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Application of Lagrange Polynomial for Interpolating Income Generation from Certain Students' Fee Structure

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Abstract
This paper presents the findings of the study that aimed to examine the changing aspects of income generation from certain students' fee structure. It used an empirical example of Higher learning institution from Tanzania to determine whether Lagrange interpolation method would emerge a useful tool for interpolating and extrapolating enrolments of undergraduate students. The functional values evaluated at all nodes on the given data set accurately match the enrolments records of each respective academic year, though with nonconformities at the additional interior nodes. The function $f_6(T)$ obtained from polynomial curve fitting using Excel spreadsheet at $R^2 = 0.9984$ replaced the Lagrange polynomial $P_6(T)$ to compute the rate of income generated from undergraduate students' fees for academic years 2012/2013 to 2018/2019. The relative errors committed by $f_6(T)$ range from the number 0.001 at the node $T_6 = 7$ to the number 0.172 at the node $T_0 = 1$. Enrolments trend however is not stable, as it fluctuates basing on the particular academic year that causes the rate of income generated from undergraduate students' fees to act wildly. Being the cause therefore oscillatory behaviour perhaps lax if enrolments grew continuously in the successive academic years, however, this would mark progressive manner if there will be reciprocal efforts to improve physical and electronic resources imbued by human resources capacity. Higher learning institutions could diversify their income generation strategies from multiplicity sources of incomes by using various techniques, including rising students' fees and downhill expenses without compromising quality of services and products, reducing dependency on Government subsidies, and strengthening the rate of students' retention in higher learning. They may also utilize to the maximum the unobserved multiplicity sources of incomes for instance; consultation fees, project funds, and new competitive learning programs to support realization of their strategic objectives in coherence with their missions and visions.
Keywords: Enrolment, Generation, Income, Interpolation, Lagrange, Polynomial
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1. Introduction

Algebraic polynomials are the most useful devices for describing real life phenomena and well-known class of functions that map real numbers into another set of real numbers. Equation (1) represents a general form of algebraic polynomial.

$$P_N(T) = \sum_{i=0}^{N} a_i T^i \,, \tag{1}$$

Whereas, $N > 0 \in ¢^+$ and a_i are constants for all $i \in ¢$ such that $0 \le i \le N$. For large N, the sequential polynomial $\{P_N(T)\}$ will either converge to function $f(T)$ or diverge away from $f(T)$. If $\{P_N(T)\}$ converges then $\lim_{N\to\infty} P_N(T) = f(T)$ for every real number T, so $f(T) \approx P_N(T)$. The polynomial (1) approximating the function $f(T)$ can be represented using Weierstrass Approximation Theory, which states that; for all $\varepsilon > 0$ and $T \in [a,b]$ there exists a polynomial $P_N(T)$ such that $|f(T) - P_N(T)| < \varepsilon$ if function $f(T)$ is continuous on a closed and bounded interval $[a,b]$ (Kabareh, Mageto & Muema, 2017).

The idea of using polynomials for approximating a function has its base on the fact that their derivatives and indefinite integrals are also polynomials and are computed easily, thus, they are convenient for understanding real world phenomena, mostly those that are without doubt described using continuous functions (Burden & Faires, 2013).

Lagrange interpolation method is one of the methods for finding polynomials of degree N that passes through $N+1$ data points (Kaw & Keteltas, 2009). Other methods of polynomial interpolation like Newton' Gregory Finite Differences (i.e. Forward, Backward and Central) are convenient for equally spaced data (Trefethen & Weideman, 1991). Newton's Divided Differences, Vandermonde, Linear Interpolants, Piecewise Linear Interpolants, likewise Lagrange interpolants are useful even for unequally spaced data. Splines interpolants (i.e. linear, quadratic and cubic splines) work effectively with interpolation because they do not use higher degrees monomials and thus, they are not predisposed to oscillatory behaviour, which is common for polynomials of higher degrees (Tal-Ezer, 1991).

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Unlike the spline interpolants, all other methods described herein produce exactly the same polynomial with the exception of round off-errors. The only difference between the two is on efficiency (i.e. less work may perhaps be required) or stability (i.e. round off error may perhaps be minimum) or both. Newton's Divided Differences scheme is considerably faster when compared with others simply because it is stable and allows additional data to existing one without requiring re-starting it, unlike the Lagrange approach which is stable but less efficient (Fritz, 1995).

Advantage of Lagrange interpolation method in comparison with standard polynomials representation counts on its simplicity. Ordinarily, Lagrange interpolating polynomial can be determined even without the need of solving a system of simultaneous equations as in the case with Vandermonde or Newton's Linear Interpolants in which the Vandermonde matrix or system of linear equations might demand re-construction or re-solving (Winrich, 1969).

