A New Six Point Finite Difference Scheme for Nonlinear Waves Interaction Model

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Abstract: In the paper, the coupled 1D Klein-Gordon-Zakharov system (KGZ-equations in short) is considered as the model equation for wave-wave interaction in ionic media. A finite difference scheme is derived for the model equations. A new six point scheme, which is equivalent to the multi-symplectic integrator, is derived. The numerical simulation is also presented for the model equations.
Keywords: Coupled 1D Klein-Gordon-Zakharov system; Energy conservation; Six-point scheme

1. **Introduction**

Nonlinear evolution equations have a major role in various scientific fields, such as fluid mechanics, plasma physics, optical fibres, solid state physics, chemical physics, and geochemistry. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations. In recent years, solving nonlinear evolution equations has become a valuable task in many scientific areas including applied mathematics as well as the physical sciences and engineering. For this purpose, some accurate methods have been presented, such as Inverse scattering transform method[1], Bäcklund transformation method[2], Jacobi elliptic function method[3], F-expansion method[4], Hirota's bilinear method[5], the extended hyperbolic functions method[6, 7], Homotopy perturbation method[8], Bifurcation method [9–11] and so on.

Wave-wave interaction is an important problem for both physical and mathematical reasons. Physically, the wave-wave interaction or the wave collisions are common phenomena in science and engineering for both solitary and non-solitary waves. Mathematically solitary wave collision is a major branch of nonlinear wave interaction in ionic media.

We consider as the model equation the following Klein-Gordon-Zakharov (KGZ) equations

$$\begin{aligned} &E_{tt} - E_{xx} + E + nE = 0, \\ &n_{tt} - n_{xx} - |E|_{xx}^{2} = 0 \end{aligned} \tag{1}$$

This system is describe [12, 13] the interaction of a Langmuir wave and an ion sound wave in plasma. More precisely, *E* is the fast scale component of the electric field, where η denotes the deviation of ion density, in the paper, we discretize the system with finite difference schemes to show the multi-symplectic structure of CNLKGZ system.

2. A difference scheme for CNLKGZ system

We consider the following generalized CNLKGZ system

$$E_{tt} - E_{xx} + E + nE = 0,$$
$$n_{tt} - n_{xx} - |E|_{xx}^{2} = 0$$

Where $E = p(x,t) + iq(x,y), \ \ \eta = \mu(x,t) + i\xi(x,y)$ (2)

We have

$$p_{xx} - p_{tt} = (p(1+\mu) - q\xi) \qquad q_{xx} - q_{tt} = (p\xi + q(1+\mu)) \tag{3}$$
$$\mu_{tt} - \mu_{xx} - \left((p)_{xx}^{2} + (q)_{xx}^{2}\right) = 0 \qquad \xi_{tt} - \xi_{xx} = 0$$

Introducing the canonical momenta

$$p_x = a,\ q_x = b,\ p_t = c,\ q_t = d,\ \mu_x = e,\ \xi_x = f,\ \mu_t = g,\ \xi_t = h,\ u = \left(p^2\right)_x,\ v = \left(q^2\right)_x$$

The above system can be written in the following form

$$Kz_t + Lz_x = \nabla_z S(z) \tag{4}$$

with independent variable $(t, x) \in R^2$ and state variable $z \in R^d$, $d \geq 2$. Here $K, L \in R^{dxd}$ are two skew-symmetric matrices and $S : R^d \to R$ is a scalar-valued smooth function. ∇_z is the standard gradient in R^d. For $S(z)$ and $\nabla_z S(z)$, the system is multi-symplectic in the sense that K is a skew-symmetric matrix representative of the t direction and L is a skew-symmetric matrix representative of the x direction. S represents a Hamiltonian function [14, 15, 16]. The equation (4) is multi-symplectic in nature with the state variables

$$z = \left(p, q, a, b, c, d, \mu, \xi, e, f, g, h, u, v, p^2, q^2\right)^T \in R^{16}$$

