TO DEVELOP EFFICIENT SCHEME FOR SOLVING INITIAL VALUE PROBLEM IN ORDINARY DIFFERENTIAL EQUATION

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ABSTRACT

In this paper, a new scheme of Runge-Kutta (RK) type has been developed while evaluating two slope functions per step and maintaining the third order accuracy of the scheme. Local truncation error is obtained with the help of principal term which is obtained via multi variable Taylor series. It has been shown that the convergence order of the scheme is three and its stability polynomial is also derived. Some numerical examples are taken in order to compare the developed scheme with other existing schemes. It is observed that the developed scheme is better than other selected existing schemes and this comparison has been performed on the basis of slope evaluations per integration step, error analysis and computer time consumed by the scheme under consideration.

KEY WORDS: Initial value problems, Runge-Kutta Scheme, Autonomous and non-autonomous differential equations, Zero stability.

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1. INTRODUCTION

It is observed that ordinary differential equations play vital role in real life situations. Various mathematical models are expressed in terms of ordinary differential equations with one or more than one initial conditions such as mass-spring-damper model, RLC series circuit, beam equation, SIR models in epidemiology, simple pendulum, Vander-pol oscillator, kinetic reactions in chemistry, particle’s trajectory, one-dimensional fluid flow equations, and many others. Such various applications of applied problems depending upon ordinary differential equations can be found in research works found in [1-10]. Ordinary differential equations can be solved analytically and numerically but in most of the situations analytical methods fail to acquire the desired solution and this happens in cases when we come across nonlinear terms in the model under consideration. Therefore, numerical schemes which are much handy as compared to analytical schemes come to the rescue. Numerical schemes have more significance than analytical schemes as these schemes can be used to find the solution of any

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ordinary differential equation whether it be autonomous, non-autonomous, linear or non-linear, provided that the ordinary differential equation in hand has unique solution in the given time interval.

Nowadays, ordinary differential equations are an integral part of applied and pure mathematics. As we know, many application problems which arise in day to day life require perfect solution in order to check the consistency of mathematical model. Unfortunately, every problem cannot be solved by analytical schemes. Therefore, it is necessary to have numerical schemes which give us the solution of any type of mathematical model which is based upon ordinary differential equations. In order to get an efficient numerical scheme, we propose a new scheme which has less computational complexity but at the same time it has third order accuracy which is considered to be a good indicator for numerical scheme to be used to serve practical purposed.

The general form of developed scheme for numerical solution of autonomous and non-autonomous initial value problems is given as

$$y'(x) = f\left(x, y(x)\right), \qquad y(x_0) = y_0; \quad x \in \left[x_0, X\right]. \tag{1}$$

In past literature, some authors have worked on deriving RK type schemes which are explicit and require fewer number of slope evaluations. We observed that in standard RK schemes, number of function evaluations depend upon its order. In third order (RK-3) schemes, three number of function evaluations are required per integration step. Most of authors wish to increase the efficiency with lower number of function evaluations. As a result, Rabiei, Faranak et al [11] authors proposed a new method of three explicit Improved Runge-Kutta (IRK) type schemes for solving first order ordinary differential equations. These schemes are two step in nature and requiring lower number of stages as compared to the classical Runge-kutta method. Zaib-Un-Nisa et al [12] proposed Improved Euler method with view to attain greater accuracy and efficiency. The performance of this method is based on autonomous initial value problems of ordinary differential equation. Zaib-Un-Nisa et al [13] constructed second numerical scheme for the solution ordinary differential equation and has analyzed on the basis bound on local truncation error and step size. Rabiei, F. and Ismail [14] developed the set of explicit third order Improved Runge-Kutta method in two stage and has lower number of function evaluations than existing method. Ochoche, A. [15] In this paper authors further modified Improved modified Euler method which is particularly used for autonomous Initial value problem. Ochoche, A [16] proposed a new method an improved approximation named as modified Euler method for obtaining solution of Initial value problems. Rabiei, F., Ismail, F., Norazak, S., & Emadi, S. [17] proposed a new method Improved Runge-Kutta method Nystrom (IRKM) method for solving second order ordinary differential equations. This method

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require lower number of function evaluation per step and numerical results are gives to illustrate the efficiency of proposed method and compared with existing method.

