frac{1}{\beta}} \Rightarrow dx = -\frac{1}{\alpha\beta} \left(\frac{y}{\alpha}\right)^{-\frac{1}{\beta}-1} dy$ ,

limits:  $x = m \Rightarrow y = \alpha m^{-\beta}$ ,  $x = g \Rightarrow y = \alpha g^{-\beta}$ ,

$$I_1 = -\frac{1}{\beta} \alpha^{\frac{r}{\beta}-1} \int_{\alpha m^{-\beta}}^{\alpha g^{-\beta}} y^{-\frac{r}{\beta}} e^{-y} dy,$$

$$I_1 = -\frac{1}{\beta} \alpha^{\frac{r}{\beta}-1} \left[ \Gamma\left(1 - \frac{r}{\beta}, \alpha m^{-\beta}\right) - \Gamma\left(1 - \frac{r}{\beta}, \alpha g^{-\beta}\right) \right]. \quad (5)$$

Following the above procedure, we obtain the simplified form of  $I_2$

$$I_2 = -\frac{1}{\beta} (2\alpha)^{\frac{r}{\beta}-1} \left[ \Gamma\left(1 - \frac{r}{\beta}, 2\alpha m^{-\beta}\right) - \Gamma\left(1 - \frac{r}{\beta}, 2\alpha g^{-\beta}\right) \right], \quad (6)$$

hence the  $r$ -th moment of DTTF distribution is obtained by placing equation (5) and equation (6) in equation (4)

$$\mu'_r = \begin{cases} -\frac{\alpha^{\frac{r}{\beta}}(1+\lambda)}{C} \left[ \Gamma\left(1 - \frac{r}{\beta}, \alpha m^{-\beta}\right) - \Gamma\left(1 - \frac{r}{\beta}, \alpha g^{-\beta}\right) \right] \\ + \frac{2\lambda\alpha^{\frac{r}{\beta}}}{C} \left[ \Gamma\left(1 - \frac{r}{\beta}, 2\alpha m^{-\beta}\right) - \Gamma\left(1 - \frac{r}{\beta}, 2\alpha g^{-\beta}\right) \right] \end{cases}, \text{ for } r < \beta, \quad (7)$$

where  $x > 0$ ,  $\alpha, \beta > 0$ ,  $|\lambda| < 1$ ,  $C = [(1+\lambda)A - \lambda A^2] - [(1+\lambda)B - \lambda B^2]$ ,  $A = e^{-\alpha g^{-\beta}}$  and  $B = e^{-\alpha m^{-\beta}}$ ,  $m$  and  $g$  are lower and upper truncation points.

**3.2.  $r$ -th negative moments** of DTTF distribution are achieved by replacing  $r$  by  $-r$  in equation (7)

$$\mu'_{-r} = \begin{cases} -\frac{\alpha^{-\frac{r}{\beta}}(1+\lambda)}{C} \left[ \Gamma\left(1 + \frac{r}{\beta}, \alpha m^{-\beta}\right) - \Gamma\left(1 + \frac{r}{\beta}, \alpha g^{-\beta}\right) \right] \\ + \frac{2\lambda\alpha^{-\frac{r}{\beta}}}{C} \left[ \Gamma\left(1 + \frac{r}{\beta}, 2\alpha m^{-\beta}\right) - \Gamma\left(1 + \frac{r}{\beta}, 2\alpha g^{-\beta}\right) \right] \end{cases}, \quad (8)$$

where  $x > 0$ ,  $\alpha, \beta > 0$ ,  $|\lambda| < 1$ ,  $C = [(1+\lambda)A - \lambda A^2] - [(1+\lambda)B - \lambda B^2]$ ,  $A = e^{-\alpha g^{-\beta}}$  and  $B = e^{-\alpha m^{-\beta}}$ ,  $m$  and  $g$  are lower and upper truncation points.

**3.3. Fractional positive moments** of DTTF distribution is obtained by replacing  $r$  with  $(m/n)$  in equation (7)

$$\mu'_{\left(\frac{m}{n}\right)} = \int_m^g x^{\left(\frac{m}{n}\right)} f(x) dx$$

simplification provide

$$\mu'_{\left(\frac{m}{n}\right)} = \left\{ \begin{array}{l} -\frac{\alpha^{\frac{m}{n\beta}}(1+\lambda)}{C} \left[ \Gamma\left(1 - \frac{m}{n\beta}, \alpha m^{-\beta}\right) - \Gamma\left(1 - \frac{m}{n\beta}, \alpha g^{-\beta}\right) \right] \\ +\frac{2\lambda\alpha^{\frac{m}{n\beta}}}{C} \left[ \Gamma\left(1 - \frac{m}{n\beta}, 2\alpha m^{-\beta}\right) - \Gamma\left(1 - \frac{m}{n\beta}, 2\alpha g^{-\beta}\right) \right] \end{array} \right\}. \quad (9)$$

**3.4. Fractional negative moments** of DTTF distribution, just replace  $(m/n)$  with  $(-m/n)$  in equation (7), we get

$$\mu'_{\left(-\frac{m}{n}\right)} = \left\{ \begin{array}{l} -\frac{\alpha^{-\frac{m}{n\beta}}(1+\lambda)}{C} \left[ \Gamma\left(1 + \frac{m}{n\beta}, \alpha m^{-\beta}\right) - \Gamma\left(1 + \frac{m}{n\beta}, \alpha g^{-\beta}\right) \right] \\ +\frac{2\lambda\alpha^{-\frac{m}{n\beta}}}{C} \left[ \Gamma\left(1 + \frac{m}{n\beta}, 2\alpha m^{-\beta}\right) - \Gamma\left(1 + \frac{m}{n\beta}, 2\alpha g^{-\beta}\right) \right] \end{array} \right\}. \quad (10)$$

**3.5. Lower incomplete moments** of DTTF distribution, we replace the upper limit  $g$  to  $w$  in equation (7)

$$M_r(w) = E_{X \leq w}(x^r) = \int_m^w x^r f(x) dx$$

hence reduced and simplified form of lower incomplete moments is given by

$$M_r(w) = \left\{ \begin{array}{l} -\frac{\alpha^{\frac{r}{\beta}}(1+\lambda)}{C} \left[ \Gamma\left(1 - \frac{r}{\beta}, \alpha m^{-\beta}\right) - \Gamma\left(1 - \frac{r}{\beta}, \alpha w^{-\beta}\right) \right] \\ +\frac{2\lambda\alpha^{\frac{r}{\beta}}}{C} \left[ \Gamma\left(1 - \frac{r}{\beta}, 2\alpha m^{-\beta}\right) - \Gamma\left(1 - \frac{r}{\beta}, 2\alpha w^{-\beta}\right) \right] \end{array} \right\}. \quad (11)$$

**3.6. Upper incomplete moments** of DTTF distribution are obtained by incorporating equation (7)

$$M_s(w) = E_{X > w}(x^r) = \int_w^g x^r f(x) dx$$

hence upper incomplete moments of DTTF distribution is followed by

$$M_s(w) = \left\{ \begin{array}{l} -\frac{\alpha^{\frac{r}{\beta}}(1+\lambda)}{C} \left[ \Gamma\left(1 - \frac{r}{\beta}, \alpha w^{-\beta}\right) - \Gamma\left(1 - \frac{r}{\beta}, \alpha g^{-\beta}\right) \right] \\ +\frac{2\lambda\alpha^{\frac{r}{\beta}}}{C} \left[ \Gamma\left(1 - \frac{r}{\beta}, 2\alpha w^{-\beta}\right) - \Gamma\left(1 - \frac{r}{\beta}, 2\alpha g^{-\beta}\right) \right] \end{array} \right\}. \quad (12)$$

**3.7. Factorial moments** of DTTF distribution, we achieve by equation (7)

$$E[X]_n = \sum_{r=0}^n \varphi_r \mu'_r$$

$$E[X]_n = \sum_{r=0}^n \varphi_r \left\{ \begin{array}{l} -\frac{\alpha^{\frac{r}{\beta}}(1+\lambda)}{C} \left[ \Gamma\left(1-\frac{r}{\beta}, \alpha m^{-\beta}\right) - \Gamma\left(1-\frac{r}{\beta}, \alpha g^{-\beta}\right) \right] \\ +\frac{2\lambda\alpha^{\frac{r}{\beta}}}{C} \left[ \Gamma\left(1-\frac{r}{\beta}, 2\alpha m^{-\beta}\right) - \Gamma\left(1-\frac{r}{\beta}, 2\alpha g^{-\beta}\right) \right] \end{array} \right\} \quad (13)$$

where

$[X]_i = X(X+1)(X+2)\dots(X+i-1)$ ,  $\varphi_r$  is the Stirling number of first kind,  $C = [(1+\lambda)(A-B)] - [\lambda(B-B^2)]$ ,  $A = e^{-\alpha g^{-\beta}}$  and  $B = e^{-\alpha m^{-\beta}}$ ,  $m$  and  $g$  are lower and upper truncation points.

