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Common Fixed Point Theorem in 2-Menger Space via (S-B) Property

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Abstract
In this paper, first we prove a common fixed point theorem using weakly compatible mapping in 2- Menger space which generalize the well known results. Secondly, we prove a common fixed point theorem using (S-B) property along with weakly compatible maps. (S-B) property defined by Sharma and Bamoria [16] via implicit relation.
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1. INTRODUCTION AND PRELIMINARIES

In 1922, Banach proved the principal contraction result [4]. As we know, there have been published many works about fixed point theory for different kinds of contractions on some spaces such as quasi-metric spaces, cone metric spaces, convex metric spaces, partially ordered metric spaces, G-metric spaces, partial metric spaces, quasi-partial metric spaces, fuzzy metric spaces and Menger spaces.

The study of 2-metric spaces was initiated by Gahler[7] and some fixed point theorems in 2-metric spaces were proved in [8],[9], [10] and [15]. In 1987, Zeng [23] gave the generalization of 2-metric to Probabilistic 2-metric as follows;

A probabilistic metric space shortly PM-Space, is an ordered pair (X, F) consisting of a non empty set X and a mapping F from $X \times X$ to L, where L is the collection of all distribution functions (a distribution function F is non decreasing and left continuous mapping of reals in to $[0,1]$ with properties, $\inf F(x) = 0$ and $\sup F(x) = 1$).

1. The value of F at $(x, y) \in X \times X$ is represented by $F_{x,y}$. The function $F_{x,y}$ are assumed satisfy the following conditions;
2. (FM-0) $F_{x,y}(t) = 1$, for all $t > 0$, iff $x = y$;
3. (FM-1) $F_{x,y}(0) = 0$, if $t = 0$;
4. (FM-2) $F_{x,y}(t) = F_{y,x}(t)$;
5. (FM-3) $F_{x,y}(t) = 1$ and $F_{y,z}(s) = 1$ then $F_{x,z}(t + s) = 1$.
6. A mapping $T: [0,1] \times [0,1] \to [0,1]$ is a t-norm, if it satisfies the following conditions;
7. (FM-4) $T(a, 1) = a$ for every $a \in [0,1]$;
8. (FM-5) $T(0, 0) = 0$,
9. (FM-6) $T(a, b) = T(b, a)$ for every $a, b \in [0,1]$;
10. (FM-7) $T(c, d) \geq T(a, b)$ for $c \geq a$ and $d \geq b$
11. (FM-8) $T(T(a, b), c) = T(a, T(b, c))$ where $a, b, c, d \in [0,1]$.
12. A Menger space is a triplet (X, F, T), where (X, F) is a PM-Space, X is a non-empty set and a $t-$ norm satisfying instead of (FM-8) a stronger requirement.
13. (FM-9) $F_{x,z}(t + s) \geq T\left(F_{x,y}(t), F_{y,z}(s)\right)$ for all $x \geq 0, y \geq 0$.
14. For a given metric space (X, d) with usual metric d, one can put $F_{x,y}(t) = H(t - d(x, y))$ for all $x, y \in X$ and $t > 0$. where H is defined as:
$$H(t) = \begin{cases} 1 & \text{if } s > 0, \\ 0 & \text{if } s \leq 0. \end{cases}$$
and t-norm T is defined as $T(a, b) = \min\{a, b\}$.

For the proof of our result we required the following definitions.

Definition 1.1 :-A triangular norm $*$(shortly t-norm) is a binary operation on the unit interval $[0,1]$ such that for all $a, b, c, d \in [0,1]$ the following conditions are satisfied:

(1) $a * 1 = a$,
(2) $a * b = b * a$,
(3) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$,

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(4) $a * (b * c) = (a * b) * c$.

Examples of t-norms are $a * b = \min\{a, b\}$, $a * b = ab$ and $a * b = \max\{a + b - 1, 0\}$.

Definition 1.2 :- Let $(X, F, *)$ be a Menger space and $*$ be a continuous t-norm.

(a) A sequence $\{x_n\}$ in X is said to be converge to a point x in X (written $x_n \to x$) iff for every $\varepsilon > 0$ and $\lambda \in (0, 1)$, there exists an integer $n_0 = n_0(\varepsilon, \lambda)$ such that $F_{x_n, x}(\varepsilon) > 1 - \lambda$ for all $n \geq n_0$.

(b) A sequence $\{x_n\}$ in X is said to be Cauchy if for every $\varepsilon > 0$ and $\lambda \in (0, 1)$, there exists an integer $n_0 = n_0(\varepsilon, \lambda)$ such that $F_{x_n, x_{n+p}}(\varepsilon) > 1 - \lambda$ for all $n \geq n_0$ and $p > 0$.

