Quasi-Newton Embedded Augmented Lagrangian Algorithm for Discretized Optimal Proportional Control Problems

Olotu Olusegun (Corresponding author)
Department of Mathematical Sciences,
Federal University of Technology,
P. M. B. 704, Akure
Tel: +2348033509040 E-mail: segolotu@yahoo.ca

Dawodu Kazeem Adebowale
Department of Mathematical Sciences,
Federal University of Technology,
P. M. B. 704, Akure
Tel: +2348032541380 E-mail: dawodukazeem@yahoo.com

Abstract

In developing a robust algorithm for solving a class of optimal control problems in which the control effort is proportional to the state of the dynamic system, a typical model was studied which generates a constant feedback gain , an estimate of the Riccati for large values of the final time. Involving the third Simpson's Rule, a discretized unconstrained non-linear problem via the Augmented Lagrangian Method was obtained. This problem was consequently subjected to the Broydon-Fletcher-Goldberg-Shannon(BFGS) based on Quasi-Newton algorithm. The positive- definiteness of the estimated quadratic control operator was analyzed to guarantee its invertibility in the BFGS. Numerical examples were considered, tested and the results responded much more favourably to the analytical solution with linear convergence.

Keywords: Proportional control, feedback gain, Augmented Lagrangian Method, Discretization, BFGS , Simpson's Rule and Quasi-Newton Method

1. Introduction

The whole idea of developing a robust algorithm for solving optimal control problem with the augmented lagrangian method emanated from the basic idea of penalty function method. An extensive work with substantial influence on present day developments in multiplier method was by Poljak [11] who analyzed the rate of convergence of the quadratic penalty function method as an extension of the quadratic method of multipliers that was first proposed independently by Hestenes [6] and Powell [12]. It employs the direct method of generating the optimizer of the objective function without necessarily going through the rigorous indirect methods of calculus of variation, where the necessary and sufficient conditions must be derived and results expressed in differential-algebraic equation. This approach requires the transformation of the continuous-time optimal control problem into a discretized unconstrained NonLinear programing problem amenable to well proven numerical methods having the generation of sparse discretized matrices that prompt the convergence of the developed scheme. Betts [2] earlier gave a more practical method for solving the optimal control problems using the nonlinear programming model through the Newton-based iterations with a finite set of variables and constraints.

Many papers have recently been written by researchers in this field including Olotu & Olorunshola [10], Adekunle & Olotu [1], Olotu [3],Gollman et al [5], and host of others. They all seem to form their finite unconstrained programming equation using the exterior penalty method amenable to the conjugate gradient algorithm except for Olotu and Akeremale [9] using the Augmented Lagrangian Method which will form the focus of this paper, since it has faster convergence with higher accuracy when compared with the Exterior penalty method. The relevance of the real symmetric positive definite properties of the quadratic operator was tested in line with Ibiejugba and Onumanyi [7] for a well-conditioned scheme.

2. General formulation of the problem

The optimal control problem is modeled in order to find the state and control trajectories that optimize (minimize) the objective function (performance index) of the following problem.

$$Min\, J(x,w) = {}^{1}\!/_{2} \int_0^T [F(t, x(t), w(t)]dt \tag{1}$$

$$subject\ to \left[\begin{array}{l} \dot{x}(t) = g(t, x(t), w(t)),\ t \in [0, T] \\ x(0) = x_0,\ w(t) = m\,x(t),\ for\ p, q, a, b, m \in \mathbb{R}\ (real) > a \end{array} \right. \tag{2}$$

Where x and u are the state and control trajectories respectively describing the system. The numerical solution to the optimal control problem is a direct approximate method requiring the parameterization of each control using a set of nodal points which then become the variable in the resulting parameter optimization problem. This approach centres on the conversion of the continuous-time optimal control problem into a discretized Non-Linear Programming (NLP) problem via the augmented multiplier method which makes it amenable to the Quasi-Newton Algorithm so as to compute the near optimal control trajectories given that the feedback gain is a constant. This numerical result is then compared with that obtained from the indirect analytical method of calculus of variation. This indirect analytical approach to the optimal proportional control problem that results to a 2-point boundary value problem (BVP) arising from the Euler-Lagrange requires the application of the first order optimality conditions of the optimal control theory to obtain the optimal proportional constant (see subsection 5.0).