So the $\nabla_z S(z) = \left(kp, kq, a, b, c, d, \mu, \xi, e, f, g, h, u, v, p^2, q^2\right)^T$

$$kp = (p(1+\mu) - q\xi); \qquad kq = (p\xi + q(1+\mu))$$

We get

$$a_x - c_t = (p(1+\mu) - q\xi) \qquad b_x - d_t = (p\xi + q(1+\mu))$$
$$p_x = a \qquad q_x = b,\ p_t = c,\ q_t = d$$
$$g_t - e_x - (u_x + v_x) = 0 \qquad h_t - f_x = 0$$
$$\mu_x = e, \qquad \xi_x = f,\ \mu_t = g,\ \xi_t = h,$$
$$u = \left(p^2\right)_x, \qquad v = \left(q^2\right)_x$$

and the pair of skew symmetric matrix K and L are

$$K=\begin{pmatrix}
0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}$$

and

$$L=\begin{pmatrix}
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&-1&0&0&0&-1&-1&0&0\\
0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix}$$

Using midpoint difference scheme to discretize multi-symplectic CNLKGZ system, we can get

$$\frac{a_{l+1}^{n+1/2}-a_l^{n+1/2}}{\Delta x}-\frac{c_{l+1/2}^{n+1}-c_{l+1/2}^{n}}{\Delta t}=\left(\widehat{p}(1+\widehat{\mu})-\widehat{q}\widehat{\xi}\right) \tag{5}$$

$$\frac{b_{l+1}^{n+1/2}-b_l^{n+1/2}}{\Delta x}-\frac{d_{l+1/2}^{n+1}-d_{l+1/2}^{n}}{\Delta t}=\left(\widehat{q}(1+\widehat{\mu})+\widehat{p}\widehat{\xi}\right) \tag{6}$$

$$\frac{p_{l+1}^{n+1/2}-p_l^{n+1/2}}{\Delta x}=a_{l+1/2}^{n+1/2} \tag{7}$$

$$\frac{q_{l+1}^{n+1/2}-q_l^{n+1/2}}{\Delta x}=b_{l+1/2}^{n+1/2} \tag{8}$$

$$\frac{p_{l+1/2}^{n+1}-p_{l+1/2}^{n}}{\Delta t}=c_{n+1/2}^{n+1/2} \tag{9}$$

$$\frac{q_{l+1/2}^{n+1}-q_{l+1/2}^{n}}{\Delta t}=d_{n+1/2}^{n+1/2} \tag{10}$$

$$\frac{g_{l+1/2}^{n+1}-g_{l+1/2}^{n}}{\Delta t}-\frac{e_{l+1}^{n+1/2}-e_{l}^{n+1/2}}{\Delta x}-\left(\frac{u_{l+1}^{n+1/2}-u_{l}^{n+1/2}}{\Delta x}+\frac{v_{l+1}^{n+1/2}-v_{l}^{n+1/2}}{\Delta x}\right)=0 \tag{11}$$

$$\frac{h_{l+1/2}^{n+1}-h_{l+1/2}^{n}}{\Delta t}-\frac{f_{l+1}^{n+1/2}-f_{l}^{n+1/2}}{\Delta x}=0 \tag{12}$$

$$\frac{\mu_{l+1}^{n+1/2}-\mu_{l}^{n+1/2}}{\Delta x}=e_{l+1/2}^{n+1/2} \tag{13}$$

$$\frac{\xi_{l+1}^{n+1/2}-\xi_{l}^{n+1/2}}{\Delta x}=f_{l+1/2}^{n+1/2} \tag{14}$$

$$\frac{\mu_{l+1/2}^{n+1}-\mu_{l+1/2}^{n}}{\Delta t}=g_{l+1/2}^{n+1/2} \tag{15}$$

$$\frac{\xi_{l+1/2}^{n+1}-\xi_{l+1/2}^{n}}{\Delta t}=h_{l+1/2}^{n+1/2} \tag{16}$$

$$\frac{\left(\left(p^{2}\right)_{x}\right)_{l+1}^{n+1/2}-\left(\left(p^{2}\right)_{x}\right)_{l}^{n+1/2}}{\Delta x}=u_{l+1/2}^{n+1/2} \tag{17}$$

$$\frac{\left(\left(q^{2}\right)_{x}\right)_{l+1}^{n+1/2}-\left(\left(q^{2}\right)_{x}\right)_{l}^{n+1/2}}{\Delta x}=v_{l+1/2}^{n+1/2} \tag{18}$$

$$\hat{\mu}=\mu_{l+1/2}^{n+1/2},\hat{q}=q_{l+1/2}^{n+1/2},\hat{p}=p_{l+1/2}^{n+1/2},\hat{\xi}=\xi_{l+1/2}^{n+1/2}$$