Among other advantages of Lagrange interpolation method is that, it is easy to calculate coefficients of the polynomial and it explicitly shows the relationship between independent and dependent variables. Thus, it simplifies computations of the functional values and the values of independent variables that correspond to any given functional values (Fritz K, 2008).

Nevertheless, weaknesses of Lagrange interpolation method are that, it necessitates considerable amount of efforts to determine coefficients of polynomial (Stoeck & Abramek, 2014), particularly, when there is at least five data points in a given data set. Polynomials of higher degrees are normally susceptible to oscillatory behaviours and failure to produce noble estimates.

A problematic with Lagrange interpolation method is that, any alteration process is tiresome; for example, if one decides to introduce in additional data, all interpolating coefficients would demand re-computation for extra time. In such cases therefore, Lagrange interpolation method may become expensive for usage to calculate values of the polynomial at various nodes (Lambers, 2016).

A study on new neural network learning used Lagrange interpolation method to develops the weighting calculation in the back propagation training (Hussein, 2011). Another study on internal combustion engine intake noises with moving piston and a value by (Kim. Y, 2003) used Lagrange interpolation method to analyse the process of reducing noises to the minimum level by replacing the noisy pixels. A study on performance of fuel engine by (Stoeck & Abramek, 2014) used Lagrange interpolation method to determine performance characteristics of the fuel engine. Lagrange interpolation method effectively estimated data not contained in the plan of the experiment and it did not pose any trouble from technical point of view. The findings indicated that additional nodes caused problems, because use of higher degree polynomials over the whole data tends to jeopardize anticipated results. On the other hand, the second order Lagrange interpolating polynomial used by (Kaw, 2009) worked conveniently in chemical reactions and determined properly the values of specific heat at some temperatures as well as their absolute relative errors.

As a typical example, a nation of Libya once used Lagrange interpolation method and Newton Divided Differences to predict the changing numbers of families groups in Zliten town in Murqub district of Libya. The study recommended that Lagrange interpolation method and Newton Divided Differences could serve to predict the number of families groups for some years in future (Aleyan, 2015). Another study conducted in Kenya used the population data from Kenya National Bureau of Statistics (KNBS) to approximate the population total. The study approximated the population of 2009 accurately as it was in the national census results. Again, it makes a projection that Kenyan population for the year 2019 census will be forty-eight million five hundred and thirty-three thousand five hundred and eighty-seven (Kabareh, Mageto & Muema, 2017).

1.1 Description of a Typical Enrolments Problem

Enrolments are subject to various influences that can ultimately place limits on their growth. The maximum level of enrolments that a Higher learning institution can sustain depends on its carrying capacity that encompass; infrastructure, physical and electronic resources such as well-equipped classrooms, laboratories, and workshops; animated library and campus services, and good network access. Enrolments also depends on the availability of adequate number of skilled human resources both academic and non-academic staff. However, enrolments that exceed these limits might endanger quality of both services and products of the higher learning institution.

NIT puts considerable efforts to maintain enrolments level and sustain quality of its services and products. Its enrolments level indicates a significant growth trend when traced back from 2012/2013 to 2018/2019. Different approaches have been employed so far in the course to guess enrolments level, including use of common sense basing on experiences of past enrolments records and institutional capacity, subject to availability of suitable human resources, and teaching and learning facilities. Figure 1 illustrates polynomial curve fitting on estimated enrolments data of 2012/2013 to 2018/2019 by using Excel spreadsheet.

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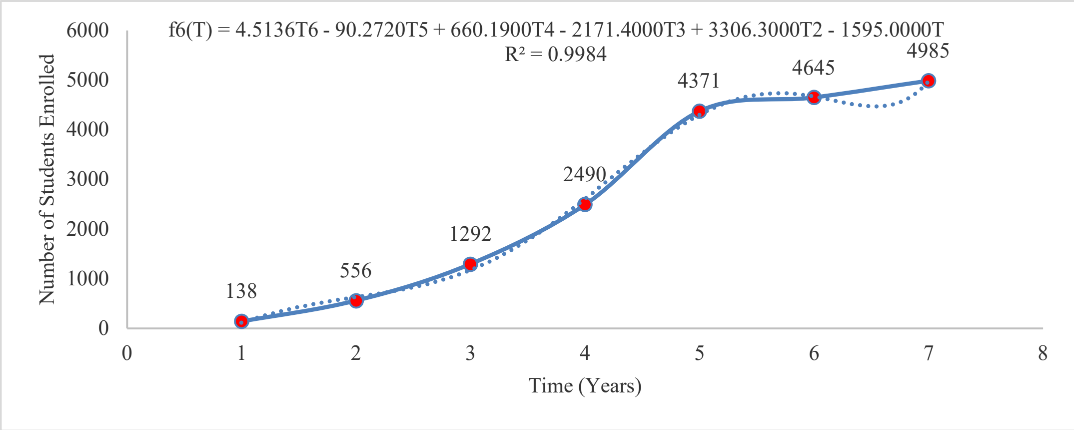

Figure1: Polynomial Curve Fitting on Estimated Enrolments Data of 2012/2013 to 2018/2019

NIT has already deployed diverse sources of incomes in the course to create organized strategies for income generation. The diverse sources include; Government subsidies, loans and grants (e.g. project funds) and multiplicity of internal sources. The internal sources are students' fees, charges from vehicle inspection and campus services, consultancy fees, and sales of engineered components and products.

