On Discrete Wrapped Cauchy Model

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Abstract

Modeling angular data throws many challenges in practical situations. Good number of circular / semicircular models are developed for modeling continuous circular / angular data. Scant attention was paid in analysis of discrete angular data, in particular, construction of discrete angular models for fitting angular data is not touched so far. Hence an attempt is made to develop method for constructing Discrete l - axial models and study their population characteristics.

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1. Introduction

In some of the cases the directional / angular data does not require full circular models for modeling, this fact is noted in Guardiola (2004), Jones (1968) and Byoung et al (2008). For example, in search of a nesting site on dry land sea turtles come out from the ocean, a random variable with points on a semicircle is well adequate for modeling such data. Also, if an aircraft is lost and its departure and its initial headings are identified, a semicircular random variable is adequate for modeling such axial / angular data. And some more examples of semicircular data are presented in Ugai et al (1977). Phani et al (2013) derived certain semicircular distributions by employing inverse stereographic projection.

Good number of circular / semicircular models are developed for modeling continuous circular / angular data. Dattatreya rao et al (2007) constructed wrapped versions of Lognormal, Logistic, Weibull and Extreme - value distributions. Scant attention was paid in analysis of discrete angular data, in particular, construction of discrete angular models for fitting angular data is not touched so far. Hence an attempt is made to develop method for constructing Discrete l - axial models and study their population characteristics.

For statistical analysis of discrete directional data existing linear distributions will not yield suitable results. Srihari et al (2018a; 2018b; 2018c) derived the Discrete Wrapped Exponential Distribution, the Wrapped Negative Binomial model and the Wrapped Logarithmic Distribution respectively. Here an attempt is made to construct a new discrete circular model by applying the method of discretization for existing continuous circular model. By considering the Wrapped Cauchy Distribution, discretization is employed to transform it as Discrete Wrapped Cauchy distribution which is a circular model. The probability mass function, distribution function and characteristic function of the new discrete circular model are obtained and population characteristics are studied . Discrete Wrapped l - axial Cauchy distribution by extending the Discrete Wrapped Cauchy distribution is derived

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which is suitable for modeling discrete axial data.

2. Circular Distributions

A circular distribution is a probability distribution whose total probability is defined on the unit circle. Since each point on the unit circle represents a direction, it is a way of assigning probabilities to different directions. The range of a circular random variable θ measured in radians, can be taken as $[0, 2\pi)$ or $[-\pi, \pi)$.

Circular distributions are of two types, they may be discrete, assigning probabilities to a finite number of directions or continuous assigning probabilities to a infinite number of directions. One of the construction methods of circular models is wrapping a linear distribution around a unit circle. Here Wrapping methodology for the construction of Discrete Circular model is discussed.

2.1. Discrete Wrapped Circular Random Variables

If *X* is a discrete random variable on the set of integers, then reduction modulo $2\pi m\left(m \in \mathbf{Z}^{+}\right)$ wraps the integers on to the group of m^{th} roots of unity which is a sub group of unit circle.

$$\theta = 2\pi x\left(\bmod 2\pi m\right)$$

Since θ contains a finite number of elements they are denoted by $\theta = \left\{\frac{2\pi r}{m} \middle/ r = 0,1,2,\ldots m-1\right\}$ which is lattice on the unit circle.

2.2. Probability Mass Function

Suppose if θ is a wrapped discrete circular random variable then probability mass function of θ is denoted by $Pr\left(\theta = \frac{2\pi r}{m}\right)$ which satisfies the following properties

1. $Pr\left(\theta = \frac{2\pi r}{m}\right) \geq 0$

2. $\sum_{r=0}^{m-1} Pr\left(\theta = \frac{2\pi r}{m}\right) = 1$

3. $Pr(\theta) = Pr(\theta + 2\pi l)$ For any integer l (i.e Pr is periodic)

2.3. Distribution Function

Suppose if θ is a wrapped discrete circular random variable then distribution function of θ is denoted by $F_w(\theta)$ and it is defined as

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$$F_w(\theta)=\sum_{r=0}^{k} Pr\left(\theta=\frac{2\pi r}{m}\right) \qquad \text{where } k=0,1,\ldots m-1$$

In mathematics, discretization is the process of transforming continuous functions, models, variables, and equations into its discrete counterparts. This process is usually carried out to model a given discrete phenomenon depending on the practical situation.

3. Discretization of a Continuous Distribution

In this paper, we adopt discretization from Min – Zhen Wang and Kunio Shimizu (2014) to construct discrete circular models by transforming existing continuous circular models. The process is as follows

Consider a continuous circular distribution with probability density function (pdf) $f(\theta)$ on the circle to construct a probability mass function $Pr\left(\theta=\frac{2\pi r}{m}\right)$ on a set of the equally spaced points for $r=0,1,2\ldots m-1$ with a fixed integer m equal or greater than 2.