(c) A Menger space in which every Cauchy sequence is convergent is said to be complete.

Remark 1.3:- If $*$ is a continuous t-norm, it follows from $(FM - 4)$ that the limit of sequence in Menger space is uniquely determined.

Definition 1.4:- Self maps A and B of a Menger space $(X, F, *)$ are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, i.e. if $Ax = Bx$ for some $x \in X$ then $ABx = BAx$.

Weakly Compatible Maps

In 1982, Sessa [17], weakened the concept of commutativity to weakly commuting mappings. Afterwards, Jungck [4] enlarged the concept of weakly commuting mappings by adding the notion of compatible mappings. In 1991, Mishra [16] introduced the notion of compatible mappings in the setting of probabilistic metric space.

Definition 1.5 :- Self maps A and B of a Menger space $(X, F, *)$ are said to be compatible if $F_{ABx_m, BAx_n}(t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n \to x$, $Bx_n \to x$ for some x in X as $n \to \infty$.

Definition 1.6:- Let S and T be weakly compatible of a Menger space $(X, M, *)$ and $Su = Tu$ for some u in X then

$$STu = TSu = SSu = TTu.$$

Definition 1.7:- (Implicit Relation) Let ϕ_4 be the set of real and continuous function from $\left(R^{+}\right)^4 \to R$ so that

(i) ϕ is non-increasing in $2^{nd}, 3^{rd}$ argument and

(ii) For $u, v \geq 0$ $\phi(u, v, v, v) \geq 0 \Rightarrow u \geq v$

Example 1.8:- Let $X = [0, 3]$ be equipped with the usual metric $d(x, y) = |x - y|$ Define $f, g: [0,3] \to [0,3]$ by

$$f(x) = \begin{cases} x & \text{if } x \in [0,1), \\ 3 & \text{if } x \in [1,3]. \end{cases}$$

And

$$g(x) = \begin{cases} 3 - x & \text{if } x \in [0,1), \\ 3 & \text{if } x \in [1,3]. \end{cases}$$

Then for any $x \in [1,3]$, x is a coincidence point and $fgx = gfx$, showing that f, g are weakly compatible maps on $[0, 3]$.

Lemma 1.9:- Let $(X, M, *)$ be a Menger space. Then for all $x, y \in X$, $M(x, y, .)$ is a non-decreasing function.

Lemma 1.10:- Let $(X, M, *)$ be a Menger space. If there exists $k \in (0, 1)$ such that for all $x, y \in X$

$$M_{x,y}(t) \geq M_{x,y}(t) \quad \forall\, t > 0$$

then $x = y$.

Lemma 1.11:- Let $\{x_n\}$ be a sequence in a Menger space $(X, M, *)$. If there exists a number $k \in (0, 1)$ such that

$$M_{x_{n+2}, x_{n+1}}(kt) \geq M_{x_{n+1}, x_n}(t) \ \forall\, t > 0 \text{ and } n \in N.$$

Then $\{x_n\}$ is a Cauchy sequence in X.

Lemma 1.12:- The only t-norm $*$ satisfying $r * r \geq r$ for all $r \in [0, 1]$ is the minimum t-norm, that is $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$.

Lemma 1.13:- Let $(X, M, *)$ be a Menger space and $\forall\ x, y \in X, t > 0$ and if for a number $k \in (0,1)$, $M(x, y, kt) \geq M(x, y, t)$ then $x = y$.

Example 1.14:- Let (X, d) be a metric space. Define $a * b = \min\{a, b\}$ and $M_{x,y}(t) = \frac{t}{t + d(x,y)}$, for all $x, y \in X$.and all $t > 0$. Then $(X, M, *)$ is a Menger space. It is called the Menger space induced by d.

Remark 1.15:- If self maps A and B of a Menger space $(X, F, *)$ are compatible then they are weakly compatible.

2. MAIN RESULT

Now we prove the following results:

Theorem 2.1: Let $(X, M, *)$ be a common fixed point theorem in 2- Menger space with compatible maps. Let

A, B, S and T be mappings of X into itself satisfying following conditions:

(2.1) $AX \subset TX$ and $BX \subset SX$

(2.2) $\{A, S\}$ or $\{B, T\}$ satisfy the (S-B) property

(2.3) there exists a constant $q \in (0,1)$ such that $x, y, a \in X$ and $t > 0$,

$$\alpha\left(M_{Ax,By,a}(qt) * \frac{M_{Sx,Ty,a}(t) + M_{Ax,Sx,a}(t)}{2} * \frac{M_{By,Ty,a}(t) + M_{Ax,Ty,a}(t)}{2}\right) \geq 0 \quad \textbf{(2.1.1)}$$

(2.4) If the pairs $\{A, S\}$ or $\{B, T\}$ are weakly compatible

(2.5) One of $A(X), B(X), S(X)$ or $T(X)$ is closed subset of X.