Usually, income generation from students' fees is not stable as it varies depending on fee's structure, number of students enrolled in a particular learning program and the ability of students to pay for full tuition fees and timely. The higher the number of enrolled students the more the income is generated if students pay full tuition fees promptly. As a result, a combination of numerous sources would serve to ensure improvement and sustainability in generating funds for facilitating institutional strategic activities. For that reasoning, the rationale aim of this study was to examine the changing aspects of income generation from certain students' fee structure using Lagrange interpolation method.

1.2 Statement of the Mathematical Problem

Let equation (2) represents the Lagrange interpolating polynomial that describes enrolments of students into undergraduate learning programs.

$$P_N(T) = \sum_{i=0}^{N} L_i(T) f(T_i) \,, \tag{2}$$

Whereas N stands for the degree of the polynomial approximating the function $f(T)$ that passes through $N+1$ data points given by $(T_0, f(T_0))$, $(T_1, f(T_1)) \ldots (T_{N-1}, f(T_{N-1}))$ and $(T_N, f(T_N))$. Normally, the bases functions for Lagrange interpolating polynomial given by (3) either switches *"On"* or switches *"Off"* depending on how the indices i and j relate. Equation (3) defines the term $L_i(T)$, also known as switching function given by equation (4).

$$L_i(T) = \prod_{j=0, j\neq i}^{N} \left(T - T_j\right) / \left(T_i - T\right)_j \,, \tag{3}$$

$$L_i(x) = \begin{cases} 0 & if \quad i \neq j \\ 1 & if \quad i = j \end{cases} \tag{4}$$

The main assignment was to compute the coefficients of $L_i(T)$ for constructing polynomial $P_N(T)$ at some nodes T_i whereas $i \in ¢$ such that $0 \le i \le N$ and then use the resulting polynomial to interpolate income generated from certain students' fee structure for academic years 2012/2013 to 2018/2019.

The specific assignments were to:

i) Construct Lagrange interpolating polynomial that describes enrolments trend and interpolate enrolments level of undergraduate students for the academic years 2012/2013 to 2018/2019,
ii) Extrapolate future enrolments plan of undergraduate students for the academic years 2019/2020 and 2020/2021, respectively, and
iii) Examine the changing aspects of income generated from certain students' fee structure for academic years 2012/2013 to 2018/2019.

The study used these three hypotheses:

i) Lagrange interpolating polynomial accurately interpolates enrolments of undergraduate students for the academic years 2012/2013 to 2018/2019,
ii) Lagrange interpolating polynomial accurately extrapolates future enrolments plan of undergraduate students for the academic years 2019/2020 and 2020/2021, respectively, and
iii) The rate of income generated from certain students' fee structure for the academic years 2012/2013 to 2018/2019 surges significantly with enrolments growth.

1.3 Significance of the Study

The study has these substantial significance, which include:

i) Improved physical and electronic resources like well-equipped classrooms and laboratories, animated library and campus services as well as good network access,
ii) Improved human resources capacity like having in place adequate number of skilled academic and non-academic staff, and
iii) Having in place both adequate infrastructure and comprehensive short and long terms strategic plans.

1.4 Scope and Limitations

The study extracted the data from students' enrolments records of academic years 2012/2013 to 2018/2019. The reason as to why this based on numerous management aspects whereas, the First Five Year Rolling Strategic Plan ended in 2010/2011. Various tremendous strategic changes were introduced in the successive plan, which started in 2011/2012 and ended in 2015/2016 and thereafter improved into the ongoing Five Year Rolling Strategic Plan of 2016/2017 to 2020/2021 (NIT Planning and Investment Unit, 2012&2017). The students' fee structure of Tsh. 1, 500,000 used in this study is as good as that of the other public and private higher learning institutions available across the country.

On the other hand, high competition from other higher learning institutions available across Tanzania emerged as one of the limitations of this study, which affects enrolments trend. Some selected applicants fail to enrol into learning programs because of various aspects including; failure to secure tuition fees and some of them droop out from studies on basis of different grounds, including financial, social, and structural bonds (U-Planner, 2018). A transformation shifts from admission process by using the National Council for Technical Education (NACTE) and Tanzania Commission for Universities-Central Admission Systems (CASs) to individual institution's admission system may have direct affect to enrolments scheme. All these are presumably to affect accuracy of income generation estimates to deviate away from the pre-set expectations.

