Mathematical Theory and Modeling www.iiste.org
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online) DOI: 10.7176/MTM
Vol.9, No.7, 2019

Polarity in signed symmetric group and signed transformation semigroup

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Abstract

Let PSS_n , $PST_n$and $PSSing_n$ be polarity of symmetric group, polarity of signed transformation semigroup and polarity of signed singular mapping respectively from $X_n \longrightarrow X_n^*$ where $X_n = \{1,2,3,\cdots,n\}$ and $X_n^* = \{-n,\cdots,-3,-2.-1.0,1,2,3,\cdots,n\}$, and $X_n \subset X_n^*$. The aim of thus paper is to determine the order of PSS_n , $PST_n$and $PSSing_n$.

Keywords:polarity, semigroup, signed symmetric group, signed transformation semigroup, signed singular mapping

DOI: 10.7176/MTM/9-7-03
Publication date: July 31st 2019

1. Introduction

The study of symmetric groups, alternating groups and dihedral groups has made a significant contribution to group theory, so has the study of various subsemigroups of T_n, P_n and I_n (Bashar 2010; Howie 2002; Laradji and Umar 2004; Umar 1992). Extensive research work have been done in the area of semigroup see Higgins 1992),(Laradji and Umar 2004), (Adeniji2012), (Adeshola2013). (Bakare and Makanjuola 2013), (Mogbonju2015).(James and Kerber1981) defined permutation group on set $X_n \longrightarrow X_n^*$and ST_n (the signed transformation group) is a semigroup analogue of T_n and SS_n(signed symmetric group) is the units of S_n. However (Mogbonj2015) studied signed transformation semigroup of full partial and partial 1 – 1. He defined a signed transformation as the set of all mapping from $\alpha: dom(\alpha) \subseteq X_n \longrightarrow Im(\alpha) \subset X_n^*$ where $X_n = \{1,2,3,\cdots,n\}$and $X_n^* = \{-n,\cdots,-3,-2.-1.0,1,2,3,\cdots,n\}$. $Dom(\alpha)$stands for the domain of α while $lm(\alpha)$ as lmage of α. (Mogbonju2015) studied the order, idempotent, nilpotent and the chain decomposition of of full and patial transformation semigroup. Also (Mogbonju2019) studied the polarity in signed order preserving, order decreasing and both order preserving and order decreasing semigroup respectively.

The semigroup of singular self-mapping of X_n was defined by Howie(1996) as $Sing_n = T_n \setminus S_n = \{\alpha \in T_n: lm(\alpha) \leq n\}$.(Mogbonju2019) also studied the order, idempotent and chain decomposition of signed singular mappings semigroup on a set $\alpha: dom(\alpha) \subseteq X_n \longrightarrow Im(\alpha) \subset X_n^*$.

The following known results and theorem from (Mogbonju2015) are very crucial to this work.

Theorem 1.1[Mogbonju (2015)]Theorem 3.2.1]. Let $S = SS_n$ and if $\alpha \in SS_n, n \geq 1\ then\ |S\ | = 2n! - (2^n - 2)n!$

Theorem 1.2[Mogbonju (2015)]Theorem 3.2.2]. Let $S = ST_n$, then $|S\ | = 2n^n + n^n(2^n - 2)$

Theorem 1.3[Mogbonju (2015)]Theorem 3.6.1]. Let $S = Sing_n\ \ if\ \alpha \in S, n \geq 0, then\ |S| = 2^n(n^n - n!)$

Theorem 1.4[Mogbonju (2015)]Theorem 3.6.1]. Let $S = SPT_n \setminus SS_n\ then\ |S| = (2n + 1)^n - 2^n n!$

2. Methodology

Let PSS_n, PST_n and $PSSing_n$ be polarity of signed symmetric group, polarity of signed transformation semigroup and polarity of signed singular mapping defined on a set $\alpha: X_n \longrightarrow X_n^*$ where $\alpha: dom(\alpha) \subseteq X_n \longrightarrow Im(\alpha) \subset X_n^*$

2.1 Elements in SS_n, ST_n and $SSing_n$

The set of elements in SS_2 is as follows:

$$|SS_2| = \left\{\begin{pmatrix}1 & 2\\-1 & -2\end{pmatrix},\begin{pmatrix}1 & 2\\-2 & -1\end{pmatrix},\begin{pmatrix}1 & 2\\-1 & 2\end{pmatrix},\begin{pmatrix}1 & 2\\-2 & 1\end{pmatrix},\begin{pmatrix}1 & 2\\1 & -2\end{pmatrix},\begin{pmatrix}1 & 2\\2 & -1\end{pmatrix},\begin{pmatrix}1 & 2\\1 & 2\end{pmatrix},\begin{pmatrix}1 & 2\\2 & 1\end{pmatrix} = 8\right\}$$