$Pr\left(\theta=\frac{2\pi r}{m}\right)=f\left(\frac{2\pi r}{m}\right)\Big/\sum_{r=0}^{m-1} f\left(\frac{2\pi r}{m}\right)$ is probability mass function(pmf) of respective discrete circular model. $m=1$ is excluded as it is degenerate.

4. Discrete Wrapped Cauchy Distribution

The probability density function of wrapped Cauchy distribution is defined as

$$f(\theta)=\frac{1}{2\pi}\frac{1-\rho^2}{1+\rho^2-2\rho\cos(\theta-\mu)} \quad \text{where } \mu,\sigma \text{ are parameters}$$
$$\text{and } \rho=e^{-\sigma}(\sigma>0) \text{ and } 0\le\theta,\mu<2\pi \tag{4.1}$$

Now the probability mass function of the Discrete Wrapped Cauchy Distribution is defined as

$$g_W(\theta)=Pr\left(\theta=\frac{2\pi r}{m}\right)=\frac{\frac{1}{2\pi}\frac{1-\rho^2}{1+\rho^2-2\rho\cos(\theta-\mu)}}{\sum_{r=0}^{m-1}\frac{1}{2\pi}\frac{1-\rho^2}{1+\rho^2-2\rho\cos(\theta-\mu)}}$$

$$= \frac{\frac{1}{1+\rho^2-2\rho\cos(\theta-\mu)}}{\sum_{r=0}^{m-1}\frac{1}{1+\rho^2-2\rho\cos(\theta-\mu)}} \tag{4.2}$$

Consider $\sum_{r=0}^{m-1}\frac{1}{1+\rho^2-2\rho\cos(\theta-\mu)} = \frac{1}{1+\rho^2-2\rho\cos\mu} + \frac{1}{1+\rho^2-2\rho\cos\left(\frac{2\pi}{m}-\mu\right)} + ...$

$$+\frac{1}{1+\rho^2-2\rho\cos\left(\frac{2\pi(m-1)}{m}-\mu\right)}$$

$$=\frac{m\left(1-\rho^{2m}\right)}{\left(1-\rho^2\right)\left(1+\rho^{2m}-2\rho^m\cos m\mu\right)}$$

Hence $g_W(\theta) = Pr\left(\theta = \frac{2\pi r}{m}\right) = \frac{1}{m}\frac{\left(1-\rho^2\right)\left(1+\rho^{2m}-2\rho^m\cos m\mu\right)}{\left(1-\rho^{2m}\right)\left(1+\rho^2-2\rho\cos(\theta-\mu)\right)}$