3. Materials and Methods

Consider an optimal control problem constrained with a linear regulator system described as

$$Min\, J(x,w) = {}^{1}\!/_{2} \int_0^T \left[px^2(t) + qw^2(t) \right] dt \tag{3}$$

$$subject\ to \left[\begin{array}{l} \dot{x}(t) = ax(t) + bw(t)\ \ t \in [0,T] \\ x(0) = x_0,\ w(t) = mx(t) \end{array} \right. \tag{4}$$

$$where\ p, q, a, b, m \in \mathbb{R}\ (real)\ and\ p, q > 0$$

We discretize the performance index of the continuous-time problem to generate large sparse discretized matrices using the *composite Simpson's rule* **[3]** of the form.

$$\int_0^n \mathbf{f[X(t)]dt} = \frac{\mathbf{h}}{\mathbf{3}} \left[\mathbf{f[x(t_0)] + f[x(t_n)] + 2\sum_{k=1}^{\frac{n}{2}-1} f[x(t_{2k})] + 4\sum_{k=1}^{\frac{n}{2}} f[x(t_{2k-1})]} \right] - \left(\frac{\mathbf{n}}{\mathbf{180}} \right) \mathbf{h^4 f^4}(\xi) \tag{5}$$

Where $x(t_j) = x_j$, $f \in C'[t_0, t_n]$, n be even, $h = \frac{t_n - t_0}{n}$ and $x_j = x_0 + jh$ for each $j = 0,1,2,...,n$.For

$E = \frac{(p+qm^2)}{3}$, $w(t) = m\,x(t)$, $h = \frac{T}{n}$ *and* $\bar{p} = 2h/3$, the discretised performance index becomes

$$J(x.w) = \tfrac{1}{2}\int_0^T [px^2(t) + qw^2(t)]dt = \tfrac{E}{2}\int_0^T x^2(t)dt \approx \frac{\mathbf{E\bar{p}}}{\mathbf{2}}\left[\frac{\mathbf{x_0^2}}{\mathbf{2}} + \sum_{k=1}^{\frac{n}{2}-1}(\mathbf{2x_{2k-1}^2} + \mathbf{x_{2k}^2}) + \frac{\mathbf{x_n^2}}{\mathbf{2}}\right] \qquad (6)$$

$$= \tfrac{1}{2}(x_1, x_2, ..., x_n)\begin{pmatrix} E\bar{p} & 0 & 0 & . & . & 0 & 0 \\ 0 & \frac{E\bar{p}}{2} & 0 & . & . & . & . \\ 0 & 0 & E\bar{p} & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & 0 \\ 0 & 0 & 0 & . & . & 0 & \frac{E\bar{p}}{4} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ . \\ x_n \end{pmatrix} + \frac{E\bar{p}x_0^2}{4} = \tfrac{1}{2}Z^T VZ + C \qquad (7)$$

Where $Z = (x_1, x_2, ... x_n)$ is a n -dimensional unit vector and $V = [v_{ij}]$ is a $n \times n$ dimensional coefficient matrix defined below as

$$V = [v_{ij}] = \begin{cases} E\bar{p} & i = j(odd) \\ \frac{E\bar{p}}{2} & i = j(even) \\ \frac{E\bar{p}}{4} & i = j = n \\ 0 & elsewhere \end{cases} \qquad (8)$$

and

$$C = \frac{E\bar{p}x_0^2}{4} \qquad (9)$$

The constraint of the optimal proportional control problem can also be discretized with the *2-step third order Simpson's rule* **[3]** as

$$\mathbf{x_{k+2} - x_k = \frac{h}{3}\{(f(x_{k+2}) + 4(x_{k+1}) + (x_k)\} + O(h^4)} \quad (10)$$