Indeed, A, B, S and T have a unique common fixed point in X.

Proof. Suppose that $\{B, T\}$ satisfies the (S-B) property. Then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

Since $BX \subset SX$, there exists in X a sequence $\{y_n\}$ such that $Bx_n = Sy_n$.

Hence $\lim_{n\to\infty} Sx_n = z$.

Let us show that $\lim_{n\to\infty} Ay_n = z$.

Now by equation (2.1.1), we have

$$\alpha\left(M_{Ay_n,Bx_n,a}(qt) * \frac{M_{Sy_n,Tx_n,a}(t) + M_{Ay_n,Sy_n,a}(t)}{2} * \frac{M_{Bx_n,Tx_n,a}(t) + M_{Ay_n,Tx_n,a}(t)}{2}\right) \geq 0$$

$$\alpha\left(M_{Ay_n,Bx_n,a}(qt) * \frac{M_{Bx_n,Tx_n,a}(t) + M_{Ay_n,Bx_n,a}(t)}{2} * \frac{M_{Bx_n,Tx_n,a}(t) + M_{Ay_n,Tx_n,a}(t)}{2}\right) \geq 0$$

Since $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n$

$\therefore M(Bx_n, Tx_n, t) = 1$

So taking $\lim n \to \infty$

$$\alpha\left(M_{Ay_n,Bx_n,a}(qt) * \frac{1 + M_{Ay_n,Bx_n,a}(t)}{2} * \frac{1 + M_{Ay_n,Bx_n,a}(t)}{2}\right) \geq 0$$

ϕ is non-increasing in $2^{nd}, 3^{rd}$ argument

$$\alpha\left(M_{Ay_n,Bx_n,a}(qt) * M_{Ay_n,Bx_n,a}(t) * M_{Ay_n,Bx_n,a}(t)\right) \geq 0$$

By the definition (1.7)

$$M_{Ay_n,Bx_n,a}(qt) \geq M_{Ay_n,Bx_n,a}(t)$$

Since M is continuous function

$$\lim_{n\to\infty} M_{Ay_n,Bx_n,a}(qt) \geq \lim_{n\to\infty} M_{Ay_n,Bx_n,a}(t)$$

By lemma (1.13)

$\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Bx_n$ and we deduce that

$\lim_{n\to\infty} Ay_n = z$

Suppose SX is a closed subset of X.

Then $z = Su$ for some $u \in X$.

Subsequently we have,

$\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sy_n = Su$.

By (2.3), we have

$$\alpha\left(M_{Au,Bx_n,a}(qt) * \frac{M_{Su,Tx_n,a}(t) + M_{Au,Su,a}(t)}{2} * \frac{M_{Bx_n,Tx_n,a}(t) + M_{Au,Tx_n,a}(t)}{2}\right) \geq 0$$

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$$\alpha\left(M_{Au,Bx_n,a}(qt) * \frac{M_{Su,Tx_n,a}(t) + M_{Au,Su,a}(t)}{2} * \frac{M_{Bx_n,Tx_n,a}(t) + M_{Au,Tx_n,a}(t)}{2}\right) \geq 0$$

Taking $\lim n \to \infty$, we have

$$\alpha\left(M_{Au,Su,a}(qt) * \frac{M_{Su,Su,a}(t) + M_{Au,Su,a}(t)}{2} * \frac{M_{Su,Su,a}(t) + M_{Au,Su,a}(t)}{2}\right) \geq 0$$

$$\alpha\left(M_{Au,Su,a}(qt) * \frac{1 + M_{Au,Su,a}(t)}{2} * \frac{1 + M_{Au,Su,a}(t)}{2}\right) \geq 0$$