2. Derivation of the Proposed Scheme

The present motivation has been obtained from the research work carried out in [Third Order Improved Runge-Kutta Method for Solving Ordinary Differential Equation]
Consider the following structure of the proposed numerical scheme:

$$y_{n+1} = y_n + h\left(\beta_1 s_1 - \beta_{-1} s_{-1} + \beta_2\left(s_2 - s_{-2}\right)\right). \tag{2}$$

Where

$$s_1 = g\left(x_n\,, y_n\right)$$

$$s_{-1} = g\left(x_{n-1}\,, y_{n-1}\right)$$

$$s_2 = g\left(x_n + c_2 h\,,\, y_n + a_{21} s_1 h\right)$$

$$s_{-2} = g\left(x_{n-1} + c_2 h\,,\, y_{n-1} + a_{21} s_{-1} h\right)$$

Here s_1, s_{-1}, s_2, s_{-2} are slopes which expand through the multivariable Taylor's series that is after expansion of s_1, s_{-1}, s_2, s_{-2} take this form

$$s_1 = g\left(x_n, y_n\right).$$

$$\begin{aligned} s_{-1} = {} & g\left(x_n, y_n\right) + \left(-g_x - g_y g\right)h \\ & + \frac{h^2}{2!}\left[g_{xx} + 2g_{xy}g + g_{yy}g + \quad g_y^2 g + g_y g_x\right] \\ & + \frac{h^3}{3!}\begin{bmatrix} -g_{xxx} - 3g_{xxy}g - 3g_{xyy}g^2 - g_{yyy}g^3 - 4g^2 g_{yy}g_y - g_y^3 g \\ -g_x g_y^2 - 5g_{xy}g_y g - 3g_{yy}g_x g - g_{xx}g_y - 3g_{xy}g_x \end{bmatrix} + o(h^4). \end{aligned}$$

$$\begin{aligned} s_2 = {} & g\left(x_n, y_n\right) + \left(g_x c_2 + g_y c_2 g\right)h \\ & + \frac{h^2}{2!}\left[g_{xx}{c_2}^2 + 2g_{xy}g\,c_2^2 + g_{yy}g\,c_2^2\right] \\ & + \frac{h^3}{3!}\left[g_{xxx}{c_2}^3 + 3g_{xxy}g\,c_2^3 + 3g_{xyy}g^2\,c_2^3 + g_{yyy}g^3\,c_2^3\right] + o(h^4). \end{aligned}$$

$$\begin{aligned} s_{-2} = {} & g\left(x_n, y_n\right) + \left(-g_x - g_y g + c_2 g_y g + c_2 g_x\right)h \\ & + \frac{h^2}{2!}\begin{bmatrix} g_{yy}g^2c_2^2 + 2g_{xy}g\,c_2^2 - 2g_{yy}g^2c_2 - 4g_{xy}gc_2 + g^2 g_{yy} + g_{xx}c_2^2 \\ -2g_y^2 gc_2 - 2g_x g_y c_2 + g_y^2 g + 2g_{xy}g - 2g_{xx}c_2 + g_x g_y + g_{xx} \end{bmatrix} + o\left(h^3\right). \end{aligned}$$

Substituting s_1, s_{-1}, s_2, s_{-2} in eq(2) after this we equate with Taylor series

$$y_{n+1} = y_n + hg_n\left(\beta_1 - \beta_{-1}\right) + h^2\left(g_x + g_y g\right)\left(\beta_{-1} + \beta_2\right) + \frac{1}{2}h^3\begin{bmatrix} g_{xx}(-\beta_{-1} + 2\beta_2 c_2 - \beta_2) + \frac{1}{2}g_{xy}g(-2\beta_{-1} + 4\beta_2 c_2 - 2\beta_2) \\ +g_{yy}g^2(-\beta_{-1} + 2\beta_2 c_2 - \beta_2) + g_y{}^2 g(-\beta_{-1} + 2\beta_2 c_2 - \beta_2) \\ +g_y g_x(-\beta_{-1} + 2\beta_2 c_2 - \beta_2) \end{bmatrix}. \quad (3)$$