For $A = a + bm < 0$, $K = \frac{(Ah+3)}{(Ah-3)}$ *and* $L = \frac{4Ah}{(Ah-3)}$, the discretized constraint becomes

$$\dot{x}(t) = ax(t) + bw(t) \approx \mathbf{X_{k+2} = -KX_k - LX_{k+1}} \quad (11)$$

$$setting \begin{cases} k=0, LX_1 + X_2 = -KX_0 = -KC_0 \\ k=1, KX_1 + LX_2 + X_3 = 0 \\ k=2, KX_2 + LX_3 + X_4 = 0 \\ \quad . \\ k=n-2, KX_{n-2} + LX_{n-1} + X_n = 0 \end{cases} \quad (12)$$

$$= \begin{pmatrix} L & 1 & 0 & . & . & . & 0 \\ K & L & 1 & . & . & . & 0 \\ 0 & K & L & 1 & . & . & 0 \\ . & 0 & K & . & . & . & 0 \\ . & . & . & . & . & . & 0 \\ . & . & . & K & L & 1 & 0 \\ 0 & 0 & 0 & 0 & K & L & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ . \\ \\ . \\ . \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} -KC_0 \\ 0 \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}, \quad JZ = H \quad (13)$$

Where J is a $(n-1)\times n$ sparse coefficient tri-diagonal matrix defined by

$$J = [j_{ij}] = \begin{cases} L & 1 \le i \le n-1 & j = i \\ K & 2 \le i \le n-1 & j = i-1 \\ 1 & 1 \le i \le n-1 & j = i+1 \\ 0 & elsewhere \end{cases} \quad (14)$$

$Z = (x_1, x_2 \ldots\ldots\ldots\ldots x_n)^T$ is a n-dimensional column vector , $H = h_{i1}$ is a $(n-1)\times 1$ row vector equal to $\mathbf{-KC_0}$ for $i = 1$ and **0** for $2 \le i \le n-1$. The combination of equations (7 &13) by the parameter optimization gives the constrained discretized non-linear programming (quadratic) problem stated below:

$$\min F(Z) = Z^T VZ + C \ subject\ JZ = H \quad (15)$$

Where Z is a column vector of dimension $n\times 1$ for $Z^T = (x_{1,}x_2, ..., x_n)$, V is a sparse tri-diagonal matrix of dimension $n\times n$, J is a sparse coefficient matrix of dimension $(n-1)\times n$ and H is a row vector of dimension $(n-1)\times 1$.

The paper reviewed by Fiacco et al **[4]** on the application of the Augmented Lagrangian Method earlier proposed by Powell and Hestenes requires that the penalty term is added not only to the objective function but also to the lagrangian function to give

$$L_\rho(Z,\lambda,\mu) = {}^1\!/_2\, Z^T VZ + C + \lambda^T |JZ - H| + \frac{\mu}{2}\|JZ - H\|^2 \tag{16}$$

Expanding equation (16), we obtain a quadratic programming problem (17) which can be solved by the Quasi-Newton Method (QNM).

$$L_\rho(Z,\lambda,\mu) = {}^1\!/_2\, Z^T\left[V + \left(\mu J^T J\right)\right]Z + \left(\lambda^T J - \mu H^T J\right)Z + \left(\frac{\mu}{2}H^T H - \lambda^T H + C\right) \tag{17}$$

$$L_\rho(Z,\lambda,\mu) = {}^1\!/_2\, Z^T V_\rho Z + J_\rho^T Z + C_\rho \tag{18}$$

Where $Dim(V_\rho) = n\times n$, $Dim(J_\rho^T) = 1\times n$, and $Dim(C_\rho) = 1\times 1$ represent the dimensions of the various coefficients (discretized matrices stated below) of the lagrangian function.

$$V_\rho = \left[V + \left(\mu J^T J\right)\right], J_\rho^T = \left(\lambda^T J - \mu H^T J\right) and\; C_\rho = \left(\frac{\mu}{2}H^T H - \lambda^T H + C\right) \tag{19}$$

Lemma 3.1: The constructed quadratic operator $V_\rho = \left[V + \left(\mu J^T J\right)\right]$ of the formulated lagrangian function is real-symmetric and positive definite. See proof in **[9].**

For the purpose of avoiding an ill-conditioning in the scheme, it is required that the quadratic operator $V_\rho = \left[V + \left(\mu J^T J\right)\right]$ be real-symmetric and positive definite so as to make the unconstrained NLP problem amenable to any proven numerical method **[7]**.In this case, BFGS embedded Quasi-Newton Algorithm (inner loop) and the feasibility condition of the Augmented Lagrangian Method (outer loop) as defined in the formulated algorithm are evolved below for the optimal proportional control problem.