2. Materials and Methods

The study used documentary review to extract relevant information and records from both print and electronic sources. It also examined Lagrange interpolation method with other related polynomial interpolation methods in terms of efficiency and stability or strengths and weakness.

3. Construction of Lagrange Interpolating Polynomial, Evaluation of Functional Values and Errors

This section covers construction of Lagrange interpolating polynomial that describes enrolments of undergraduate students into learning programs, evaluation of functional values and errors. Consider equation (5) representing the usual Lagrange polynomial that passes through the points $(T_i, I(T_i))$ whereas T_i stands for an arbitrary node, $I(T_i)$ for the corresponding enrolments level and $i \in \mathbb{Z}$ such that $0 \le i \le N$.

$$P_N(T) = \prod_{i=0}^{N}(T - T_i) \ , \qquad (5)$$

The polynomial (5) has a zero at each node T_i in its horizontal axis that we may desire to interpolate. Therefore, we may use the set of equations given by equation (6) to construct the basis polynomial function given by equation (7) from equation (5).

$$\left\{L_0(T)=\frac{P_N(T)}{T-T_0}, L_1(T)=\frac{P_N(T)}{T-T_1}, L_2(T)=\frac{P_N(T)}{T-T_2}, \cdots, L_{N-1}(T)=\frac{P_N(T)}{T-T_{N-1}}, L_N(T)=\frac{P_N(T)}{T-T_N}\right\} \qquad (6)$$

It follows from the set (6) at the node T_0 we obtain (7).

$$L_0(T)=(T-T_1)(T-T_2)(T-T_3)(T-T_4)(T-T_5)(T-T_6)\cdots(T-T_{N-1})(T-T_N) \qquad (7)$$

Equation (7) has the property of switching itself to zero at some nodes $T_i = i+1$ for all $i \in \mathbb{Z}$ such that $1 \le i \le N$ and to non-zero only at $T = T_0$ .This means, it switches *"On"* at the first node but it switches *"Off"* at all other nodes. The bases functions $L_i(T)$ for all $i \in \mathbb{Z}$ such that $1 \le i \le N$ work in a similar manner, consequently, the expression to the right hand side of equation (8) that represents the general Lagrange polynomial follows.

$$P_N(T)=I(T_0)\frac{L_0(T)}{L_0(T_0)}+I(T_1)\frac{L_1(T)}{L_1(T_1)}+I(T_2)\frac{L_2(T)}{L_2(T_2)}+\cdots+I(T_N)\frac{L_N(T)}{L_N(T_{N-1})}+I(T_{N+1})\frac{L_{N+1}(T)}{L_{N+1}(T_{N+1})} \qquad (8)$$

The function (8) passes through all $N+1$ nodes and at every node, if plugged in the value of $T_i = i+1$, the functions $L_i(T)$ switches *"On"* at each one node in time and switches *"Off"* at all other nodes in that particular time. The coefficients of (8) have tendency of forcing the expression to the right hand side of this function to equal the corresponding $I(T_i)$ coordinate that represents enrolments level at that particular node when chosen arbitrarily at a point in time.

By plugging T_5 into (8) and simplify we obtain equation (9) consequently a desired point $(T_5, I(T_5))$.

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$$P_N(T_5) = I(T_0)\frac{0}{L_0(T_0)} + I(T_1)\frac{0}{L_1(T_1)} + I(T_2)\frac{0}{L_2(T_2)} + \text{L} + I(T_{N-1})\frac{0}{L_{N-1}(T_{N-1})} + I(T_N)\frac{0}{L_N(T_N)}$$

$$P_N(T_5) = I(T_5)\frac{L_5(T_5)}{L_5(T_5)} = I(T_5), \quad (9)$$

The polynomial (8) when expressed explicitly it interpolates enrolments at every node on the given data set.

3.1 An Empirical Example of Enrolments System

The third column of Table 1 indicates the enrolments data of undergraduate students of academic years 2012/2013 to 2018/2019. Let T_0 represents node 1, T_1 node 2, T_2 node 3, T_3 node 4, T_4 node 5, T_5 node 6, and T_6 node 7, respectively. Equation (10) represents the Lagrange interpolating polynomial in short hand by considering these seven successive nodes.