ϕ is non-increasing in $2^{nd}, 3^{rd}$ argument

$$\alpha\left(M_{Au,Su,a}(qt) * M_{Au,Su,a}(t) * M_{Au,Su,a}(t)\right) \geq 0$$

By the definition (1.7)

$$M_{Au,Su,a}(qt) \geq M_{Au,Su,a}(t)$$

Thus by lemma (1.13)
We have $Au = Su$.
The weak compatibility of A and S implies that $ASu = SAu$ and then $AAu = ASu = SAu = SSu$.
On the other hand,
Since $AX \subseteq TX$, there exists a point $v \in X$ such that $Au = Tv$. We claim that $Au = Bv$ using (2.3); we have

$$\alpha\left(M_{Au,Bv,a}(qt) * \frac{M_{Su,Tv,a}(t) + M_{Au,Su,a}(t)}{2} * \frac{M_{Bv,Tv,a}(t) + M_{Au,Tv,a}(t)}{2}\right) \geq 0$$

$$\alpha\left(M_{Au,Bv,a}(qt) * \frac{M_{Su,Au,a}(t) + M_{Au,Su,a}(t)}{2} * \frac{M_{Bv,Au,a}(t) + M_{Au,Au,a}(t)}{2}\right) \geq 0$$

$$\alpha\left(M_{Au,Bv,a}(qt) * 1 * \frac{1 + M_{Au,Bv,a}(t)}{2}\right) \geq 0$$

ϕ is non-increasing in $2^{nd}, 3^{rd}$ argument

$$\alpha\left(M_{Au,Bv,a}(qt) * M_{Au,Bv,a}(t) * M_{Au,Bv,a}(t)\right) \geq 0$$

By the definition (1.7)

$$M_{Au,Bv,a}(t) \geq M_{Au,Bv,a}(t)$$

Therefore by lemma, we have
$Au = Bv$
Thus $Au = Su = Tv = Bv$.
The weak compatibility of B and T implies that $BTv = TBv$ and $TTv = TBv = BTv = BBv$.
Let us show that Au is a common fixed point of A, B, S and T.
In view of (2.3) we have

$$\alpha\left(M_{AAu,Bv,a}(qt) * \frac{M_{SAu,Tv,a}(t) + M_{AAu,SAu,a}(t)}{2} * \frac{M_{Bv,Tv,a}(t) + M_{AAu,Tv,a}(t)}{2}\right) \geq 0$$

$$\alpha\left(M_{AAu,Au,a}(qt) * \frac{M_{AAu,Au,a}(t) + M_{AAu,AAu,a}(t)}{2} * \frac{M_{Au,Au,a}(t) + M_{AAu,Au,a}(t)}{2}\right) \geq 0$$

$$\alpha\left(M_{AAu,Au,a}(qt) * \frac{1 + M_{AAu,Au,a}(t)}{2} * \frac{1 + M_{AAu,Au,a}(t)}{2}\right) \geq 0$$

ϕ is non-increasing in $2^{nd}, 3^{rd}$ argument

$$\alpha\left(M_{AAu,Au,a}(qt) * M_{AAu,Au,a}(t) * M_{AAu,Au,a}(t)\right) \geq 0$$

By the definition (1.7)

$$M_{AAu,Au,a}(qt) \geq M_{AAu,Au,a}(t)$$

Therefore by lemma, we have
$Au = AAu = SAu$ and Au is a common fixed point of A and S.
Similarly, we can validate that Bv is a common fixed point of B and T.
Since $Au = Bv$, we achieve that Au is point of A, B, S and T,

which is called common fixed point..
If $Au = Bu = Su = Tu = u$ and $Av = Bv = Sv = Tv = v$.
Then by (2.3), we have

$$\alpha\left(M_{Au,Bv,a}(qt) * \frac{M_{Su,Tv,a}(t) + M_{Au,Su,a}(t)}{2} * \frac{M_{Bv,Tv,a}(t) + M_{Au,Tv,a}(t)}{2}\right) \geq 0$$

$$\alpha\left(M_{u,v,a}(qt) * \frac{M_{u,v,a}(t) + M_{u,u,a}(t)}{2} * \frac{M_{v,v,a}(t) + M_{u,v,a}(t)}{2}\right) \geq 0$$

$$\alpha\left(M_{u,v,a}(qt) * \frac{1 + M_{u,v,a}(t)}{2} * \frac{1 + M_{u,v,a}(t)}{2}\right) \geq 0$$

ϕ is non-increasing in $2^{nd}, 3^{rd}$ argument

$$\alpha\left(M_{u,v,a}(qt) * M_{u,v,a}(t) * M_{u,v,a}(t)\right) \geq 0$$

By the definition (1.7)

$$M_{u,v,a}(t) \geq M_{u,v,a}(t)$$

Therefore by lemma, we have $u = v$ and the common fixed point is a unique.
This explanation is verified the theorem. Hence A, B, S and T have a unique common fixed point in X .

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