Generally, Taylor's series for a function g is as follows:

$$y(t_j + \Delta t) = y_n + \Delta t g + \frac{1}{2}\Delta t^2\left(g_x + g_y g\right) + \frac{1}{6}\Delta t^3\left(g_{xx} + 2g_{xy}g + g_{yy}g^2 + g_y^2 g + g_y g_x\right) + \frac{1}{24}\Delta t^4\begin{bmatrix} g_{xxx} + 3g_{xxy}g + 3g_{xyy}g^2 + g_{yyy}g^3 + 4g^2 g_{yy}g_y + g_y^3 g + \\ g_x g_y^2 + 5g_{xy}g_y g + 3g_{yy}g_x g + g_{xx}g_y + 3g_{xy}g \end{bmatrix} + o\left(h^5\right). \quad (4)$$

After comparison of equations, we get the non-linear system as follows.

$$\beta_1 - \beta_{-1} = 1, \quad \beta_{-1} + \beta_2 = \frac{1}{2}, \quad \beta_2 c_2 = \frac{5}{12} \quad (5)$$

There are four unknown constants on above set of equations. Since all the terms up to third power of h are cancelled therefore it is concluded that the developed scheme is third order accurate. It can be observed that there are three equations and four unknowns. Therefore it is necessary to assume c_2 to determine three other values. If we assume $c_2 = \frac{1}{7}$, then we will obtain the following RK type scheme:

$$y_{n+1} = y_n + h\left[-\frac{17}{12}s_1 + \frac{29}{12}s_{-1} + \frac{35}{12}\left(s_2 - s_{-2}\right)\right]. \quad (6)$$

$$s_1 = g\left(x_n, y_n\right), \qquad s_{-1} = g\left(x_{n-1}, y_{n-1}\right)$$

$$s_2 = g\left(x_n + \frac{1}{7}h, y_n + \frac{1}{7}k_1 h\right), \qquad s_{-2} = g\left(x_{n-1} + \frac{1}{7}h, y_{n-1} + \frac{1}{7}k_{-1}h\right)$$

3. STABILITY ANALYSIS

To check the stability of region through the test problem $\frac{dy}{dx} = \lambda y$, where λ is a complex number.

$$\frac{dy}{dx} = \lambda y(x); \quad \mathrm{Re}(\lambda) < 0.$$

$$s_1 = \lambda y_n, \qquad s_{-1} = \lambda y_{n-1} \quad (7)$$

$$s_2 = \lambda\left(1 + \frac{\lambda h}{7}\right)y_n, \qquad s_{-2} = \lambda\left(1 + \frac{\lambda h}{7}\right)y_{n-1}$$

We obtain the following stability polynomial

$$p(w,z) = w^2 + p(z)w + q(z), \quad (8)$$

where $p(z)=\frac{1}{12}\left(5z^2+18z+12\right)$, $q(z)=-\frac{1}{12}\left(5z^2+6z\right)$, and $z=h\lambda.$

4. RESULTS AND DISCUSSION

Different types of initial value problems have been examined with various step size values for testing the developed scheme. The developed scheme is compared with standard or existing schemes which are present in literature having same number function evaluations. To serve the purpose, maximum absolute error, absolute last error and CPU time at each step of integration are tabulated for all the scheme under consideration. Each data cell in every table of numerical experiments list maximum absolute error and absolute last error and CPU time values from top the bottom order. The numerical results are presented in table by utilizing these methods, methods in first row of each table. The amount of absolute maximum error, last error and CPU time are consequently presented in second, third and fourth column by using step size as h=0.1 and h=0.01.

Example: 01 $\frac{dy}{dx}=\frac{x^2}{y}$ $y(0)=1$ Exact solution $y(x)=\sqrt{\frac{2}{3}x^3+1}$

It is observed from table 1, new proposed and Huen's and Ralston scheme produce less errors (absolute maximum and last error). It means results are converging to the exact solution. It can be observed that new proposed scheme gives less absolute maximum error and last error with different step size than other two existing schemes named as Huens and Ralston scheme. It is also observed that new proposed method took much less time as comparison to other two existing schemes. Therefore, the new proposed scheme is better than Huens and Ralston schemes in terms of convergence and time.