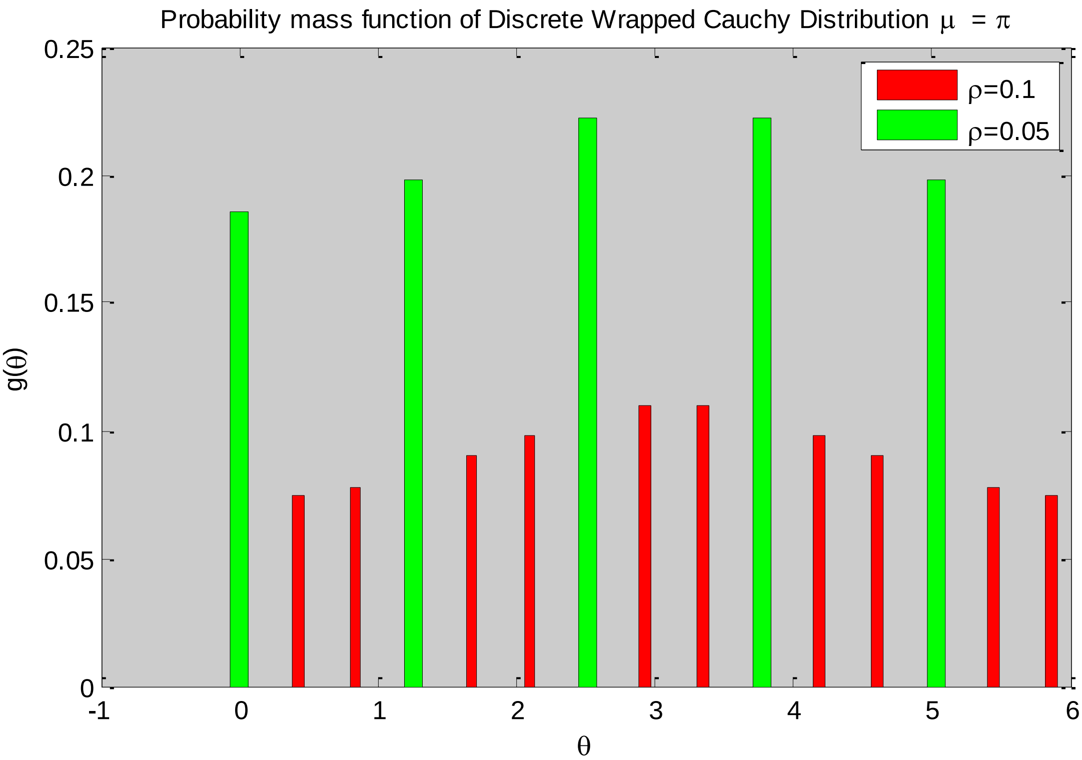

Fig 1 Graph for Probability mass function of Discrete Wrapped Cauchy Distribution

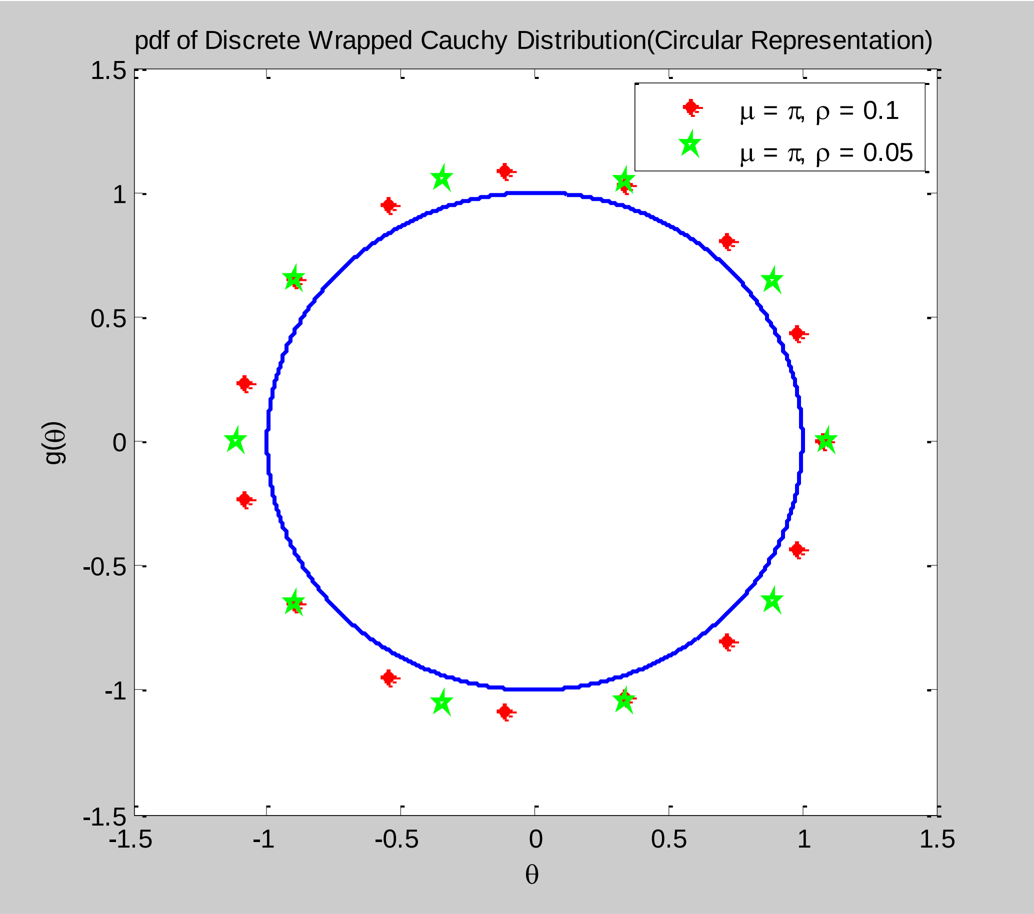

Fig 2 Graph for Probability mass function of Discrete Wrapped Cauchy Distribution (Circular representation)