**3.8. Moment generating function (m.g.f.)** of r.v.  $X$  follow to DTTF distribution is defined as

$$M_x(t) = E(e^{tx}) = \int_m^g e^{tx} f(x) dx \quad (14)$$

by using series expansion  $e^{tx} = \sum_{r=0}^{\infty} \frac{(tx)^r}{r!}$ , equation (14) can be defined as

$$M_x(t) = \int_m^g \sum_{r=0}^{\infty} \frac{(t)^r}{r!} x^r f(x) dx,$$

$$M_x(t) = \sum_{r=0}^{\infty} \frac{(t)^r}{r!} \left\{ \begin{array}{l} -\frac{\alpha^{\frac{r}{\beta}}(1+\lambda)}{C} \left[ \Gamma\left(1-\frac{r}{\beta}, \alpha m^{-\beta}\right) - \Gamma\left(1-\frac{r}{\beta}, \alpha g^{-\beta}\right) \right] \\ +\frac{2\lambda\alpha^{\frac{r}{\beta}}}{C} \left[ \Gamma\left(1-\frac{r}{\beta}, 2\alpha m^{-\beta}\right) - \Gamma\left(1-\frac{r}{\beta}, 2\alpha g^{-\beta}\right) \right] \end{array} \right\} \quad (15)$$

**3.9. Central moments** can be obtained by using a relation between ordinary and central moments. It is defined as

$$\mu_r = E[X - E(X)]^r = \sum_{j=0}^r \binom{r}{j} (-1)^j (\mu_1)^j \mu_{r-j}$$

based on equation (7), central moments of DTTF distribution

$$\mu_r = \sum_{j=0}^r \binom{r}{j} (-1)^j \left\{ \begin{array}{l} -\frac{\alpha^{\frac{1}{\beta}}(1+\lambda)}{C} \left[ \Gamma\left(1-\frac{1}{\beta}, \alpha m^{-\beta}\right) - \Gamma\left(1-\frac{1}{\beta}, \alpha g^{-\beta}\right) \right] \\ +\frac{2\lambda\alpha^{\frac{1}{\beta}}}{C} \left[ \Gamma\left(1-\frac{1}{\beta}, 2\alpha m^{-\beta}\right) - \Gamma\left(1-\frac{1}{\beta}, 2\alpha g^{-\beta}\right) \right] \end{array} \right\}^j \left\{ \begin{array}{l} -\frac{\alpha^{\frac{r-j}{\beta}}(1+\lambda)}{C} \left[ \Gamma\left(1-\frac{r-j}{\beta}, \alpha m^{-\beta}\right) - \Gamma\left(1-\frac{r-j}{\beta}, \alpha g^{-\beta}\right) \right] \\ +\frac{2\lambda\alpha^{\frac{r-j}{\beta}}}{C} \left[ \Gamma\left(1-\frac{r-j}{\beta}, 2\alpha m^{-\beta}\right) - \Gamma\left(1-\frac{r-j}{\beta}, 2\alpha g^{-\beta}\right) \right] \end{array} \right\} \quad (16)$$

**3.10. Cumulants generating function** based on a relation between ordinary moments and cumulants is defined as

$$K_r = \mu_r - \sum_{i=1}^{r-1} \binom{r-1}{i-1} K_i \mu_{r-i}$$

cumulants generating function of DTF distribution can be written as

$$K_r = \left\{ \begin{array}{l} -\frac{\alpha^{\frac{r}{\beta}}(1+\lambda)}{C} \left[ \Gamma\left(1-\frac{r}{\beta}, \alpha m^{-\beta}\right) - \Gamma\left(1-\frac{r}{\beta}, \alpha g^{-\beta}\right) \right] \\ + \frac{2\lambda\alpha^{\frac{r}{\beta}}}{C} \left[ \Gamma\left(1-\frac{r}{\beta}, 2\alpha m^{-\beta}\right) - \Gamma\left(1-\frac{r}{\beta}, 2\alpha g^{-\beta}\right) \right] \end{array} \right\} - \sum_{i=0}^{r-1} \binom{r-1}{i-1} K_i \left\{ \begin{array}{l} -\frac{\alpha^{\frac{r-i}{\beta}}(1+\lambda)}{C} \left[ \Gamma\left(1-\frac{r-i}{\beta}, \alpha m^{-\beta}\right) - \Gamma\left(1-\frac{r-i}{\beta}, \alpha g^{-\beta}\right) \right] \\ + \frac{2\lambda\alpha^{\frac{r-i}{\beta}}}{C} \left[ \Gamma\left(1-\frac{r-i}{\beta}, 2\alpha m^{-\beta}\right) - \Gamma\left(1-\frac{r-i}{\beta}, 2\alpha g^{-\beta}\right) \right] \end{array} \right\} \quad (17)$$

**3.11. Skewness** of DTF distribution is

$$\beta_1 = \frac{\left( \begin{array}{l} -\frac{\alpha^{\frac{3}{\beta}}(1+\lambda)}{C} \left[ \Gamma\left(1-\frac{3}{\beta}, \alpha m^{-\beta}\right) - \Gamma\left(1-\frac{3}{\beta}, \alpha g^{-\beta}\right) \right] \\ + \frac{2\lambda\alpha^{\frac{3}{\beta}}}{C} \left[ \Gamma\left(1-\frac{3}{\beta}, 2\alpha m^{-\beta}\right) - \Gamma\left(1-\frac{3}{\beta}, 2\alpha g^{-\beta}\right) \right] \end{array} \right)^2}{\left( \begin{array}{l} -\frac{\alpha^{\frac{2}{\beta}}(1+\lambda)}{C} \left[ \Gamma\left(1-\frac{2}{\beta}, \alpha m^{-\beta}\right) - \Gamma\left(1-\frac{2}{\beta}, \alpha g^{-\beta}\right) \right] \\ + \frac{2\lambda\alpha^{\frac{2}{\beta}}}{C} \left[ \Gamma\left(1-\frac{2}{\beta}, 2\alpha m^{-\beta}\right) - \Gamma\left(1-\frac{2}{\beta}, 2\alpha g^{-\beta}\right) \right] \end{array} \right)^3} \quad (18)$$

**3.12. Kurtosis** of DTF distribution is identified as

$$\beta_2 = \frac{\left( \begin{array}{l} -\frac{\alpha^{\frac{4}{\beta}}(1+\lambda)}{C} \left[ \Gamma\left(1-\frac{4}{\beta}, \alpha m^{-\beta}\right) - \Gamma\left(1-\frac{4}{\beta}, \alpha g^{-\beta}\right) \right] \\ + \frac{2\lambda\alpha^{\frac{4}{\beta}}}{C} \left[ \Gamma\left(1-\frac{4}{\beta}, 2\alpha m^{-\beta}\right) - \Gamma\left(1-\frac{4}{\beta}, 2\alpha g^{-\beta}\right) \right] \end{array} \right)^2}{\left( \begin{array}{l} -\frac{\alpha^{\frac{2}{\beta}}(1+\lambda)}{C} \left[ \Gamma\left(1-\frac{2}{\beta}, \alpha m^{-\beta}\right) - \Gamma\left(1-\frac{2}{\beta}, \alpha g^{-\beta}\right) \right] \\ + \frac{2\lambda\alpha^{\frac{2}{\beta}}}{C} \left[ \Gamma\left(1-\frac{2}{\beta}, 2\alpha m^{-\beta}\right) - \Gamma\left(1-\frac{2}{\beta}, 2\alpha g^{-\beta}\right) \right] \end{array} \right)^2} \quad (19)$$

where  $x > 0$ ,  $\alpha, \beta > 0$ ,  $|\lambda| < 1$ ,  $C = [(1+\lambda)A - \lambda A^2] - [(1+\lambda)B - \lambda B^2]$ ,  $A = e^{-\alpha g^{-\beta}}$  and  $B = e^{-\alpha m^{-\beta}}$ ,  $m$  and  $g$  are lower and upper truncation points.



#### 4. Reliability Measures of DTTF Distribution

In reliability engineering, reliability analysis through probability distribution is the most extensively exercised method which pays significant contribution in studying and predicting the survival or hazard life of the component during a particular interval of time.