The set of elements in ST_2 is as follows:

$$|ST_2| = \left\{\begin{matrix}\begin{pmatrix}1 & 2\\1 & -1\end{pmatrix},\begin{pmatrix}1 & 2\\-1 & 1\end{pmatrix},\begin{pmatrix}1 & 2\\-1 & -1\end{pmatrix},\begin{pmatrix}1 & 2\\1 & -2\end{pmatrix},\begin{pmatrix}1 & 2\\-1 & 2\end{pmatrix},\begin{pmatrix}1 & 2\\-1 & -2\end{pmatrix},\begin{pmatrix}1 & 2\\2 & -1\end{pmatrix},\begin{pmatrix}1 & 2\\-2 & 1\end{pmatrix},\\ \begin{pmatrix}1 & 2\\-2 & -1\end{pmatrix},\begin{pmatrix}1 & 2\\2 & -2\end{pmatrix},\begin{pmatrix}1 & 2\\-2 & 2\end{pmatrix},\begin{pmatrix}1 & 2\\2 & -2\end{pmatrix},\begin{pmatrix}1 & 2\\1 & 1\end{pmatrix},\begin{pmatrix}1 & 2\\1 & 2\end{pmatrix}\begin{pmatrix}1 & 2\\2 & 1\end{pmatrix},\begin{pmatrix}1 & 2\\2 & 2\end{pmatrix}\end{matrix}\right\} = 16$$

The set of elements in $SSing_2$ is as follows:

$$|SSing_2| = \left\{\begin{pmatrix}1 & 2\\-1 & -1\end{pmatrix},\begin{pmatrix}1 & 2\\-2 & -2\end{pmatrix},\begin{pmatrix}1 & 2\\-1 & 1\end{pmatrix},\begin{pmatrix}1 & 2\\-2 & 2\end{pmatrix},\begin{pmatrix}1 & 2\\1 & -1\end{pmatrix},\begin{pmatrix}1 & 2\\2 & -2\end{pmatrix},\begin{pmatrix}1 & 2\\1 & 1\end{pmatrix},\begin{pmatrix}1 & 2\\2 & 2\end{pmatrix}\right\} = 8$$

2.2 Polarity of elements in SS_n, ST_n and $SSing_n$

Polarity of elements in SS_n is as follows:

When $n = 1$

$$|PSS_1| = \left\{\begin{pmatrix}1\\-1\end{pmatrix}\right\} = 1$$

When $n = 2$

$$|PSS_2| = \left\{\begin{pmatrix}1 & 2\\-1 & 2\end{pmatrix},\begin{pmatrix}1 & 2\\1 & -2\end{pmatrix},\begin{pmatrix}1 & 2\\-1 & -2\end{pmatrix},\begin{pmatrix}1 & 2\\-2 & 1\end{pmatrix},\begin{pmatrix}1 & 2\\2 & -1\end{pmatrix},\begin{pmatrix}1 & 2\\-2 & -1\end{pmatrix}\right\} = 6$$

Image of element in PSS_2

$$|Im(\alpha^-)| = \left\{\begin{pmatrix}1 & 2\\-1 & -2\end{pmatrix},\begin{pmatrix}1 & 2\\-2 & -1\end{pmatrix}\right\} = 2$$

$$|Im(\alpha^*)| = \left\{\begin{pmatrix}1 & 2\\-2 & 1\end{pmatrix},\begin{pmatrix}1 & 2\\-1 & 2\end{pmatrix},\begin{pmatrix}1 & 2\\1 & -2\end{pmatrix},\begin{pmatrix}1 & 2\\2 & -1\end{pmatrix}\right\} = 4$$

Table 2.1: Values of elements in PSS_n

n	$\lvert Im(\alpha^-)\rvert$	$\lvert Im(\alpha^*)\rvert$	$\lvert PSS_n\rvert = n! + (2^n - n)n!$
1	1	$-$	1
2	2	4	6
3	6	36	42
4	24	336	360
5	120	3300	3420
6	720	44640	45360

ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online) DOI: 10.7176/MTM
Vol.9, No.7, 2019

Polarity of elements in ST_n is as follows:

When $n = 1$

$$|PTS_1| = \left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\} = 1$$

When $n = 2$

$$|\,PTS_2| = \left\{\begin{matrix} \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}. \\ \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & 2 \end{pmatrix}, \end{matrix}\right\} = 8$$

Image of element in PSS_2

$$|\,Im(\alpha^-)\,| = \left\{\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & -2 \end{pmatrix}\right\} = 4$$

$$|\,Im(\alpha^*)| = \left\{\begin{matrix} \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}, \\ \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & 2 \end{pmatrix}, \end{matrix}\right\} = 4$$

Table 2.2: Values of elements in PTS_n

n	$\mid Im(\alpha^-)\mid$	$\mid Im(\alpha^*)\mid$	$\lvert PST_n\rvert = n^n + n^n(2^n - 2)$
1	1	−	1
2	4	8	12
3	27	162	189
4	256	3584	3840
5	3125	93750	96875