Eliminate *a, and c* from (5), using (7), (9) so we can get

$$\frac{2\left(p_{l+2}^{n+1/2}-2p_{l+1}^{n+1/2}+p_{l}^{n+1/2}\right)}{(\Delta x)^{2}}-\frac{2\left(p_{n+1/2}^{n+3/2}-2p_{n+1/2}^{n+1/2}+p_{n+1/2}^{n-1/2}\right)}{(\Delta t)^{2}}-$$
$$=\left(p_{n+3/2}^{n+1/2}\left(1+\mu_{n+3/2}^{n+1/2}\right)-(q)_{n+3/2}^{n+1/2}(\xi)_{n+3/2}^{n+1/2}\right)+\left(p_{n+1/2}^{n+1/2}\left(1+\mu_{n+1/2}^{n+1/2}\right)-(q)_{n+1/2}^{n+1/2}(\xi)_{n+1/2}^{n+1/2}\right) \tag{19}$$

Eliminate *b*, and *d* from (6), using (8), (10) so we can get

$$\frac{2\left(q_{l+2}^{n+1/2}-2q_{l+1}^{n+1/2}+q_{l}^{n+1/2}\right)}{(\Delta x)^{2}}-\frac{2\left(q_{n+1/2}^{n+3/2}-2q_{n+1/2}^{n+1/2}+q_{n+1/2}^{n-1/2}\right)}{(\Delta t)^{2}}-$$
$$=\left(q_{n+3/2}^{n+1/2}\left(1+\mu_{n+3/2}^{n+1/2}\right)+(p)_{n+3/2}^{n+1/2}(\xi)_{n+3/2}^{n+1/2}\right)+\left(q_{n+1/2}^{n+1/2}\left(1+\mu_{n+1/2}^{n+1/2}\right)+(p)_{n+1/2}^{n+1/2}(\xi)_{n+1/2}^{n+1/2}\right) \tag{20}$$

Eliminate g, *e*, *u*, *and v* from (11), using (13), (15), (17), (18) so we can get

$$\left(\frac{\mu_{n+1/2}^{n+3/2}-2\mu_{n+1/2}^{n+1/2}+\mu_{n+1/2}^{n-1/2}}{(\Delta t)^2}\right)-\left(\frac{\mu_{l+2}^{n+1/2}-2\mu_{l+1}^{n+1/2}+\mu_{l}^{n+1/2}}{(\Delta x)^2}\right)$$
$$-\left(\frac{\left(\left(p^2\right)_x\right)_{l+2}^{n+1/2}-2\left(\left(p^2\right)_x\right)_{l+1}^{n+1/2}+\left(\left(p^2\right)_x\right)_{l}^{n+1/2}}{(\Delta x)^2}+\frac{\left(\left(q^2\right)_x\right)_{l+2}^{n+1/2}-2\left(\left(q^2\right)_x\right)_{l+1}^{n+1/2}+\left(\left(q^2\right)_x\right)_{l}^{n+1/2}}{(\Delta x)^2}\right)=0 \qquad (21)$$

Eliminate *h, and f* from (12), using (14), (16) so we can get

$$\left(\frac{\xi_{n+1/2}^{n+3/2}-2\xi_{n+1/2}^{n+1/2}+\xi_{n+1/2}^{n-1/2}}{(\Delta t)^2}\right)-\left(\frac{\xi_{l+2}^{n-1/2}-2\xi_{l+1}^{n-1/2}+\xi_{l}^{n-1/2}}{(\Delta x)^2}\right)=0 \qquad (22)$$