##### 4.1. Survival function

Survival or reliability function is used to measure the risk of occurrence of some event at a specific time. It is denoted by  $S(x)$ . For DTTF distribution it can be written as

$$S(x) = 1 - \left( \frac{1}{C} \left\{ [(1 + \lambda)e^{-\alpha x^{-\beta}} - \lambda e^{-2\alpha x^{-\beta}}] - [(1 + \lambda)e^{-\alpha m^{-\beta}} - \lambda e^{-2\alpha m^{-\beta}}] \right\} \right). \quad (20)$$

##### 4.2. Hazard function

Hazard function  $H(x)$  is used to measure the failure rate of some components in a particular period of time  $x$ . For DTTF distribution it is illustrated by

$$H(x) = \frac{\left( \frac{1}{C} [(1 + \lambda)\alpha\beta x^{-\beta-1} e^{-\alpha x^{-\beta}} - 2\alpha\beta\lambda x^{-\beta-1} e^{-2\alpha x^{-\beta}}] \right)}{1 - \left( \frac{1}{C} \left\{ [(1 + \lambda)e^{-\alpha x^{-\beta}} - \lambda e^{-2\alpha x^{-\beta}}] - [(1 + \lambda)e^{-\alpha m^{-\beta}} - \lambda e^{-2\alpha m^{-\beta}}] \right\} \right)}. \quad (21)$$

**4.3. Cumulative hazard function**  $H_c(x)$  of DTTF distribution is developed by equation (20)

$$H_c(x) = -\ln \left\{ 1 - \frac{1}{C} \left[ (1 + \lambda)e^{-\alpha x^{-\beta}} - \lambda e^{-2\alpha x^{-\beta}} \right] - \left[ (1 + \lambda)e^{-\alpha m^{-\beta}} - \lambda e^{-2\alpha m^{-\beta}} \right] \right\}. \quad (22)$$

**4.4. Reverse hazard function**  $H_r(x)$  of DTTF distribution is obtained by incorporating equation (1) and equation (20)

$$H_r(x) = \frac{\left( \frac{1}{C} [(1 + \lambda)\alpha\beta x^{-\beta-1} e^{-\alpha x^{-\beta}} - 2\alpha\beta\lambda x^{-\beta-1} e^{-2\alpha x^{-\beta}}] \right)}{\left( \frac{1}{C} \left\{ [(1 + \lambda)e^{-\alpha x^{-\beta}} - \lambda e^{-2\alpha x^{-\beta}}] - [(1 + \lambda)e^{-\alpha m^{-\beta}} - \lambda e^{-2\alpha m^{-\beta}}] \right\} \right)}. \quad (23)$$

**4.5. Mills ratio**  $M(x)$  of DTTF distribution is obtained by equation (1) and equation (20)

$$M(x) = \frac{1 - \left( \frac{1}{C} \left\{ [(1 + \lambda)e^{-\alpha x^{-\beta}} - \lambda e^{-2\alpha x^{-\beta}}] - [(1 + \lambda)e^{-\alpha m^{-\beta}} - \lambda e^{-2\alpha m^{-\beta}}] \right\} \right)}{\left( \frac{1}{C} [(1 + \lambda)\alpha\beta x^{-\beta-1} e^{-\alpha x^{-\beta}} - 2\alpha\beta\lambda x^{-\beta-1} e^{-2\alpha x^{-\beta}}] \right)}. \quad (24)$$

**4.6. Odd function**  $O(x)$  of DTTF distribution is achieved by equation (2) and equation (20)

$$O(x) = \frac{\left( \frac{1}{C} \left\{ [(1 + \lambda)e^{-\alpha x^{-\beta}} - \lambda e^{-2\alpha x^{-\beta}}] - [(1 + \lambda)e^{-\alpha m^{-\beta}} - \lambda e^{-2\alpha m^{-\beta}}] \right\} \right)}{1 - \left( \frac{1}{C} \left\{ [(1 + \lambda)e^{-\alpha x^{-\beta}} - \lambda e^{-2\alpha x^{-\beta}}] - [(1 + \lambda)e^{-\alpha m^{-\beta}} - \lambda e^{-2\alpha m^{-\beta}}] \right\} \right)}. \quad (25)$$

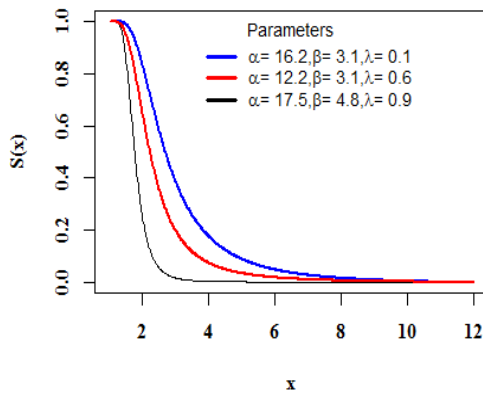
**4.7. Elasticity**  $e(x)$  of DTTF distribution is illustrated by equation (1) and equation (2)

$$e(x) = \frac{x \left( \frac{1}{C} \left[ (1 + \lambda) \alpha \beta x^{-\beta-1} e^{-\alpha x^{-\beta}} - 2 \alpha \beta \lambda x^{-\beta-1} e^{-2 \alpha x^{-\beta}} \right] \right)}{\left( \frac{1}{C} \left\{ [(1 + \lambda) e^{-\alpha x^{-\beta}} - \lambda e^{-2 \alpha x^{-\beta}}] - [(1 + \lambda) e^{-\alpha m^{-\beta}} - \lambda e^{-2 \alpha m^{-\beta}}] \right\} \right)}, \quad (26)$$

where  $x > 0$ ,  $\alpha, \beta > 0$ ,  $|\lambda| < 1$ ,  $C = [(1 + \lambda)A - \lambda A^2] - [(1 + \lambda)B - \lambda B^2]$ ,  $A = e^{-\alpha g^{-\beta}}$  and  $B = e^{-\alpha m^{-\beta}}$ ,  $m$  and  $g$  are lower and upper truncation points.

Possible shapes of survival function, hazard function, cumulative hazard function, reverse hazard function, mills ratio and odd function plots are drafted over various selected combinations of the parameters  $\alpha, \beta$  and  $\lambda$  display in Fig. 3 to Fig. 8.

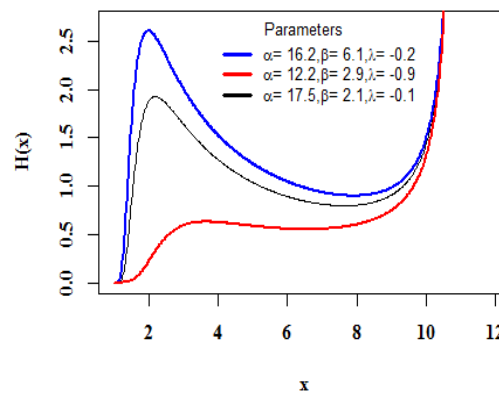
**Survival Function**



**Fig.3**

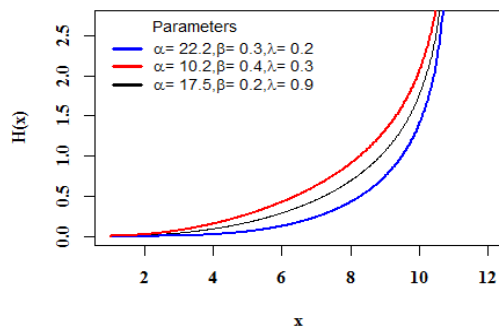
**Cumulative Hazard Function**

**Hazard Function**



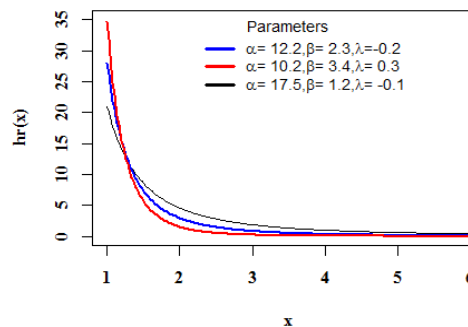
**Fig.4**

**Reverse Hazard Function**



**Fig. 5**

**Mills Ratio**



**Fig. 6**

**Odd Function**

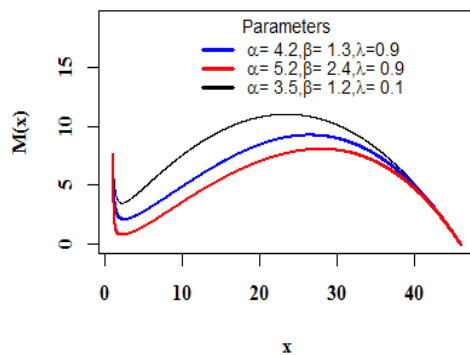


Fig. 7

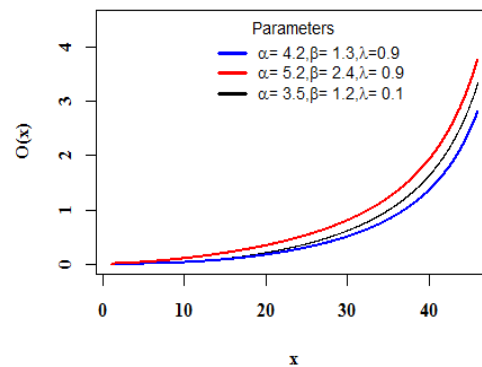


Fig. 8

## 5. Quantiles and Descriptive Statistics of DTTF Distribution

**5.1. Quantile function** can be defined as when under investigation CDF is inverted by the method of inversion. It is referred to a quantile function.