5. Distribution Function of Discrete Wrapped Cauchy Distribution

The Distribution function of Discrete Wrapped Cauchy Distribution is defined as

$$G_w(\theta) = \sum_{r=0}^{k} Pr\left(\theta = \frac{2\pi r}{m}\right) = \sum_{r=0}^{k} \left(\frac{\dfrac{1}{1+\rho^2 - 2\rho\cos\left(\dfrac{2\pi r}{m} - \mu\right)}}{\displaystyle\sum_{r=0}^{m-1} \frac{1}{1+\rho^2 - 2\rho\cos(\theta-\mu)}} \right)$$

$$\text{where } 0 \le k \le m-1 \tag{5.1}$$

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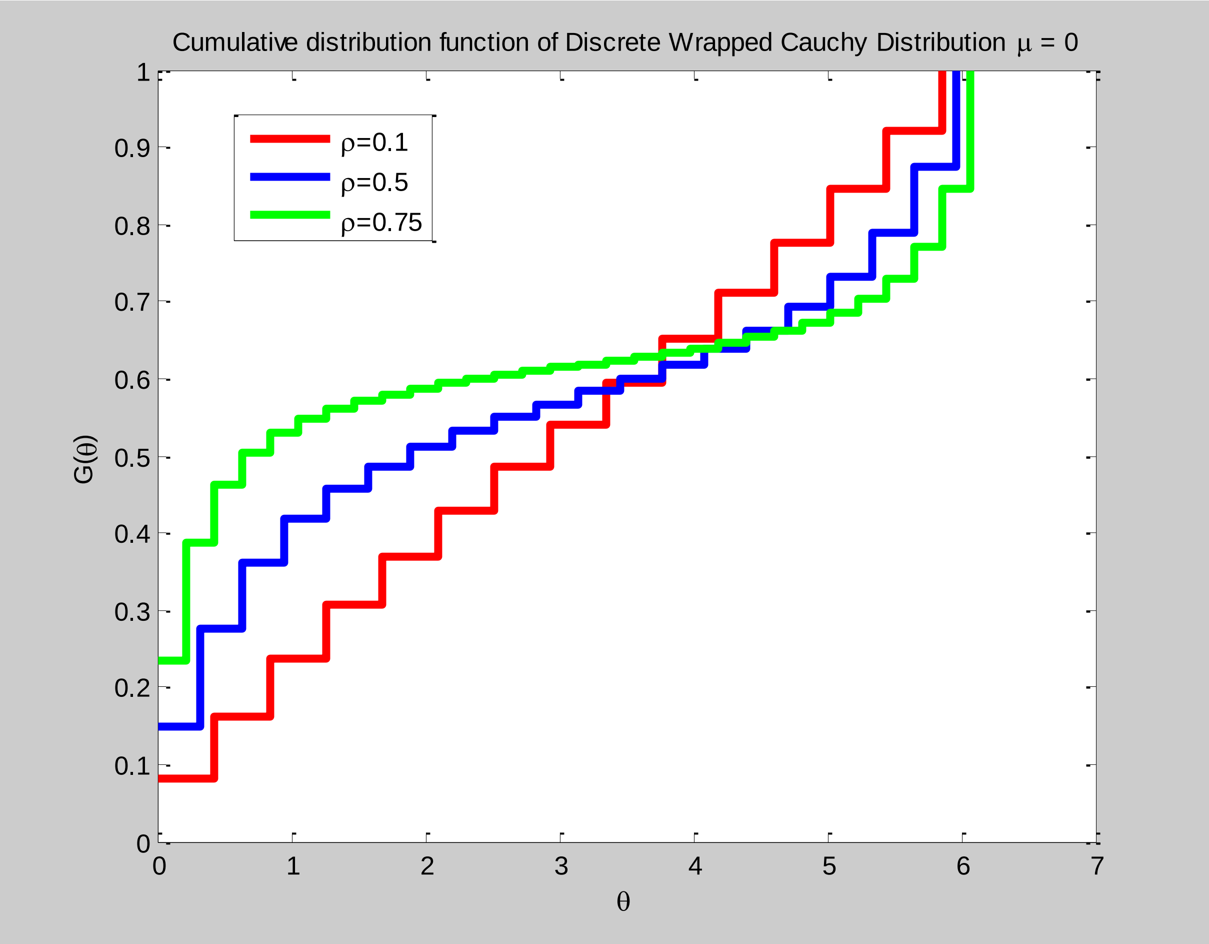

Fig 3 Graph for Cumulative distribution function of Discrete Wrapped Cauchy Distribution

6. Characteristic function of the Discrete Wrapped Cauchy Distribution

The characteristic function of the Discrete Wrapped Cauchy Distribution is defined as $\varphi_\theta(p) = E(e^{ip\theta})$ where $p \in ¢$

$$= \sum_{r=0}^{m-1} e^{ip\theta} Pr\left(\theta = \frac{2\pi r}{m}\right)$$

$$= \frac{1}{m}\frac{\left(1-\rho^2\right)\left(1+\rho^{2m}-2\rho^m\cos m\mu\right)}{\left(1-\rho^{2m}\right)}\sum_{r=0}^{m-1}\frac{e^{ip\theta}}{1+\rho^2-2\rho\cos(\theta-\mu)},\quad m \in ¥ \qquad (6.1)$$