$$P_7(T) = \sum_{i=0}^{6} L_i(T) I(T_i), \quad (10)$$

Whereas, $L_i(T) = \prod_{\substack{j=0 \\ j\neq i}}^{6} (T - T_j)/(T_i - T_j)$ (11)

$$L_0(T) = [(T-2)(T-3)(T-4)(T-5)(T-6)(T-7)]/720 \quad (12)$$

$$L_1(T) = -[(T-1)(T-3)(T-4)(T-5)(T-6)(T-7)]/120 \quad (13)$$

$$L_2(T) = [(T-1)(T-2)(T-4)(T-5)(T-6)(T-7)]/48 \quad (14)$$

$$L_3(T) = -[(T-1)(T-2)(T-3)(T-5)(T-6)(T-7)]/36 \quad (15)$$

$$L_4(T) = [(T-1)(T-2)(T-3)(T-4)(T-6)(T-7)]/48 \quad (16)$$

$$L_5(T) = -[(T-1)(T-2)(T-3)(T-4)(T-5)(T-7)]/120 \quad (17)$$

$$L_6(T) = [(T-1)(T-2)(T-3)(T-4)(T-5)(T-6)]/120 \quad (18)$$

$$P_6(T) = 138L_0(T) + 556L_1(T) + 1292L_2(T) + 2490L_3(T) + 4371L_4(T) + 4645L_5(T) + 4985L_6(T) \quad (19)$$

Consequently, equation (19) represents the Lagrange interpolating polynomial that describes enrolments of undergraduate students for the academic years 2012/2013 to 2018/2019 whereas equations 12 to 18 define the bases functions $L_i(T)$ for all $i \in ¢ \quad 0 \le i \le 6$.

3.2 Interpolation of Functional Values

We used the Lagrange polynomial (19) to interpolate enrolments at the interior nodes $T_1 = 2$, $T_3 = 4$ and $T_5 = 6$. Functional values evaluated at these nodes are $P(T_1 = 2) = 556$, $P(T_3 = 4) = 2490$ and $P(T_5 = 6) = 4645$, respectively. By considering, staggered enrolments system used at the institute in the past years and take into accounts some additional interior nodes defined by $T_{i+1/4}$, $T_{i+1/2}$ and $T_{i+3/4}$ for some $i \in ¢ \ni 0 \le i \le 6$ .The functional values evaluated between the nodes $T_3 = 4$ and $T_4 = 5$ are $P(4.25) = 2960$, $P(4.5) = 3460$ and $P(4.75) = 3950$ whereas functional values between the nodes $T_4 = 5$ and $T_5 = 6$ are $P(5.25) = 4680$, $P(5.5) = 4840$ and $P(5.75) = 4820$. In addition, the functional values evaluated between the nodes $T_5 = 6$ and $T_6 = 7$ are $P(6.25) = 4390$, $P(6.5) = 4180$ and $P(6.75) = 4270$. However, the institute phased out staggered enrolments system in the academic year 2017/2018 so this is not the case as of now.

The functional values evaluated at the nodes $T_1 = 2$, $T_3 = 4$ and $T_5 = 6$ match the enrolments records in the respective academic years, similarly this applies for the functional values evaluated at the nodes $T_0 = 1$, $T_2 = 3$ $T_4 = 5$ and $T_6 = 7$ which are $P(T_0 = 1) = 138$, $P(T_2 = 3) = 1292$ and $P(T_4 = 5) = 4371$. The results indicate that Lagrange polynomial effectively interpolates enrolments of undergraduate students. Figure 2 and the fifth column of Table 1 illustrate enrolments trend of undergraduate students at NIT for academic years 2012/2013 to 2018/2019 by using Lagrange interpolating polynomial.

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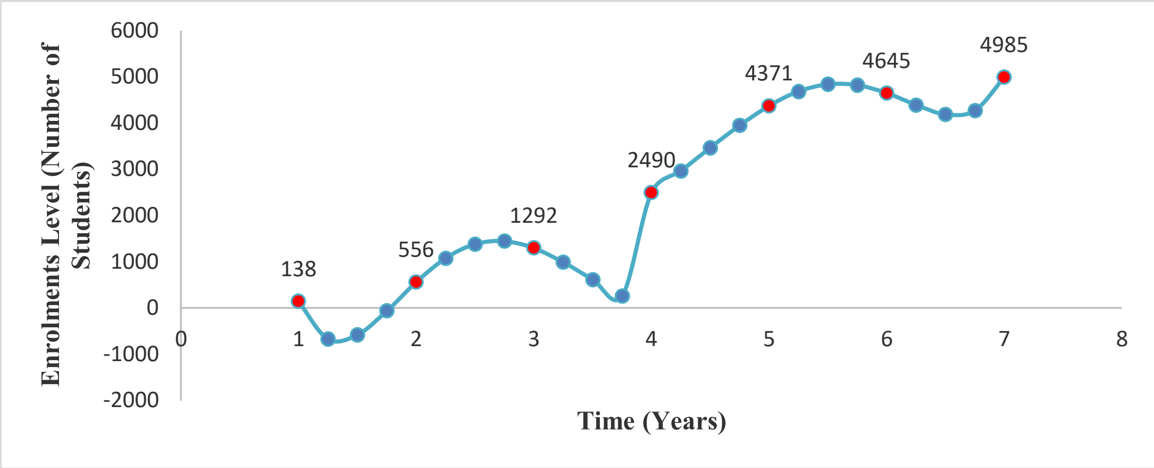