$q$ -th quantile function of DTTF distribution is given by equation (1)

$$q = F(x_q) = \frac{[(1 + \lambda)e^{-\alpha x^{-\beta}} - \lambda e^{-2\alpha x^{-\beta}}] - [(1 + \lambda)e^{-\alpha m^{-\beta}} - \lambda e^{-2\alpha m^{-\beta}}]}{[(1 + \lambda)A - \lambda A^2] - [(1 + \lambda)B - \lambda B^2]}, \quad (27)$$

equation (27) can be written as

$$q = \frac{[(1 + \lambda)y - \lambda y^2] - E}{D - E} \quad \text{for } 0 \leq q \leq 1,$$

$$\lambda y^2 - (1 + \lambda)y + E(1 - q) + qD = 0,$$

$q^{\text{th}}$  quantile function in reduced form

$$x_q = \left[ -\frac{1}{\alpha} \ln \left[ \frac{(1 + \lambda) - \sqrt{(1 + \lambda)^2 - 4\lambda[qD + E(1 - q)]}}{2\lambda} \right] \right]^{-\frac{1}{\beta}}. \quad (28)$$

for simplification we suppose  $y = e^{-\alpha x^{-\beta}}$ ,  $E = [(1 + \lambda)A - \lambda A^2]$ ,  $D = [(1 + \lambda)B - \lambda B^2]$  where  $A = e^{-\alpha g^{-\beta}}$ ,  $B = e^{-\alpha m^{-\beta}}$ ,  $m$  and  $g$  are lower and upper truncation points.

**5.2. Median** can easily achieve by placing  $q = 0.5$  in equation (28)

$$x_{0.5} = \left[ -\frac{1}{\alpha} \ln \left[ \frac{(1 + \lambda) - \sqrt{(1 + \lambda)^2 - 2\lambda[D + E]}}{2\lambda} \right] \right]^{-\frac{1}{\beta}}. \quad (29)$$

To generate random numbers, we suppose that CDF of DTTF distribution follow to uniform distribution  $u = U(0, 1)$ .

**5.3. Random numbers** of DTTF distribution is calculated by

$$x_{Rd} = \left[ -\frac{1}{\alpha} \ln \left[ \frac{(1 + \lambda) - \sqrt{(1 + \lambda)^2 - 4\lambda[uD + E(1 - u)]}}{2\lambda} \right] \right]^{\frac{1}{\beta}} \quad (30)$$

**5.4. Harmonic mean** of DTTF distribution is achieved as we replace  $r$  by  $-1$  in equation (7)

$$HM = \left\{ \begin{array}{l} -\frac{\alpha^{-\frac{1}{\beta}}(1 + \lambda)}{C} \left[ \Gamma\left(1 + \frac{1}{\beta}, \alpha m^{-\beta}\right) - \Gamma\left(1 + \frac{1}{\beta}, \alpha g^{-\beta}\right) \right] \\ + \frac{2\lambda\alpha^{-\frac{1}{\beta}}}{C} \left[ \Gamma\left(1 + \frac{1}{\beta}, 2\alpha m^{-\beta}\right) - \Gamma\left(1 + \frac{1}{\beta}, 2\alpha g^{-\beta}\right) \right] \end{array} \right\} \quad (31)$$

**5.5. Mean** of DTTF distribution is achieved, simply as we replace  $r$  by  $1$  in equation (7)

$$E(X) = \mu'_1 = \left\{ \begin{array}{l} -\frac{\alpha^{\frac{1}{\beta}}(1 + \lambda)}{C} \left[ \Gamma\left(1 - \frac{1}{\beta}, \alpha m^{-\beta}\right) - \Gamma\left(1 - \frac{1}{\beta}, \alpha g^{-\beta}\right) \right] \\ + \frac{2\lambda\alpha^{\frac{1}{\beta}}}{C} \left[ \Gamma\left(1 - \frac{1}{\beta}, 2\alpha m^{-\beta}\right) - \Gamma\left(1 - \frac{1}{\beta}, 2\alpha g^{-\beta}\right) \right] \end{array} \right\} \quad (32)$$

**5.6. Variance** of DTTF distribution may be calculated by incorporating the equation (32) and equation (7)

$$Var(X) = \left\{ \begin{array}{l} \left( \begin{array}{l} -\frac{\alpha^{\frac{2}{\beta}}(1 + \lambda)}{C} \left[ \Gamma\left(1 - \frac{2}{\beta}, \alpha m^{-\beta}\right) - \Gamma\left(1 - \frac{2}{\beta}, \alpha g^{-\beta}\right) \right] \\ + \frac{2\lambda\alpha^{\frac{2}{\beta}}}{C} \left[ \Gamma\left(1 - \frac{2}{\beta}, 2\alpha m^{-\beta}\right) - \Gamma\left(1 - \frac{2}{\beta}, 2\alpha g^{-\beta}\right) \right] \end{array} \right)^2 \\ \left( \begin{array}{l} -\frac{\alpha^{\frac{1}{\beta}}(1 + \lambda)}{C} \left[ \Gamma\left(1 - \frac{1}{\beta}, \alpha m^{-\beta}\right) - \Gamma\left(1 - \frac{1}{\beta}, \alpha g^{-\beta}\right) \right] \\ + \frac{2\lambda\alpha^{\frac{1}{\beta}}}{C} \left[ \Gamma\left(1 - \frac{1}{\beta}, 2\alpha m^{-\beta}\right) - \Gamma\left(1 - \frac{1}{\beta}, 2\alpha g^{-\beta}\right) \right] \end{array} \right)^2 \end{array} \right\} \quad (33)$$

**5.7. Mode** of DTTF distribution is calculated by taking first derivative of PDF and equate to zero. PDF of DTTF distribution rewrite for the simplification approach

$$f(x) = \frac{1}{C} \left[ (1 + \lambda)\alpha\beta x^{-\beta-1} e^{-\alpha x^{-\beta}} - 2\alpha\beta\lambda x^{-\beta-1} e^{-2\alpha x^{-\beta}} \right],$$

here we find first derivative and set it to zero

$$\frac{1}{C} \left\{ \begin{aligned} &e^{-\alpha x^{-\beta}}(1 + \lambda)[(\alpha\beta)^2 x^{(-2\beta-2)} + \alpha\beta(-\beta - 1)x^{(-\beta-2)}] \\ &-e^{-2\alpha x^{-\beta}}\lambda[(2\alpha\beta)^2 x^{(-2\beta-2)} + 2\alpha\beta x^{(-\beta-2)}(-\beta - 1)] \end{aligned} \right\} = 0,$$

since  $e^{-\alpha x^{-\beta}}$ ,  $e^{-2\alpha x^{-\beta}}$  and  $\lambda$  can not be zero for any finite value of  $x$

hence

$$[(\alpha\beta)^2 x^{(-2\beta-2)} + \alpha\beta(-\beta - 1)x^{(-\beta-2)}] = 0, \quad (34)$$

$$\hat{x}_1 = \left( \frac{\alpha\beta}{\beta + 1} \right)^{\frac{1}{\beta}}. \quad (35)$$

and

$$-e^{-2\alpha x^{-\beta}}\lambda[(2\alpha\beta)^2 x^{(-2\beta-2)} + 2\alpha\beta x^{(-\beta-2)}(-\beta - 1)] = 0, \quad (36)$$

$$\hat{x}_2 = \left( \frac{2\alpha\beta}{\beta + 1} \right)^{\frac{1}{\beta}}, \quad (37)$$

thus DTTF distribution is declared as a bimodal distribution

where  $x > 0$ ,  $\alpha, \beta > 0$ ,  $|\lambda| < 1$ ,  $C = [(1 + \lambda)A - \lambda A^2] - [(1 + \lambda)B - \lambda B^2]$ ,  $A = e^{-\alpha g^{-\beta}}$  and  $B = e^{-\alpha m^{-\beta}}$ ,  $m$  and  $g$  are lower and upper truncation points.

