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ON QUASI-INVERTIBILITY AND QUASI-SIMILARITY OF OPERATORS IN HILBERT SPACES.

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ABSTRACT

It is a well known fact in operator Theory that if A and B are operators with at least one of them invertible then AB and BA are similar operators. In this paper we prove an analogous result about quasi-invertible operators A and B. We thus show that if A and B are quasi-invertible then AB and BA are quasi-similar. We also deduce a number of corollaries about spectra and essential spectra of AB and BA.

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1. INTORDUCTION

Let H be a complex Hilbert space and B(H) denote the Banach algebra of all bounded linear operators on H. An operator $A \in B(H)$ is said to be quasi-invertible if A is both one-one and has dense range. Equivalently A is quasi-invertible if it is a quasiaffinity. Operators A and B are said to be similar if there exist an invertible operator S such that AS=SB, while A and B are said to be quasisimilar if there exist quasi-invertible of operators X and Y such that

AX = XB and BY=YA.

The concept of quasisimilarity particularly with respect to equality of spectra has been studied by a number of authors among them W.C Clary [1] who showed that quasisimilar hyponormal operators have equal spectra J.M Khalagai and B. Nyamai [5] showed that if A and B are quasisimilar operators with A dominant and B* is M-hyponormal then A and B have same spectra. J.P. William [6] and [7] showed that there are several cases which imply that A and B have equal essential spectra. For example if A and B are both hyponormal operators or are both partial isometries or quasinormal operators etc. B.P. Duggal [3] proved that if A_i i=1,2 are quasisimilar p-hyponormal operators such that U_i is unitary in the polar decomposition $A_i=U_i|A_i|$, then A_1 and A_2 have same spectra and also same essential spectra. In this paper we deduce a numbers of results in this direction concerning the operators AB and BA.

2. NOTATION AND TERMINOLOGY

Given an operator $A \in B(H)$ we denote the numerical range of A by W(A) .

Thus $W(A) = \{< Ax, x >: \|x\| = 1\}$

The spectrum of A is denoted by σ(A). Thus $\sigma(A) = \{\lambda \epsilon \mathbb{C}$: A- λI is not invertible} where $\mathbb{C}$ is the field of complex numbers. The commutator of A and B is denoted by [A,B] where

[A,B] = AB - BA

An operator A is said to be dominant, if to each $\lambda \epsilon \mathbb{C}$ there corresponds a number $M_\lambda \geq 1$ such that

$$\| (A-\lambda)^*x \| \leq M_\lambda \| (A-\lambda)x \| \quad \forall x \in H$$

M-hyponormal, if ∃ M ≥ M_λ for all λ in the definition of dominant operator.

Hyponormal, if A*A ≥ AA*

quasinormal if [A*A, A]= 0

p − hyponormal if $(A^*A)^p \geq (AA^*)^p$ for 0<p≤1

Self adjoint if A = A*

normal if [A, A*] = 0

Partial isometry if A = AA*A

Isometry if A*A = I

Unitary if A*A = AA* = I

Fredholm if its range denoted by ran A is closed and both null space, kerA and Ker A* are finite dimensional.

The essential spectrum of A is denoted by $\sigma_e(A) = \{\lambda \epsilon \not\subset : A- \lambda I$ is not Fredholm$\}$.

The following operator inclusions are proper:

Normal ⊂ hyponormal ⊂ p-hyponormal and

Hypornormal ⊂ M-hyponormal ⊂ dominant

3. RESULTS

Theorem 1

Let A,B ∈ B(H) be quasi-Invertible.
Then AB and BA are quasisimilar.

Proof

We first note that in the equations:
(AB) A = A (BA)
and
(BA) B = B (AB)
We let T =AB and S = BA
Thus we have
TA = AS
and
SB = BT
Now A and B are quasi-invertible implies T and S are quasisimilar. Hence AB and BA are quasisimilar.
We note that in view of the results in [1], [3], [5], [6] and [7] the following corollaries are immediate.

Corollary 1

Let A,B ∈ B(H), be quasi-invertible.
Then σ (AB) = σ (BA)
Under any one of the following conditions:

(i) AB and BA are hyponormal
(ii) AB is dominant and (BA)* is M-hyponornal.
(iii) AB and BA are p-hyponormal with U and V unitary in the polar decomposition AB = U│AB│ and BA = V│BA│.

Corollary 2

Let A,B $\in$ B(H) be quasi-invertible. Then σ_e (AB) = σ_e (BA) under any one of the following conditions:

(i) AB and BA are quasinormal.
(ii) AB and BA are hyponormal with either A or B compact.
(iii) AB and BA are p-hyponormal with U and V unitary in the polar decomposition AB = U$|$AB$|$ and BA = V$|$BA$|$.