Figure 2: Enrolments Trend using Lagrange Interpolating Polynomial

3.3 Extrapolation of Functional Values

Functional values evaluated at the exterior nodes $T_7 = 8 > 7$ and $T_8 = 9 > 8$ are $P(T_7 = 8) = 26560$ and $P(T_8 = 9) = 134650$. Projections at these two successive nodes act wildly; therefore, Lagrange interpolating polynomial does not accurately evaluate functional values outside the given data set. Hence, we do not recommend Lagrange interpolation method for application in the future enrolments plan of undergraduate students now or in similar scenarios.

3.4 Numerical Errors

The function $f(T)$ is not given and the Lagrange polynomial $P_6(T)$ only involves interpolation of recorded enrolments data of academic years 2012/2013 to 2018/2019. Its explicit representation depends only on the values of Lagrange polynomial at specific data points. Similarly, the explicitly representation of the error term $E_6(T)$ cannot be obtained; however, the resulting polynomial of degree 6 passes through all seven nodes of the given data set. Therefore, the polynomial $P_6(T)$ makes the most accurate approximation of the function $f(T)$ since it includes all given data points (Burden & Faires, 2013). Equation (20) represents Lagrange polynomial with the error term. Rearranging (20), we obtain equation (22), which represents the error term (Dahiya, 2014).

$$f(T) = P_6(T) + E_6(T)\ , \tag{20}$$

$$E_6(T) = L_6(T)\frac{f^{6+1}(\xi(T))}{(6+1)!}\ , T_0 = 1 \le \xi(T) \le 7 = T_6 \tag{21}$$

But $L_6(T) = (T - T_0)(T - T_1)(T - T_2)(T - T_3)(T - T_4)(T - T_5)(T - T_6)$,

So, $$E_6(T) = (T-1)(T-2)(T-3)(T-4)(T-5)(T-6)(T-7)\frac{f^7(\xi(T))}{7!} \tag{22}$$

The Lagrange polynomial $P_6(T)$ of degree 6 represents the unknown function $f(T)$ then $f^7(\xi(T)) = 0$. This means the error term $E_6(T) = 0$ for some $T = T_i$ for all $i \in ¢$ such that $0 \le i \le 6$. Therefore, if $f(T)$ exists somewhere in the real world, then $P_6(T)$ exactly represents $f(T)$ as indicated by equation (23).

Hence,

$$f(T) \approx P_6(T) = 138L_0(T) + 556L_1(T) + 1292L_2(T) + 2490L_3(T) + 4371L_4(T) + 4645L_5(T) + 4985L_6(T) \tag{23}$$

The function closer to (23) at $R? = 0.9984$ is the standard polynomial of degree 6 given by (24);

$$f_6(T) = 4.5136T^6 - 90.2720T^5 + 660.1900T^4 - 2171.4000T^3 + 3306.3000T^2 - 1595.0000T\ . \tag{24}$$

Recalling the Weierstrass Approximation Theory statement in section 1.1 and the equations (23) and (24), equation (25) follows.

$|f_6(T) - P_6(T)| < \varepsilon$, that represents $|f(T) - P_6(T)| < \varepsilon$

Hence, $$|f(T) - P_6(T)| < \varepsilon \tag{25}$$

Whereas $\varepsilon = \inf_{i \in ¢ \ni 0 \le i \le 6}\{|f_6(T) - P_6(T)|\} = 4$ as indicated in the last column of Table1.

4. Income Generation

The Institution's total income is from three main fundamental sources; Government subsidies, loans and grants from donors, development partners, and internal sources. Internal sources as described earlier in Section 1.1

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include; students' fees for short courses and long courses such as; basic and technician certificates, ordinary diplomas, undergraduate and postgraduate programs. These also add values to the other sources of incomes namely; consultation fees, sales of engineered components and products, vehicle inspection charges, owing graduation gowns and a like. Therefore, total income generation is a linear combination of incomes from multiplicity sources given by equation (26).

$$T(T) = G(T, subs) + \mathrm{Prj}(T, gnts, \mathrm{lns}) + Ins(T, cncfs, cmscgs, \mathrm{stfs}, \mathrm{vicgs}) , \quad (26)$$

Whereas, $subs$ - *government subsidies*, $gnts$ - *grants*, lns - *loans*, $cmscgs$ - *campus service charges*, $cnfs$ - *consultancy fees*, $stfs$ - *students' fees* and vicgs - *vehicle inspection charges*.