### 5.8. Entropy of DTTF distribution

The degree of disorder or unpredictability / randomness in a system is defined as entropy.

By definition, Rényi (1961) entropy is described as

$$I_\varphi(X) = \frac{1}{\varphi - 1} \log \int_0^\infty f^\varphi(x) dx \quad \text{for } \varphi > 0 \text{ and } \varphi \neq 1.$$

Rényi entropy of DTTF distribution is obtained by incorporating equation (2)

$$I_\varphi(X) = \frac{1}{\varphi - 1} \log \int_m^g \left\{ \frac{1}{C} \left[ (1 + \lambda)\alpha\beta x^{-\beta-1} e^{-\alpha x^{-\beta}} - (2\alpha\beta\lambda x^{-\beta-1} e^{-2\alpha x^{-\beta}}) \right] \right\}^\varphi dx, \quad (38)$$

let's simplify first  $f^\varphi(x)$

$$\begin{aligned} \text{but } f^\varphi(x) &= \left\{ \frac{1}{C} \left[ (1 + \lambda)\alpha\beta x^{-\beta-1} e^{-\alpha x^{-\beta}} - (2\alpha\beta\lambda x^{-\beta-1} e^{-2\alpha x^{-\beta}}) \right] \right\}^\varphi, \\ &= \left\{ \frac{1}{C} (1 + \lambda)\alpha\beta x^{-\beta-1} e^{-\alpha x^{-\beta}} \left[ 1 - \frac{2\lambda e^{-\alpha x^{-\beta}}}{1 + \lambda} \right] \right\}^\varphi, \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{C^\varphi} \left( (1 + \lambda) \alpha \beta x^{-\beta-1} \right)^\varphi \sum_{j=0}^{\infty} \left\{ (-1)^j \binom{\varphi}{j} \left( \frac{2\lambda}{1 + \lambda} \right)^j \left( e^{-\alpha j x^{-\beta} - \alpha \varphi x^{-\beta}} \right) \right\}, \\
 &= \left( \frac{(1 + \lambda) \alpha \beta}{C} \right)^\varphi \sum_{j=0}^{\infty} \left\{ (-1)^j \binom{\varphi}{j} \left( \frac{2\lambda}{1 + \lambda} \right)^j \sum_{k=0}^{\infty} \frac{(-\alpha x^{-\beta} (j + \varphi))^k}{k!} \right\}, \\
 f^\varphi(x) &= \left( \frac{(1 + \lambda) \alpha \beta}{C} \right)^\varphi \sum_{j,k=0}^{\infty} \left\{ \frac{\alpha^k (-1)^{j+k} (j + \varphi)^k}{k!} \binom{\varphi}{j} \left( \frac{2\lambda}{1 + \lambda} \right)^j \right\}, \tag{39} \\
 &\quad x^{(-\beta(\varphi+k)-\varphi)}
 \end{aligned}$$

place equation (39) in equation (38), we get

$$I_\varphi(X) = \left( \frac{(1 + \lambda) \alpha \beta}{C} \right)^\varphi \left( \frac{1}{\varphi - 1} \right) \log \int_m^g \sum_{j,k=0}^{\infty} \left\{ \frac{(-1)^{j+k} (\alpha(j + \varphi))^k}{k!} \binom{\varphi}{j} \left( \frac{2\lambda}{1 + \lambda} \right)^j \right\} dx. \tag{40}$$

Integrate equation (40) yield the simplified and reduced form of Rényi entropy of DTTF distribution is followed by

$$I_\varphi(X) = \left( \frac{1}{1 - \varphi} \right) \log \left[ \left( \frac{(1 + \lambda) \alpha \beta}{C} \right)^\varphi \sum_{j,k=0}^{\infty} \left\{ \frac{(-1)^{j+k} (\alpha(j + \varphi))^k}{k!} \binom{\varphi}{j} \left( \frac{2\lambda}{1 + \lambda} \right)^j \right\} \left( \frac{g^{(1-\beta(\varphi+k)-\varphi)} - m^{(1-\beta(\varphi+k)-\varphi)}}{1 - \beta(\varphi + k) - \varphi} \right) \right], \tag{41}$$

where  $x > 0$ ,  $\alpha, \beta > 0$ ,  $|\lambda| < 1$ ,  $C = [(1 + \lambda)A - \lambda A^2] - [(1 + \lambda)B - \lambda B^2]$ ,  $A = e^{-\alpha g^{-\beta}}$  and  $B = e^{-\alpha m^{-\beta}}$ ,  $m$  and  $g$  are lower and upper truncation points.

### 5.9. The Mellin Transformation of DTTF distribution

In the theory of statistics, the Mellin transformation is well-known since it is a distribution of the product and proportion for independent r.v.'s.

The Mellin transformation is defined as

$$M_x(n) = \mu'_{n-1} = E(x^{n-1}) = \int_0^{\infty} x^{n-1} f(x) dx$$

The Mellin transformation of DTTF distribution, we replace  $r$  by  $n-1$  in equation (7), we get

$$M_x(n) = \left\{ \begin{array}{l} -\frac{\alpha^{\frac{n-1}{\beta}}(1+\lambda)}{C} \left[ \Gamma\left(1 - \frac{n-1}{\beta}, \alpha m^{-\beta}\right) - \Gamma\left(1 - \frac{n-1}{\beta}, \alpha g^{-\beta}\right) \right] \\ + \frac{2\lambda\alpha^{\frac{n-1}{\beta}}}{C} \left[ \Gamma\left(1 - \frac{n-1}{\beta}, 2\alpha m^{-\beta}\right) - \Gamma\left(1 - \frac{n-1}{\beta}, 2\alpha g^{-\beta}\right) \right] \end{array} \right\}, \quad (42)$$

where  $x > 0$ ,  $\alpha, \beta > 0$ ,  $|\lambda| < 1$ ,  $C = [(1+\lambda)A - \lambda A^2] - [(1+\lambda)B - \lambda B^2]$ ,  $A = e^{-\alpha g^{-\beta}}$  and  $B = e^{-\alpha m^{-\beta}}$ ,  $m$  and  $g$  are lower and upper truncation points.

### 5.10. Order Statistics of DTTF distribution

In reliability analysis and life testing of a component in quality control, order statistics and its moments consider noteworthy measures. Let  $X_1, X_2, X_3, \dots, X_n$  be a random sample of size  $n$  follows to DTTF distribution and  $\{X_{(1)} < X_{(2)} < X_{(3)} < \dots < X_{(n)}\}$  be consider as order statistics. The r.v's  $X_{(i)}, X_{(1)}, \tilde{X}_{(m)}, X_{(n)}$  be the  $i$ -th, minimum, median and maximum order statistics of DTTF distribution are followed by

***i*-th Order Statistic PDF is defined as**

$$f_{x_{(i)}}(x) = \frac{n!}{(i-1)!(n-i)!} [F(x)]^{i-1} [1-F(x)]^{n-i} f(x)$$

for  $i=1,2,3, \dots, n$ .

by equation (1) and equation (2), *i*-th order statistics PDF of DTTF distribution may obtain by

$$f_{x_{(i)}}(x) = \frac{n!}{(i-1)!(n-i)!} \left[ \begin{array}{l} \left( \frac{1}{C} \left\{ [(1+\lambda)e^{-\alpha x^{-\beta}} - \lambda e^{-2\alpha x^{-\beta}}] - \right\} \right)^{i-1} \\ \left( \frac{1}{C} \left\{ [(1+\lambda)e^{-\alpha m^{-\beta}} - \lambda e^{-2\alpha m^{-\beta}}] \right\} \right) \\ \times \left( 1 - \frac{1}{C} \left\{ [(1+\lambda)e^{-\alpha x^{-\beta}} - \lambda e^{-2\alpha x^{-\beta}}] - \right\} \right)^{n-i} \\ \times \frac{1}{C} \left[ (1+\lambda)\alpha\beta x^{-\beta-1} e^{-\alpha x^{-\beta}} - 2\alpha\beta\lambda x^{-\beta-1} e^{-2\alpha x^{-\beta}} \right] \end{array} \right]. \quad (43)$$