Polarity of elements in $SSing_n$ is as follows:

When $n = 1$

$$|PSSing_1| = 0$$

When $n = 2$

$$|PSSing_2| = \left\{\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}\right\} = 6$$

Image of element in $PSSing_2$

$$|\,Im(\alpha^-)\,| = \left\{\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & -2 \end{pmatrix}\right\} = 2$$

$$|\,Im(\alpha^*)| = \left\{\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}\right\} = 4$$

ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online) DOI: 10.7176/MTM
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Table 2.3: Values of elements in $PSSing_n$

n	$\mid Im(\alpha^-) \mid$	$\mid Im(\alpha^*) \mid$	$\lvert PSSing_n \rvert = (2^n - n)(n^n - n!)$
1	0	0	0
2	2	4	6
3	21	126	168
4	229	3254	3712
5	3125	89910	96160

3. Main results

Theorem 3.1

Let $S = PSS_n$ and if $\alpha \in PSS_n$ then $|PSS_n| = n! + (2^n - n)n!$

Proof

Let $X_n = \{1,2,3,\cdots,n\}$,$X_n^* = \{-n,\cdots,-3,-2.-1.0,1,2,3,\cdots,n\}$ and $X_n \subset X_n^*$. It follows from counting argument that there are $n!$ numbers of elements for $\mid Im(\alpha^-)|$. Since the $|\, Im(\alpha)|$ is either $+i\ or - i$ where $i = 1,2,3,\ldots,n$ and by binomial theorem for a positive n where $\sum_{k=0}^{n} \binom{n}{k} = 2^n$, the $|\, Im(\alpha^*)|$ in PSS_n have $(n-1)$ groups and each group consists of $(2^n - n)n!$ elements. Multiply and summing we have $n! + (2^n - n)n!$ elements and this is equivalent to $2^n n! - n!$

Theorem 3.2

Let $S = PST_n$ and if $\alpha \in PST_n$ then $|PST_n| = n^n + n^n(2^n - 2)$

Proof

Let $\alpha: X_n \longrightarrow X_n^*$ and $\alpha(i) = \pm j$ where $i \in dom(\alpha)$and $lm(\alpha) \subset X_n^*$. If the $|\, Im(\alpha)|$ is either $i\ or - i$ for $i = 1,2,3,\ldots,n$ then the nature of $|\, Im(\alpha^*)\mid$ is such that each group consists $n^n(2^n - 2)$ elements. Since the semigroup is $1 - 1$ mapping then $|\, \alpha S| = n^n$ when $|\, Im(\alpha^*)\mid$ for each$n$. Hence by summing $|PST_n| = n^n + n^n(2^n - 2)$.

Theorem 3.3

Let $S = PSSing_n$ and if $\alpha \in PSSing_n$, then $|PSSing_n| = (2^n - 1)(n^n - n!)$

Proof

Let $\alpha \in PSSing_n$ and such that $\alpha: dom(\alpha) \subseteq X_n \longrightarrow Im(\alpha) \subset X_n^*$ and from theorem 1.3 it follows that $|SSing_n| = 2^n(n^n - n!)$ and $|\, Im(\alpha^+)\mid = n^n - n!$ and also there are n such elements having the property $|\, Im(\alpha^+)\mid = \mid Im(\alpha^-)\mid = n^n - n!.\ldots equation$ (3.1) Thus, from table 2.3 and the$|\, Im(\alpha^*)\mid$ is simplified as followed:

$$2^n(n^n - n!) - 2(n^n - n!)$$

$$2^n n^n - 2^n n! - 2n^n - 2n!$$

$$2^n n^n - 2n^n - 2^n n! - 2n!$$

$$n^n(2^n - 2) - n!\,(2^n - 2)$$

$$(2^n - 2)\,(n^n - n!)\ldots\ldots equation\ (3.2).$$

thus combining equation (3.1) and equation (3.2) yields $|PSSing_n|$ for each n

$$(n^n - n!) + (2^n - 2)(n^n - n!)$$

$$(n^n - n!)[1 + (2^n - 2)]$$

$$(n^n - n!)(1 + 2^n - 2)$$

$$(2^n - 1)(n^n - n!)$$

and hence the result $|PSSing_n| = (2^n - 1)(n^n - n!)$

4. Summary of results

The following results with sequences were obtained for all n

Let $S = PSS_n$ then $|PSS_n| = n! + (2^n - n)n!$ which generates the sequence $1,6,42,360,3420,45360, \ldots$

Let $S = PST_n$ then $|PST_n| = n^n + n^n(2^n - 2)$ which generate the sequence $1,12,189,3842,96875, \ldots$

Let $S = PSSing_n$ then $|PSSSing_n| = (2^n - 1)(n^n - n!)$ which generate the sequence $0,6,168,3712,96160, \ldots$

5. Conclusion

It is hereby recommended that the idempotent , chain decomposition of polarity of full and partial transformation can also be study.

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