The real and imaginary parts α_p and β_p respectively of the characteristic function are known as p^{th} trigonometric moments and population characteristics are evaluated using first two trigonometric moments.

$$\alpha_p = \frac{1}{m}\frac{\left(1-\rho^2\right)\left(1+\rho^{2m}-2\rho^m\cos m\mu\right)}{\left(1-\rho^{2m}\right)}\sum_{r=0}^{m-1}\frac{\cos p\theta}{1+\rho^2-2\rho\cos(\theta-\mu)},\quad m \in ¥ \qquad (6.2)$$

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$$\beta_p = \frac{1}{m}\frac{\left(1-\rho^2\right)\left(1+\rho^{2m}-2\rho^m\cos m\mu\right)}{\left(1-\rho^{2m}\right)}\sum_{r=0}^{m-1}\frac{\sin p\theta}{1+\rho^2-2\rho\cos(\theta-\mu)},\quad m\in ¥ \tag{6.3}$$

Clearly $\rho_p = \sqrt{\alpha_p^2+\beta_p^2}$ and $\mu_p = \tan^{-1}\left[\frac{\beta_p}{\alpha_p}\right]$

Now μ_1 represents mean direction and $\rho_1 (0 \le \rho_1 \le 1)$ represents concentration towards the mean direction

In general $\mu_1=\mu$ and $\rho_1=\rho$

With first two trigonometric moments population characteristics are evaluated using MATLAB.

7. Discrete Wrapped *l* - axial Cauchy Distribution

We extend the above **Discrete Wrapped Cauchy Distribution which is a circular model** to the l -axial distribution, which is applicable to any arc of arbitrary length say $2\pi/l$ for $l=1,2,...,$. So it is desirable to extend the **Discrete Wrapped Cauchy Distribution** to construct the **Discrete Wrapped *l* - axial Cauchy Distribution**, we consider the density function of **Discrete Wrapped Cauchy Distribution** and use the transformation $\phi=\frac{\theta}{l}$, $l=1,2,...,$. The probability mass function of ϕ is given by

$$g_W(\phi)=Pr\left(\phi=\frac{2\pi r}{lm}\right)=\frac{l}{m}\frac{\left(1-\rho^2\right)\left(1+\rho^{2m}-2\rho^m\cos ml\mu\right)}{\left(1-\rho^{2m}\right)\left(1+\rho^2-2\rho\cos\left(l\left(\phi-\mu\right)\right)\right)}\ ,\ m\in ¥\ ,$$

$$0\le\phi,\mu<\frac{2\pi}{l},\ \rho=e^{-\sigma}(\sigma>0)\,\text{and}\,l=1,2,.. \tag{7.1}$$

It is coined as **Discrete Wrapped *l* - axial Cauchy Distribution**

Case (1) When $l=1$, the probability density function is

$$g_W(\phi)=Pr\left(\phi=\frac{2\pi r}{m}\right)=\frac{1}{m}\frac{\left(1-\rho^2\right)\left(1+\rho^{2m}-2\rho^m\cos m\mu\right)}{\left(1-\rho^{2m}\right)\left(1+\rho^2-2\rho\cos\left(\phi-\mu\right)\right)}$$

$$m\in ¥,\ 0\le\phi<2\pi,\ \rho=e^{-\sigma}(\sigma>0) \tag{7.2}$$

It is called the **Discrete Wrapped Circular Cauchy Distribution**.

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Case (2) When $l = 2$, the probability density function is the **Discrete Wrapped Semicircular Cauchy Distribution**

$$g_W(\phi) = Pr\left(\phi = \frac{\pi r}{m}\right) = \frac{2}{m}\,\frac{\left(1-\rho^2\right)\left(1+\rho^{2m}-2\rho^m\cos 2m\mu\right)}{\left(1-\rho^{2m}\right)\left(1+\rho^2-2\rho\cos\left(2\left(\phi-\mu\right)\right)\right)}\;,\; m \in ¥\,,$$

$$0 \le \phi, \mu < \pi,\; \rho = e^{-\sigma}(\sigma > 0) \qquad (7.3)$$

Conclusion: A new discrete circular model is constructed which was not figured out so far. Its pdf , cdf and characteristic function are derived and population characteristics are studied. This discrete circular model is extended to its l – axial model.

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