Students' fees vary among offered programs, whereby, aviation programs fee's structures range between Tsh. 5,000, 000 to 6,000,000, postgraduate programs range between Tsh. 4,900,000 to 8,000,000, other long terms programs range between Tsh. 1,000,000 to 1,500,000 and short terms programs fee's structures range between Tsh. 210,000 to 600,000.

Income generation from students' fees is dynamic and subject to changing aspects of enrolments growth. Equation (27) gives the mathematical representation income generation from certain students' fee structure, which expresses its direct proportionality with enrolments growth.

$$R_6(T) \alpha P_6(T) \quad (27)$$

Equation (27) reduces to equation (28) by introducing parameter ada also known as per capita growth annual student's fee that repeals the proportionality symbol α .

$$R_6(T) = adaP_6(T) ,. \quad (28)$$

4.1 Changing Aspects of Income Generation

Equation (29) describes the rate of change of income generated from undergraduate students' fees in the academic years 2012/2013 to 2018/2019.

$$\frac{dR_6(T)}{dT} = ada\frac{dP_6(T)}{dT} , \quad (29)$$

It follows by the chain rule we obtain equation (30),

$$\frac{dP_6(T)}{dT} = \left(\frac{dP_6(T)}{dI(T)}\right)\left(\frac{dI(T)}{dT}\right) \quad (30)$$

Plugging equation (30) into (29), we obtain equation (31),

$$\frac{dR_6(T)}{dT} = ada\left(\frac{dP_6(T)}{dI(T)}\right)\left(\frac{dI(T)}{dT}\right) \quad (31)$$

Equation (32) represents (31), since $P_6(I(T))$ is a function of $I(T)$, which also depends on T .

$$\frac{dR_6(P_6(I(T)))}{dT} = ada\left(\frac{dP_6(I(T))}{dI(T)}\right)\left(\frac{dI(T)}{dT}\right), \quad (32)$$

Equation (33) represents the general form of continuous version of equation (32).

$$\frac{dR_N(P_N(I(T)))}{dT} = ada\left(\frac{dP_N(I(T))}{dI(T)}\right)\left(\frac{dI(T)}{dT}\right), \quad (33)$$

Equation (33) represents the rate of income generated from students' fees in $N+1$ successive academic years and equation (34) represents its discrete version counterpart.

$$\frac{\Delta R_N(P_N(I(T)))}{\Delta T} = ada\left(\frac{\Delta P_N(I(T))}{\Delta I(T)}\right)\left(\frac{\Delta I(T)}{\Delta T}\right) = ada\left(\frac{\Delta I(T)}{\Delta T}\right) \quad (34)$$

Knowing that, $P_6(T) \approx f_6(T)$ by recalling equations (23), (24) and (29), we estimate the rate of income generated in the academic years 2012/2013 to 2018/2019 by using the formula given by equation (35).

$$r(T) = \frac{dR_6(T)}{dT} = ada\frac{dP_6(T)}{dT} \approx ada\frac{df_6(T)}{dT} = r_6(T) ,$$

Simply, $r(T) \approx ada\left[27.0780T^5 - 453.6350T^4 + 2640.7600T^3 - 652.2000T^2 + 6612.6000T - 1595.0000\right]$ (35)

Figure 3 and the last column of Table 1 illustrate the rate of income generated from undergraduate students' fees for the academic years 2012/2013 to 2018/2019 by using students' fee structure of Tsh. 1,500, 000. The trend depicts oscillatory behaviour with the rise and fall of enrolments level in every particular academic year. This tendency would settle if enrolments grew continuously with improved physical, electronic and human resources capacity of the respective higher learning institution.

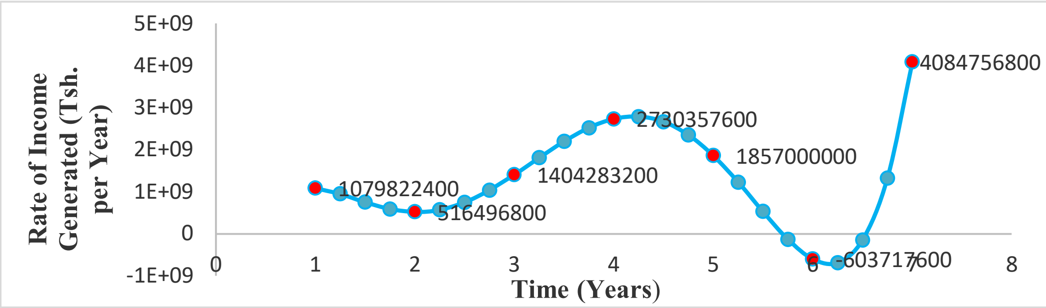

Figure 3: Rate of Income Generated from Undergraduate Students' fees for the Academic Years 2012/2013 to 2018/2019 by using Students' Fee Structure of Tsh. 1,500,000