4. The numerical Algorithm for the scheme

(1) $Compute\ given\ variables\ V, M, N, m$

(2) $Choose\ Z_{0,0} \in \mathbb{R}^{n(2m+1)}, B_0 = I, \mathrm{T}^*(tolerance)\ and\ initialize\ \mu_j > 0, \lambda_j > 0\ by\ setting\ j = 0$

$(3a)\ set\ i=0\ and\ g_0=\nabla_z L(Z_{0,0})=\nabla L_0$

$(3b)\ compute\ V_i=[\mathrm{V}+(\mu_j \mathrm{J}^{\mathrm{T}}\mathrm{J})], \mathrm{M}_i=[\lambda_j \mathrm{J}-(\mu_j \mathrm{H}^{\mathrm{T}}\mathrm{J})], N_i=\left(\frac{\mu_j}{2}H^T H-\lambda_j^{\,T} H+C\right)$

$(3c)\ set\ S_i=-[B_i]g_i\,(search direction)\ and$

$$(3d)\ compute\ \alpha_i^*=\frac{-(M_i S_i+Z_i^T V_i S_i)}{S_i^T V_i S_i}\ (steplength)$$

$(3e)\ set\ Z_{j,i+1}=Z_{j,i}+\alpha_i^* S_i\ and$

$(3f)\ compute\ g_{k+1}=\nabla_z L(Z_{j,i+1},\lambda_j,\mu_j)$

$(3g)\ if\ \left\|\nabla_z L(Z_{j,i+1},\lambda_j,\mu_j\right\| \le T^*\ goto step\ 4\ else\ go to\ (3h)$

$(3h)\ set\ q_i=g_{i+1}-g_i\ and\ p_i=z_{i+1}-z_i$

$$(3i)\ compute\ B^u{}_i=[1+\frac{q_i^T B_i q_i}{p_i^T q_i}][\frac{p_i p_i^T}{p_i^T q_i}]-[\frac{(p_i q_i^T B_i)+(B_i q_i p_i^T)}{p_i^T q_i}]\ (BFGS)$$

$(3j)\ set\ B_{i+1}=B_i+B_i^u\ and\ repeat\ steps\ 3(a-f)\ for\ next\ i=i+1$

$(4)\ If\ \left\|JZ_{j,i+1}-H\right\| \le T^*\ stop!\ \text{Choose}\ Z_{j,i+1}^*\ and\ compute\ W_{j,i+1}^*=mZ_{j,i+1}^*$

$else\ go to\ step\ (5)(outer\ convergence\ from\ lagrangian)$

$(5)\ Update\ \mu_{j+1}=\mu_0\times 2^{j+1}(penalty)\ and\ \lambda_{j+1}=\lambda_j+\mu_j\left(JZ_j-H\right)(multiplier)$

$(6)\ Go to\ step\ (3)\ for\ next\ j=j+1$

5.0 The analytical optimal proportional control formulation

Theorem 5.1

Given the optimal control $w^*(t)$ *proportional to the solution* $x^*(t)$ *of the state system at a constant rate* $m \in \mathbb{R}$ *that minimizes the performance index* $J(x,w)$ *over* T*, then there exist a unique solution that satisfies the condition* $a+bm<0$ *with the proportional control constant and optimal objective values defined below as* $m=-\frac{1}{b}\left[a+\sqrt{\frac{(pb^2+qa^2)}{q}}\right]$ *and* $J^*(m)=\frac{x_0^2(p+qm^2)}{4(a+bm)}\left[e^{2(a+bm)T}-1\right]$ *respectively.*

Proof:

Considering the optimal control model defined by equations (3) and (4) above with a proportional constant m (independent of time) as the amount of control effort proportional to the deviation in the state such that