**Minimum Order Statistic PDF is defined as**

$$f_{x_{(1)}}(x) = n [1 - F(x)]^{n-1} f(x)$$

minimum order statistic of DTTF distribution is given by

$$f_{x_{(1)}}(x) = n \left[ \begin{array}{l} \left( 1 - \frac{1}{C} \left\{ [(1+\lambda)e^{-\alpha x^{-\beta}} - \lambda e^{-2\alpha x^{-\beta}}] - \right\} \right)^{n-1} \\ \times \frac{1}{C} \left[ (1+\lambda)\alpha\beta x^{-\beta-1} e^{-\alpha x^{-\beta}} - 2\alpha\beta\lambda x^{-\beta-1} e^{-2\alpha x^{-\beta}} \right] \end{array} \right]. \quad (44)$$

**Maximum Order Statistic PDF is defined as**

$$f_{x_{(n)}}(x) = n [F(x)]^{n-1} f(x)$$

maximum order statistic of DTTF distribution is given by

$$f_{x(n)}(x) = n \left[ \begin{array}{l} \left( \frac{1}{C} \left\{ \left[ (1+\lambda)e^{-\alpha x^{-\beta}} - \lambda e^{-2\alpha x^{-\beta}} \right] - \right\} \right)^{n-1} \\ \times \frac{1}{C} \left\{ \left[ (1+\lambda)\alpha\beta x^{-\beta-1} e^{-\alpha x^{-\beta}} - 2\alpha\beta\lambda x^{-\beta-1} e^{-2\alpha x^{-\beta}} \right] \right\} \end{array} \right]. \quad (45)$$

**Median Order Statistic PDF is defined as**

$$f_m(\tilde{x}) = \frac{(2m+1)!}{m!} [1 - F(\tilde{x})]^{m-1} [F(\tilde{x})]^m f(\tilde{x})$$

$$f_m(\tilde{x}) = \frac{(2m+1)!}{m!} \left[ \begin{array}{l} \left( 1 - \frac{1}{C} \left\{ \left[ (1+\lambda)e^{-\alpha \tilde{x}^{-\beta}} - \lambda e^{-2\alpha \tilde{x}^{-\beta}} \right] - \right\} \right)^{m-1} \\ \times \left( \frac{1}{C} \left\{ \left[ (1+\lambda)e^{-\alpha \tilde{x}^{-\beta}} - \lambda e^{-2\alpha \tilde{x}^{-\beta}} \right] - \right\} \right)^m \\ \times \frac{1}{C} \left\{ \left[ (1+\lambda)\alpha\beta \tilde{x}^{-\beta-1} e^{-\alpha \tilde{x}^{-\beta}} - 2\alpha\beta\lambda \tilde{x}^{-\beta-1} e^{-2\alpha \tilde{x}^{-\beta}} \right] \right\} \end{array} \right]. \quad (46)$$

### 5.11. Joint Distribution of DTTF distribution

The joint distribution of  $i$ -th and  $j$ -th order statistics of DTTF distribution is

$$f_{(x,y)_{(i,j)}}(x,y) = E[F(x_i)]^{i-1} [1 - F(y_j)]^{n-j} [F(y_j) - F(x_i)]^{j-i-1} f(x_i)f(y_j)$$

for  $i=1,2,3,\dots,n$ ,  $j=1,2,3,\dots,n$  and  $E = \frac{n!}{(i-1)!(n-j)!(j-i-1)!}$

$$f_{(x,y)_{(i,j)}}(x,y) = E \left[ \begin{array}{l} \left[ \frac{1}{C} \left\{ \left[ (1+\lambda)e^{-\alpha x_i^{-\beta}} - \lambda e^{-2\alpha x_i^{-\beta}} \right] - F \right\} \right]^{i-1} \\ \times \left[ 1 - \frac{1}{C} \left\{ \left[ (1+\lambda)e^{-\alpha y_j^{-\beta}} - \lambda e^{-2\alpha y_j^{-\beta}} \right] - F \right\} \right]^{n-j} \\ \times \left[ \frac{1}{C} \left\{ \left[ (1+\lambda)e^{-\alpha y_j^{-\beta}} - \lambda e^{-2\alpha y_j^{-\beta}} \right] - F \right\} - \right]^{j-i-1} \\ \times \frac{1}{C} \left\{ \left[ (1+\lambda)e^{-\alpha x_i^{-\beta}} - \lambda e^{-2\alpha x_i^{-\beta}} \right] - F \right\} \\ \times \frac{1}{C} \left[ (1+\lambda)\alpha\beta x_i^{-\beta-1} e^{-\alpha x_i^{-\beta}} - 2\alpha\beta\lambda x_i^{-\beta-1} e^{-2\alpha x_i^{-\beta}} \right] \\ \times \frac{1}{C} \left[ (1+\lambda)\alpha\beta y_j^{-\beta-1} e^{-\alpha y_j^{-\beta}} - 2\alpha\beta\lambda y_j^{-\beta-1} e^{-2\alpha y_j^{-\beta}} \right] \end{array} \right] \quad (47)$$

where  $x > 0$ ,  $\alpha, \beta > 0$ ,  $|\lambda| < 1$ ,  $C = [(1+\lambda)A - \lambda A^2] - [(1+\lambda)B - \lambda B^2]$ ,  $A = e^{-\alpha x^{-\beta}}$  and  $B = e^{-\alpha y^{-\beta}}$ ,  $m$  and  $g$  are lower and upper truncation points.



## 6. Estimation of Parameters, Simulation Study and Application of DTTF Distribution

### 6.1. Estimation of DTTF distribution

Parameters of the DTTF distribution are derived by the method of maximum likelihood. Here equation (2) is presented in a simplified way

$$f(x) = \frac{1}{C} \alpha \beta x^{-\beta-1} e^{-\alpha x^{-\beta}} \{(1 + \lambda) - 2\lambda e^{-\alpha x^{-\beta}}\},$$

likelihood function of DTTF distribution can be written as

$$L = \frac{\alpha^n \beta^n}{C^n} \prod_{i=1}^n (x_i)^{-\beta-1} \prod_{i=1}^n (e^{-\alpha x_i^{-\beta}}) \prod_{i=1}^n \{(1 + \lambda) - 2\lambda e^{-\alpha x_i^{-\beta}}\}, \quad (48)$$

log of equation (48) provides the log-likelihood function of DTTF distribution

$$l = LL = n[\ln \alpha + \ln \beta - \ln C] - (\beta + 1) \sum \ln x - \alpha \sum x^{-\beta} + \sum \ln [1 + \lambda - 2\lambda e^{-\alpha x^{-\beta}}], \quad (49)$$

partial derivatives of equation (49) w.r.t  $\alpha, \beta, \lambda$  and  $C$  yield

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} - \frac{n}{C} \frac{\partial C}{\partial \alpha} - \sum x^{-\beta} + \sum \left[ \frac{2\alpha \lambda e^{-\alpha x^{-\beta}} x^{-\beta}}{1 + \lambda - 2\lambda e^{-\alpha x^{-\beta}}} \right], \quad (50)$$

$$\frac{\partial l}{\partial \beta} = \frac{n}{\beta} - \frac{n}{C} \frac{\partial C}{\partial \beta} - \sum \ln x + \alpha \sum x^{-\beta} \ln x + \sum \left[ \frac{2\alpha \lambda e^{-\alpha x^{-\beta}} x^{-\beta} \ln x}{1 + \lambda - 2\lambda e^{-\alpha x^{-\beta}}} \right], \quad (51)$$

$$\frac{\partial l}{\partial \lambda} = -\frac{n}{C} \frac{\partial C}{\partial \lambda} + \sum \left[ \frac{1 - 2e^{-\alpha x^{-\beta}}}{1 + \lambda - 2\lambda e^{-\alpha x^{-\beta}}} \right], \quad (52)$$

where

$$\frac{\partial C}{\partial \alpha} = (1 + \lambda) \left[ -g^{-\beta} e^{-\alpha g^{-\beta}} + m^{-\beta} e^{-\alpha m^{-\beta}} \right] - \lambda \left[ -2g^{-\beta} e^{-2\alpha g^{-\beta}} + 2m^{-\beta} e^{-2\alpha m^{-\beta}} \right], \quad (53)$$

$$\frac{\partial C}{\partial \beta} = (1 + \lambda) \left[ \alpha e^{-\alpha g^{-\beta}} g^{-\beta} \ln g - \alpha e^{-\alpha m^{-\beta}} m^{-\beta} \ln m \right] - \lambda \left[ 2\alpha e^{-2\alpha g^{-\beta}} g^{-\beta} \ln g - 2\alpha e^{-2\alpha m^{-\beta}} m^{-\beta} \ln m \right], \quad (54)$$

$$\frac{\partial C}{\partial \lambda} = \left[ e^{-\alpha g^{-\beta}} - e^{-\alpha m^{-\beta}} \right] - \left[ e^{-2\alpha g^{-\beta}} - e^{-2\alpha m^{-\beta}} \right]. \quad (55)$$