Table 1: Enrolment Estimates, Relative Errors, and the Rate of Income Generated from Undergraduate Students' fees for the Academic Years 2012/2013 to 2018/2019 by using Students' Fee Structure of Tsh. 1,500,000

Enrolment Year	Node (T_i)	$I(T_i)$	$f_6(T_i)$	$P(T_i)$	Rerrorf (T_i)	RerrorP6 (T_i)	r (T_i) (Tsh. per Year)	$\lvert f_6(T)-P_6(T)\rvert$
2012/2013	1	138	114	138	0.172	0.000	1079822400	24
	1.25		285	-677			946110351.6	962
	1.5		426	-585			750233850	1011
	1.75		537	-65			587047692.2	602
2013/2014	2	556	627	556	(0.128)	0.000	516496800	71
	2.25		716	1072			568376657.8	356
	2.5		824	1378			747093750	554
	2.75		971	1443			1036425998	472
2014/2015	3	1292	1174	1292	0.092	0.000	1404283200	118
	3.25		1441	988			1807467464	453
	3.5		1775	612			2196433650	1163
	3.75		2170	255			2520049805	1915
2015/2016	4	2490	2609	2490	(0.048)	0.000	2730357600	119
	4.25		3071	2961			2787332770	110
	4.5		3528	3461			2663645550	67
	4.75		3948	3948			2349421111	0
2016/2017	5	4371	4301	4371	0.016	0.000	1857000000	70
	5.25		4560	4680			1225698577	120
	5.5		4706	4835			526569450	129
	5.75		4737	4817			-132838082.8	80
2017/2018	6	4645	4672	4645	(0.006)	0.000	-603717600	27
	6.25		4557	4387			-691245117.2	170
	6.5		4477	4185			-149818650	292
	6.75		4559	4268			1321702223	291
2018/2019	7	4985	4989	4985	(0.001)	0.000	4084756800	4
2019/2020	8		16405	26563			3.77E+10	10158
2020/2021	9		70252	134648			1.4E+11	64396

Source: Estimated from Enrolments Data of 2012/2013 to 2018/2019

4.2 Marginal Income

Equation (36) represents the continuous version of the marginal income generated from undergraduate students' fees and equation (37) gives the discrete version counterpart.

$$\frac{dR_N(P_N(I(T)))}{dP_N(I(T))} = ada\left(\frac{dT}{dI(T)}\right)\left(\frac{dI(T)}{dT}\right) = ada \quad (36)$$

$$\frac{\Delta R_N(P_N(I(T)))}{\Delta P_N(I(T))} = ada\left(\frac{\Delta T}{\Delta I(T)}\right)\left(\frac{\Delta I(T)}{\Delta T}\right) = ada \quad (37)$$

Equations (36) and (37) each indicate that, marginal income generation from undergraduate students' fees is equal to the unit cost of training an undergraduate student. These expenses at NIT are Tsh. 1,500,000 in every undergraduate learning programs except that found in the School of Aviation Technology. In this regards, the marginal income generation may be equal to per capita growth annual student's fee (Tsh. 1,500,000 only). In other words, if undergraduate students enrol into learning programs, then the institute shall collect from them tuition fees equal to Tsh. 1,500,000 each. This amount will add value into the total income of the higher learning institution. Higher learning institutions may diversify various strategies of income generation efforts not by relying on raising students' fees but also reducing unnecessary expenditure while improving quality of products and campus services delivery. Additionally, seek out funds from unobserved sources such as consultation services, project funds (e.g. loans or grants) and reduce dependency on Government subsidies by introducing new competitive learning programs, and improving financial, social, and structural bonds for strengthening the rate of students' retention. All these will support smooth operationalization of the institutional strategic objectives. In addition, the new learning programs will nurture enrolments level and so each program will add an additional income of at least Tsh 1,500,000 to the total income of the higher learning institution from every new student that enrols.

5. Conclusion and Recommendations

Lagrange interpolating polynomial emerged a useful tool for interpolating enrolments of undergraduate students; however, it does not suggest an alternative for extrapolation of future enrolments level of undergraduate students. Therefore, we recommend it for application to examine the changing aspects of income generation from students' fees in similar setups available in the other higher learning institutions across the country. Higher learning institutions could diversify their income generation strategies from multiplicity sources of incomes by using various techniques, including rising students' fees and downhill expenses without compromising quality of services and products, reducing dependency on Government subsidies, and strengthening the rate of students' retention in higher learning. They may also utilize to the maximum the unobserved multiplicity sources of incomes for instance; consultation fees, project funds, and new competitive learning programs to support realization of their strategic objectives in coherence with their missions and visions.

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