$w(t) = mx(t)$ given $x(t_0) = x_0,\ t \in [0,T],$

Therefore, $\dot{x}(t) = \frac{dx}{dt} = ax(t) + bw(t) = (a+bm)x(t) \qquad a,b,m \in \mathbb{R}$ (20)

The solution of equation (20) is $x(t) = x_0 \exp(a+bm)t = x_0 \exp(At)$ (21)

Applying the ***Euler-lagrange equation*** (E-L) to equation (3) above as a *necessary condition*,

$$E-L \Rightarrow \partial F/\partial x - \frac{d}{dt}[\partial F/\partial \dot{x}] = 0 \quad where\ F = \frac{1}{2}\left\{px^2 + q/b^2 [\dot{x} - ax]^2\right\}$$

$$[p + \frac{qa^2}{b^2}]x - \frac{q}{b^2}\ddot{x} = 0 \Rightarrow \ddot{x} - \frac{(pb^2 + qa^2)}{q}x = 0 \equiv \ddot{x} - r^2 x = 0 \tag{22}$$

$where\ r = \pm\sqrt{\frac{(pb^2+qa^2)}{q}} = [a+bm] = A.$ *Substituting* equation (21) into euation (22)

Substituting equation (21) into equation (22), the solution of equation (22) is expressed as a linear combination given below:

$x(t) = k_1 \exp(-rt) + k_2 \exp(rt) = x_0 \exp(At)\ \ for\ k_1 = x_0, k_2 = 0\ and\ A \le 0$ since $r > 0.$ Putting the state equation (21) into the objective function (performance index) gives

$$Min\ J(m) = J^*(x,w) = 1/2\int_0^T \left(px^2(t) + qw^2(t)\right)dt = \frac{x_0^{\ 2}(p+qm^2)}{2}\int_0^T \left(e^{2(a+bm)t}\right)dt$$

$$\Rightarrow J^*(m) = J^*(x,w) = \frac{x_0^2(p+qm^2)}{4(a+bm)}\left[e^{2(a+bm)T} - 1\right]\ at\ optimum \tag{23}$$

The imposition of the restriction $2(a+bm) < 0$ on the performance index $J'(m)$ is pertinent to control its exponential growth for infinitely large values of t so as to *guarantee the existence, convergence and asymptotic stability* of the solution as iterated by Morton **[8]**. By the optimality (sufficiency) condition,

$$\frac{\partial^2 J^*(m)}{\partial^2 m} = \frac{-2(p+qm^3)}{(a+bm)^3} > 0\ iff\ \left.\frac{\partial^2 J^*(m)}{\partial^2 m}\right|_{m=-1/b[a+r]} > 0 \Rightarrow \tag{24}$$

Minimum

Summarily,

$$\begin{cases} m = -\frac{1}{b}\left[a + \sqrt{\frac{(pb^2+qa^2)}{q}}\right] & \text{Proportional control Constant} \\ x(t) = x_o e^{(a+bm)t} \quad t\in[0,T] & \text{State solution (trajectory)} \\ w(t) = mx(t) = \left\{-\frac{1}{b}(a+\sqrt{\frac{(pb^2+qa^2)}{q}}\right\}x(t) & \text{Optimal Proportional control law} \\ J^*(m) = \frac{x_0^2(p+qm^2)}{4(a+bm)}\left[e^{2(a+bm)T}-1\right] & \text{Optimal objective value} \end{cases} \tag{25}$$

Where m is the proportional control constant, w (t) and x(t) are the control and state variables respectively for the proportional feedback control law that optimize the objective function.

6. Numerical examples and presentation of results

Example (6.1**)** *consider a one-dimensional optimal control problem*

$$Min\ J(x,w) = \tfrac{1}{2}\int_0^5 \left[2x^2(t) + w^2(t)\right]dt \tag{26}$$

Subject to $\dot{x}(t) = 2x(t) + 3w(t), \quad x(0)=1, \quad 0 \le t \le 5$

The *analytical objective value* from the proportional control result with the given parameters $p=2, q=1, a=2, b=3\,and\ x_0=1$ is given as $J_A = \mathbf{0.37168976}$.