Table 1. Error and CPU values for Example 1

Step Size/ Method	Proposed Method	Huen's Method	Ralston Method
0.1	2.8296e-04	1.3631e-03	4.1378e-04
	2.8296e-04	1.3631e-03	4.1378e-04
	4.6875e-02	1.5625e-02	0.00002e+00
0.01	2.9482e-07	1.2980e-05	3.8995e-06
	2.9482e-07	1.2980e-05	3.8995e-06
	2.8125e-01	0.0000e+00	0.0000e+00

Example 02 $\frac{dy}{dx}=x-y$ $y(0)=1$ Exact solution $y(x)=x+2e^{-x}-1$

It can be seen from table 2, new proposed, Huen's and Ralston scheme produce less errors (absolute maximum and last error). It means results are converging to the exact solution. It can be observed that Raltson's and Heun's Scheme give almost same errors (absolute maximum and last error) and new proposed scheme gives less absolute maximum error and last error with different step size than these two existing scheme. Therefore, the new proposed scheme is better than Huens and Ralston scheme in terms of convergence and time.

Table 2. Error and CPU values for Example 2

Step Size/ Method	Proposed Method	Huen's Method	Ralston Method
0.1	1.2223e-04	1.3231e-03	1.3231e-03
	1.2223e-04	1.3231e-03	1.3231e-03
	3.1250-02	0.0000e+00	0.0000e+00
0.01	1.2261e-07	1.2355e-05	1.2355e-05
	1.2261e-07	1.2355e-05	1.2355e-05
	2.6563e-01	0.0000e+00	0.0000e+00

Example 03 $\frac{dy}{dx}=\frac{\sin(x)}{y}$ $y(0)=1$ Exact solution $y(x)=\sqrt{3-2\cos(x)}$

It can be seen from table 3, new proposed, Huen's and Ralston scheme produce less errors (absolute maximum and last error). It means results are converging to the exact solution. It can be observed that Raltson's and Heun's Scheme give almost same errors (absolute maximum and last error) and new proposed scheme gives less absolute maximum error and last error with different step size than these two existing scheme. Therefore, the new proposed scheme is better than Huens and Ralston scheme in terms of convergence and time.

Table 3. Error and CPU values for Example 3

Step Size/ Method	Proposed Method	Huen's Method	Ralston Method
0.1	1.3055e-04	2.3495e-04	2.4995e-04
	1.3055e-05	2.3495e-04	2.4995e-04
	4.6875e-02	0.0000e+00	0.0000e+00
0.01	1.4937e-07	2.7244e-06	2.2459e-06
	6.8062e-08	2.7244e-06	2.2459e-06
	2.9688e-01	0.0000e+00	0.0000e+00

5. CONCLUSION

In this research study, a new numerical scheme has been developed by reducing number of slope evaluations per integration step. The developed scheme can be used to solve first and higher order ordinary differential equations with initial conditions. The developed scheme has third order accuracy and it is explicit in nature. Its error analysis is carried out via multivariable Taylor series. The scheme has much smaller error than existing schemes selected for comparison. As soon as we take step size larger (0.01) then maximum error and last error of the scheme remains below the errors produced by other schemes of same function evaluations. Proposed scheme yielded successful and better results in comparison of other second order numerical schemes and it provides well-organized way to estimate numerical solutions to initial value problems and systems of initial value problems**.** It is noticed that the accuracy of the developed scheme increases with decrease in the step size *h* value.

Examples in this research paper proved that the proposed scheme is more accurate and effective than some existing standard methods based upon the results obtained in the Tables 1 to 3 above in which the maximum error, last error and CPU time values related to all above mentioned schemes are better in case of our newly developed scheme. Hence the proposed scheme performs best among the existing methods. Based upon the three numerical problems solved above, it can be concluded the proposed scheme is powerful and effective for finding numerical solution of initial value problem in the fields of applied and pure mathematics.

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