One can find ML estimates  $(\hat{\alpha}, \hat{\beta}, \hat{\lambda})$  by solving these systems of non-linear equations simultaneously present in equation (50), (51), (52). Numerical results are developed by incorporating Statistical software R under the package AdequacyModel. Moreover, the second-order derivatives are required for hypothesis testing and interval estimation. For this, we require  $(3 \times 3)$  Fisher information matrix  $K(\varphi)$ .

$$K(\varphi) = \begin{bmatrix} \frac{\partial^2 l}{\partial \alpha^2} & \frac{\partial^2 l}{\partial \alpha \partial \beta} & \frac{\partial^2 l}{\partial \alpha \partial \lambda} \\ \frac{\partial^2 l}{\partial \beta \partial \alpha} & \frac{\partial^2 l}{\partial \beta^2} & \frac{\partial^2 l}{\partial \beta \partial \lambda} \\ \frac{\partial^2 l}{\partial \lambda \partial \alpha} & \frac{\partial^2 l}{\partial \lambda \partial \beta} & \frac{\partial^2 l}{\partial \lambda^2} \end{bmatrix}. \quad (53)$$

Since  $m$  and  $g$  are the lower and upper truncation points of density function of DTF distribution, as a result minimum and maximum value of the sample will be considered the estimate of  $m$  and  $g$ .

### 6.3. Simulation Study of DTF distribution

The study of simulation is conducted to assess the behavior of a finite sample. For DTF distribution we conduct a small scale experiment and different finite sample of sizes  $n=100, 200, 300, 400, 500$  and  $600$  are generated from equation (30). The performance of the estimates are (present in table 1 and table 2) evaluated based on Standard Errors (S.Es). Moreover, numerous statistics are calculated present in table 3 and table 4. Furthermore, 1000 of times simulation is performed to achieve the results and for this entire situation R (Statistical software) is incorporated.

**Table 1**  
**MLEs and Standard Errors in parenthesis at various sample sizes**  
*for  $\alpha=2.5, \beta =2.5, \lambda= 0.5$  as left and right truncation points are  $m=1.1, g=15.5$*

| Parameters      | Sample Size        |                    |                    |                    |                    |                    |
|-----------------|--------------------|--------------------|--------------------|--------------------|--------------------|--------------------|
|                 | 100                | 200                | 300                | 400                | 500                | 600                |
| $\hat{\alpha}$  | 1.6113<br>(0.9466) | 2.6744<br>(0.6982) | 2.7767<br>(0.5433) | 2.2997<br>(0.4665) | 2.3069<br>(0.3721) | 2.4979<br>(0.3378) |
| $\hat{\beta}$   | 2.0335<br>(1.1295) | 2.6820<br>(0.4573) | 2.8559<br>(0.3819) | 2.5394<br>(0.4685) | 2.4008<br>(0.3789) | 2.4793<br>(0.3599) |
| $\hat{\lambda}$ | 0.7236<br>(0.5923) | 0.3078<br>(0.4533) | 0.3445<br>(0.3275) | 0.4092<br>(0.4335) | 0.4772<br>(0.3147) | 0.4900<br>(0.2858) |

**Table 2**  
**MLEs and Standard Errors in parenthesis at various sample sizes**  
*for  $\alpha=3.5, \beta =1.5, \lambda= 0.5$  as left and right truncation points are  $m=1.1, g=75.5$*

| Parameters      | Sample Size        |                    |                    |                    |                    |                    |
|-----------------|--------------------|--------------------|--------------------|--------------------|--------------------|--------------------|
|                 | 100                | 200                | 300                | 400                | 500                | 600                |
| $\hat{\alpha}$  | 2.7837<br>(0.7406) | 4.1905<br>(0.7912) | 3.8676<br>(0.4956) | 3.3731<br>(0.4175) | 3.6364<br>(0.3429) | 3.5754<br>(0.3191) |
| $\hat{\beta}$   | 1.3710<br>(0.4501) | 1.9228<br>(0.2920) | 1.7402<br>(0.2114) | 1.5809<br>(0.2371) | 1.4952<br>(0.1936) | 1.5589<br>(0.1791) |
| $\hat{\lambda}$ | 0.5876<br>(0.5634) | 0.1601<br>(0.5247) | 0.3015<br>(0.3200) | 0.3414<br>(0.3873) | 0.4198<br>(0.2923) | 0.4075<br>(0.2667) |

**Table 3**  
**Descriptive Statistics at various sample sizes**  
 for  $\alpha=2.5, \beta =2.5, \lambda= 0.5$  as left and right truncation points are  $m=1.1, g=15.5$

| Descriptive Statistics | Sample Size |        |         |         |         |         |
|------------------------|-------------|--------|---------|---------|---------|---------|
|                        | 100         | 200    | 300     | 400     | 500     | 600     |
| CV%                    | 260.90      | 278.80 | 195.20  | 156.40  | 155.30  | 154.30  |
| Skewness               | 1.7150      | 1.7320 | 2.7250  | 4.8520  | 4.2340  | 4.7170  |
| Kurtosis               | 5.7280      | 5.8970 | 12.5700 | 39.1800 | 30.5400 | 36.5700 |
| AIC                    | 162.30      | 361.20 | 478.60  | 678.40  | 886.80  | 1036    |
| -Log-likelihood        | 78.150      | 177.60 | 236.30  | 336.20  | 440.40  | 515.0   |

**Table 4**  
**Descriptive Statistics at various sample sizes**  
 for  $\alpha=3.5, \beta =1.5, \lambda= 0.5$  as left and right truncation points are  $m=1.1, g=75.5$

| Descriptive Statistics | Sample Size |         |         |         |         |         |
|------------------------|-------------|---------|---------|---------|---------|---------|
|                        | 100         | 200     | 300     | 400     | 500     | 600     |
| CV%                    | 136.70      | 124.30  | 94.80   | 69.92   | 71.55   | 69.50   |
| Skewness               | 2.1550      | 2.6620  | 3.8050  | 6.8350  | 5.9070  | 6.6070  |
| Kurtosis               | 7.6720      | 12.3500 | 21.8500 | 67.1200 | 52.7800 | 62.5200 |
| AIC                    | 381.60      | 734.20  | 1134.0  | 1560.0  | 1995.0  | 2360.0  |
| -Log-likelihood        | 187.80      | 364.20  | 564.10  | 776.90  | 994.60  | 1177.0  |

Table 1 and tables 2 represent the MLEs along with the standard errors in the parenthesis. We observe the decreasing behavior of standard errors as the size of the sample increases. Furthermore, Table 3, 4 represent various statistics computed on different simulated sample sizes. One can see in table 3, 4 the decreasing trend of coefficient of variation (CV) when the size of sample increases. Moreover, the additional statistics including skewness, kurtosis, Akaike Information Criterion (AIC) and negative Log-likelihood, increase with increase of sample size. All above conditions in the support of proposed model to declare that MLE estimates of DTF distribution are consistent in their performance and work quit well.

### 6.3. Application of DTF distribution

In this section, flexibility and potentiality of DTF distribution is demonstrated by experiencing and integrating two suitable lifetime datasets. The first dataset presented by Ghitany *et al.* (2008), entitled waiting time (in minutes) before the customer received service in a bank and the second dataset presented by Nadarajah (2007a) entitled, the daily ozone measurements in New York, May-September 1973. The results of DTF distribution and its competing models (Double Truncated Transmuted Fréchet (DTrF), Transmuted Fréchet (TF) and Fréchet (F)) are evaluated based on  $-LL$ , Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC) and Kolmogorov-Smirnov (K-S). Furthermore, least values of declared statistics lead to consider a model as a reasonable fit for the dataset.

**Competing Models (x > 0):**

Double Truncated Fréchet (DTrF) Distribution: Abid (2016)

$$G(x) = \frac{e^{-\alpha x^{-\beta}} - e^{-\alpha m^{-\beta}}}{e^{-\alpha g^{-\beta}} - e^{-\alpha m^{-\beta}}}, \text{ for } \alpha, \beta > 0, m < x \text{ and } g > x,$$

Transmuted Fréchet (TF) Distribution: Mahmoud and Mandouh (2013)

$$G(x) = (1 + \lambda)e^{-\alpha x^{-\beta}} - \lambda e^{-2\alpha x^{-\beta}}, \text{ for } \alpha, \beta > 0 \text{ and } |\lambda| < 1,$$

Fréchet (F) Distribution: Fréchet (1927)

$$G(x) = e^{-\alpha x^{-\beta}}, \text{ for } \alpha, \beta > 0.$$

**6.3.1 Data Set-1:** Waiting time (in minutes) before the customer received service in a bank discussed by Ghitany *et al.* (2008).