Multiply (20) with *i.* and adding Eq. (19) then we can get

$$\frac{2\left(E_{l+2}^{n+1/2}-2E_{l+1}^{n+1/2}+E_{l}^{n+1/2}\right)}{(\Delta x)^2}-\frac{2\left(E_{n+1/2}^{n+3/2}-2E_{n+1/2}^{n+1/2}+E_{n+1/2}^{n-1/2}\right)}{(\Delta t)^2}=$$
$$E_{n+3/2}^{n+1/2}\left(1+\eta_{n+3/2}^{n+1/2}\right)+E_{n+1/2}^{n+1/2}\left(1+\eta_{n+1/2}^{n+1/2}\right) \qquad (23)$$

Multiply (22) with *i.* And adding Eq. (21) then we can get

$$\left(\frac{\eta_{n+1/2}^{n+3/2}-2\eta_{n+1/2}^{n+1/2}+\eta_{n+1/2}^{n-1/2}}{(\Delta t)^2}\right)-\left(\frac{\eta_{l+2}^{n+1/2}-2\eta_{l+1}^{n+1/2}+\eta_{l}^{n+1/2}}{(\Delta x)^2}\right)-\frac{\left(\left|E^2\right|_x\right)_{l+2}^{n+1/2}-2\left(\left|E^2\right|_x\right)_{l+1}^{n+1/2}+\left(\left|E^2\right|_x\right)_{l}^{n+1/2}}{(\Delta x)^2}=0 \qquad (24)$$

3. Numerical simulation

In this section, we present the numerical result of the Coupled Klein-Gordon-Zakharov system using the multi-symplectic integrator. Now we consider the CNLKGZ system

$$E_{tt}-E_{xx}+E+nE=0,$$
$$n_{tt}-n_{xx}-|E|_{xx}^{\;2}=0$$

with the initial value

$$u(x,o)=\sqrt{2}r_1\sec h(r_1x+\frac{1}{2}D_0)e^{tV_0x/4}$$
$$v(x,o)=\sqrt{2}r_1\sec h(r_2x+\frac{1}{2}D_0)e^{-tV_0x/4} \qquad (25)$$

We take the time step $\Delta t = 0.02$ and a space step $\Delta x = 0.02$, $-0.3 \le x \le 0.3$, $D_0 = 25$, $r_1 = r_2 = 1$ and $V_0=1$. In Fig. 1, the computation is done for $0 \le t \le 2$.

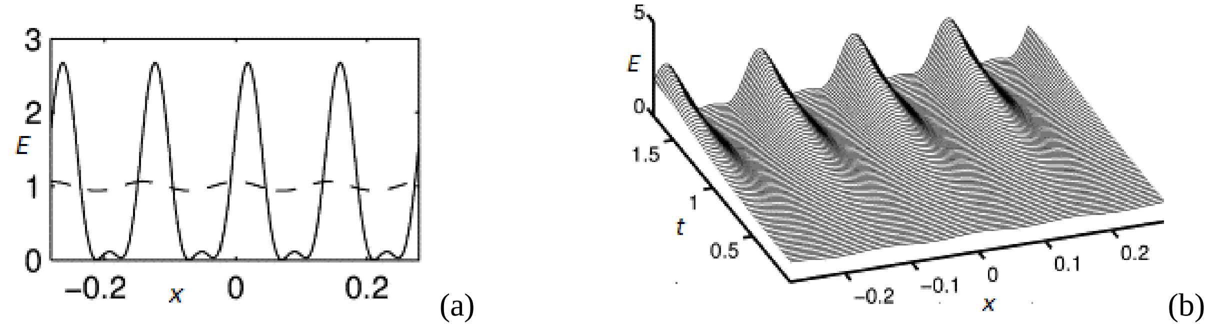

Fig. 1. Simulation results of the interaction of the two waves

4. **Conclusion:** In this paper, the multi-symplectic formulation for the coupled Klein-Gordon-Zakharov system is presented. Numerical simulations are also reported. We observe that the multi-symplectic scheme well simulates the evolution of the solitons. It has advantage for the long time computing accuracy and preserving the energy conservation property.

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