Corollary 3

If A $\in$ B(H) is quasi-invertible then we have that

$$\sigma \text{(AA*)} = \sigma \text{(A*A)} \quad and \qquad \sigma_e \text{(AA*)} = \sigma_e \text{(A*A)}$$

Proof

We first note that if A is quasi-invertible then A* is also quasi-invertible. Hence by theorem 1 above AA* and A*A are quasi-similar. But AA* $\geq$ 0 and A*A $\geq$ 0. Hence by part (i) of Corollary 1 and part (i) of Corollary 2 above we have respectively that

$$\sigma \text{(AA*)} = \sigma \text{(A*A)}$$

and

$$\sigma_e \text{(AA*)} = \sigma_e \text{(A*A)}$$

For an operator B$\in$ B(H), we say that B is consistent in invertibility (with respect to multiplication) or briefly that B is a CI operator if for each A$\in$ B (H), AB and BA are invertible or non-invertible together. Thus B is a CI operator if σ(AB) = σ(BA). It is well known result that if B is invertible then for any A$\in$ B(H) we have AB = B^{-1} (BA) B. Thus AB and BA are similar operators and hence σ(AB) = σ(BA). W. Gong and D. Han [4] proved among other results that an operator

B$\in$ B (H) is CI operator iff

σ(B*B) = σ(BB*)

We use this result to deduce a number of results on CI operators. Firstly the following corollary provides an alternative proof to corollary 1.3 of [4].

Corollary 4

Let B be quasi-invertible.

Then B is a CI operator.

Proof

We note from corollary 3 above that since B is quasi-inevertible we have that

σ(B*B) = σ(BB*)

Hence B is a CI operator.

Corollary 5

Let B$\in$ B (H) be such that O$\notin$ W(B). Then both B^* and B are CI operators.

Proof

We first note that if $O \notin W(B)$ then both B and B* are quasi-invertible.

Hence by corollary 4 above B and B* are CI operators.

Theorem 2

If B is an M-hyponormal operator satisfying the equation

$BX = XB^*$

Where X is quasi-invertible then B is a CI operator.

Proof

Since B is M-hypononormal

$BX = XB^*$ implies

$B^*X = XB$

Taking adjoints we have:

$BX^* = X^*B^*$ and $B^*X^* = X^*B$

Now using the equations above we have:

$B^*BX = B^* X B^* = XBB^*$ and $BB^*X^* = B X^* B = X^*B^*B$

i.e BB* and B*B are quasi-similar since X* is also quasi-invertible.

Thus $\sigma(BB^*) = \sigma(B^*B)$ implying B is a CI operator.

Corollary 6

If an M-hyponormal operator B is quasi-similar to its adjoint B* then B is a CI operator.

Proof

In this case there exist quasi-invertible operators X and Y such that

$BX = XB^*$ and $B^*Y = YB$

Thus the proof is immediate by theorem 2

The following result due to Duggal [2] is required in the proof of our next theorem.

Theorem P

Let $A: H_1 \rightarrow H_1$, $B: H_2 \rightarrow H_2$ and

$X: H_2 \rightarrow H_1$ be operators such that

$AX = XB$

Where H_1 and H_2 are Hilbert spaces.

If A is dominant and B* is M-hyponormal them

$A^*X = XB^*$

Theorem 3

Let $A,B,X \in B(H)$ be such that

$BX = XA$, where B is dominant, A* is M-hyponormal and X is quasi-invertible. If B is a CI operator, then A is also a CI operator.

Proof

In this case,

$BX = XA$ implies $B^*X = XA^*$ Taking adjoints we also have: $A^*X^* = X^*B^*$

and

$AX^* = X^*B$

Now using these equations we have

$B^*BX = B^*XA = XA^*A$

and

$A^*AX^* = A^*X^*B = X^*B^*B$

i.e B*B and A* A are quasi-similar and hence

$\sigma(B^*B) = \sigma(A^*A)$

Similarly we have that

$BB^*X = BXA^* = XAA^*$

and

$AA^*X^* = A X^*B^* = X^*BB^*$

i.e BB* and AA* are quasisimilar and hence

$\sigma(BB^*) = \sigma(AA^*)$

Now if B is a CI operator then we have that

$\sigma(B^*B) = \sigma(BB^*) = \sigma(AA^*) = \sigma(A^*A)$

Hence A is also a CI operator.

Corollary 7

If a dominant operator B is quasi similar to any operator A with A* M-hyponormal, then

B is a CI operator implies A is also a CI operator.

Proof

In this case, there exist quasi-invertible operators X and Y such that

BX = XA and AY = YB

The proof of theorem 3 above can now be traced to give the result.

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