**Table 3**  
**Descriptive Statistics**

| Mean   | Median | S.D    | Variance | Skewness | Kurtosis |
|--------|--------|--------|----------|----------|----------|
| 9.8770 | 8.1    | 7.2370 | 52.3700  | 1.4730   | 5.5400   |

**Table 4**  
**Parameter Estimates and Information Criterion**

$m = 0.8$  and  $g=38.5$  be the lower and upper truncation points

| Models | Coefficients<br>(Standard Error) |                       |                        | Information Criterion |               |               |             |
|--------|----------------------------------|-----------------------|------------------------|-----------------------|---------------|---------------|-------------|
|        | $\hat{\alpha}$                   | $\hat{\beta}$         | $\hat{\lambda}$        | -LL                   | AIC           | BIC           | K-S         |
| DTTF   | <b>3.99</b><br>(0.94)            | <b>1.01</b><br>(0.12) | <b>-0.77</b><br>(0.19) | <b>319.49</b>         | <b>644.99</b> | <b>651.24</b> | <b>0.07</b> |
| DTrF   | 5.52<br>(0.79)                   | 0.87<br>(0.11)        | -                      | 320.92                | 645.84        | 651.50        | 0.08        |
| TF     | 4.93<br>(0.91)                   | 1.28<br>(0.09)        | -0.76<br>(0.15)        | 330.63                | 667.27        | 675.09        | 0.10        |
| F      | 6.53<br>(0.89)                   | 1.16<br>(0.07)        | -                      | 334.38                | 672.76        | 677.97        | 0.12        |

**6.3.2. Data Set-2:** The daily Ozone measurements in New York, May-September1973 presented by Nadarajah (2007a).

**Table 5**  
**Descriptive Statistics**

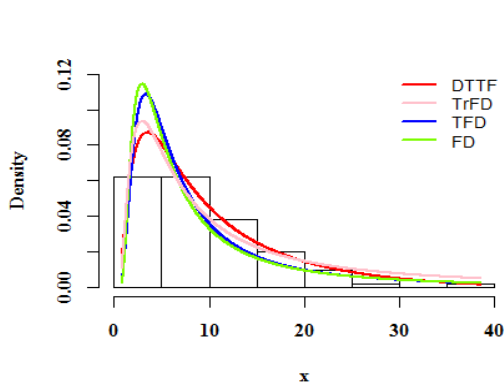
| Mean  | Median | S.D   | Variance | Skewness | Kurtosis |
|-------|--------|-------|----------|----------|----------|
| 42.13 | 31.50  | 32.99 | 1088     | 1.2260   | 4.1840   |

**Table 6**  
**Parameter Estimates and Information Criterion**

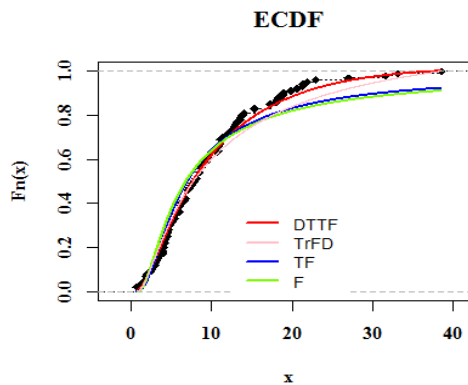
$m = 1$  and  $g=168$  be lower and upper truncation points

| Models   | Coefficients<br>(Standard Error) |                       |                        | Information Criterion |                |                |             |
|----------|----------------------------------|-----------------------|------------------------|-----------------------|----------------|----------------|-------------|
|          | $\hat{\alpha}$                   | $\hat{\beta}$         | $\hat{\lambda}$        | $-LL$                 | AIC            | BIC            | K-S         |
| DTTF     | <b>8.48</b><br>(1.98)            | <b>0.82</b><br>(0.09) | <b>-0.92</b><br>(0.08) | <b>542.98</b>         | <b>1091.96</b> | <b>1100.22</b> | <b>0.07</b> |
| DT $r$ F | 11.92<br>(2.27)                  | 0.71<br>(0.08)        | -                      | 545.50                | 1094.99        | 1100.51        | 0.08        |
| TF       | 13.17<br>(2.56)                  | 1.07<br>(0.06)        | -0.92<br>(0.08)        | 560.12                | 1126.24        | 1134.51        | 0.12        |
| F        | 17.99<br>(2.99)                  | 0.97<br>(0.06)        | -                      | 566.38                | 1136.75        | 1142.26        | 0.14        |

**PDF and CDF plots drafted over empirical histogram for dataset -1**



**Fig. 9**



**Fig. 10**

**PDF and CDF plots drafted over empirical histogram for dataset-2**

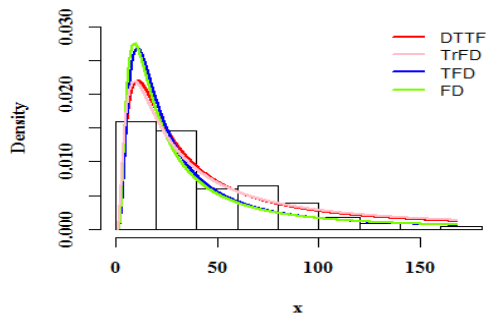


Fig. 11

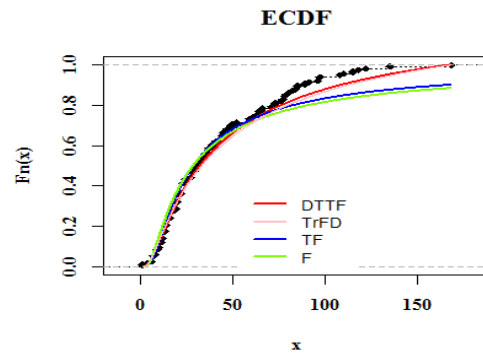


Fig. 12

**Total Test Time (TTT) plots for dataset-1 and dataset-2**

**TTT Plot for dataset- 1**

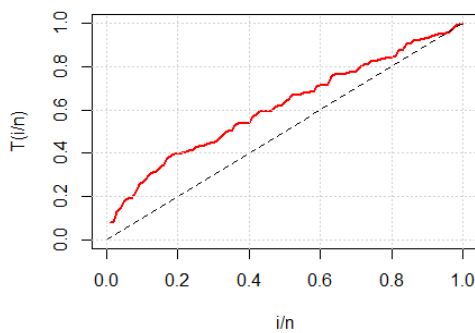


Fig. 13

**TTT Plot for dataset- 2**

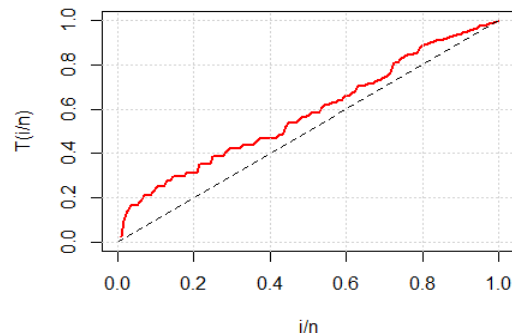


Fig. 14

Various descriptive and empirical results are presented in table 1 to table 6. Table 4, 6 present the empirical results of fitted models comprising Double Truncated Transmuted Fréchet (DTrF), Double Truncated Fréchet (DTTrF), Transmuted Fréchet (TF) and Fréchet (F) distribution. The Statistical software R is incorporated for the results present in table 1 to 6. Since the minimum results of  $-LL$ , AIC, BIC or K-S is the criteria to select the better fit model and results (in table 4, 6) are in the support of DTrF distribution. Consequently, we do not hesitate anymore to declare that DTrF distribution is a better fitted model on both the datasets as compared to its competing models. Furthermore, one can see the empirically fitted PDF (Fig. 9 and Fig. 11) and CDF (Fig. 10 and Fig. 12) plots of DTrF distribution display the close fit to the empirical histogram. Fig.13 and Fig. 14 represents the plots of total test time (TTT), proposed by Aarset (1987) may be used as a tool for obtaining empirical behavior of failure rate of the DTrF distribution.

**7. Conclusion**

The present study is conducted to provide supportive and more fixable results than its competing models on skewed datasets. Diverse probabilistic and reliability measures along with Rényi entropy and order statistics are developed and discussed. The method of MLE is suggested to derive the estimates and execution of the estimates is assessed by simulation study. The application of DTrF distribution is illustrated by two real-time datasets.

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