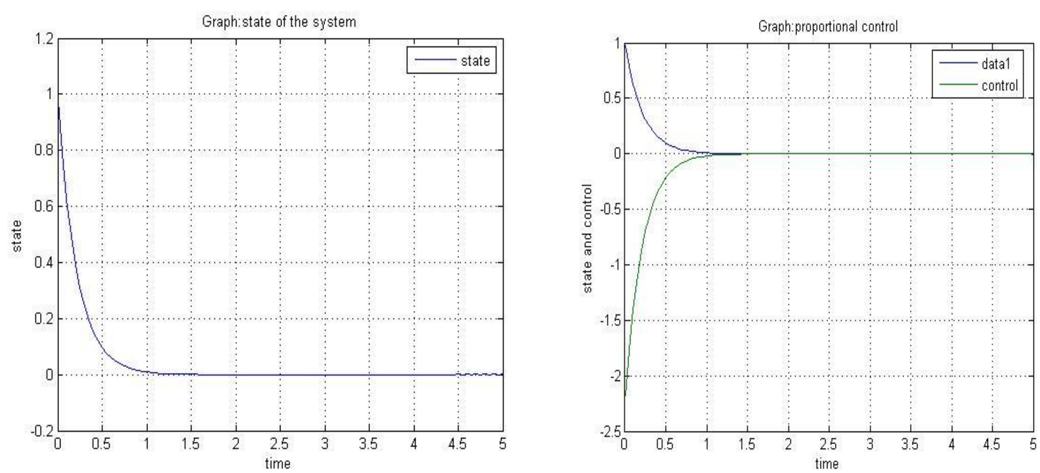
The numerical objective value from the *Quasi-Newton based augmented lagrangian method using MATLAB subroutine is* $J_N = \mathbf{0.37178228}$. Here we take $\mu = 1000,\ \varepsilon = 10^{-5},\ h = 0.05$ *for* large $T = 5$ as shown in the selected values of the parameters (**X_N** ,**W_N** ,**X_A** ,**W_A** ,**E_X** and **E_W**) representing the state , control and errors for both the numerical and analytical results respectively as outlined in the table 1 below:

Table 1: Comparison of analytical and numerical results from the newly developed scheme

t	**X_N**	**W_N**	**X_A**	**W_A**	**E_X**	**E_W**

0.0000	1.0000	-2.2301	1.0000	-2.2301	0.0000	0.0000
0.0500	0.7909	-1.7638	0.7909	-1.7638	0.0000	0.0000
0.1000	0.6256	-1.3952	0.6256	-1.3952	0.0000	0.0000
0.1500	0.4948	-1.1035	0.4953	-1.1046	-0.0005	0.0011
0.2000	0.3914	-0.8729	0.3916	-0.8733	-0.0002	0.0004
0.2500	0.3096	-0.6904	0.3096	-0.6904	0.0000	0.0000
0.3000	0.2448	-0.5459	0.2448	-0.5459	0.0000	0.0000
0.3500	0.1937	-0.4320	0.1937	-0.4320	0.0000	0.0000
0.4000	0.1532	-0.3417	0.1532	-0.3417	0.0000	0.0000
0.4500	0.1211	-0.2701	0.1212	-0.2703	-0.0001	0.0002
0.5000	0.0958	-0.2136	0.0958	-0.2136	0.0000	0.0000
1.0000	0.0092	-0.0205	0.0093	-0.0207	-0.0001	0.0002
2.0000	0.0001	-0.0002	0.0001	-0.0002	0.0000	0.0000
3.0000	0.0001	-0.0002	0.0000	0.0000	0.0001	-0.0002
4.0000	0.0003	-0.0007	0.0000	0.0000	0.0003	-0.0007
5.0000	0.0014	-0.0031	0.0000	0.0000	0.0014	-0.0031

optimal state and control trajectories for example 6.1



Example (6.2): *consider a one-dimensional optimal control problem*

$$Min\ J(x,w) = \frac{1}{2}\int_0^{10}\left[x^2(t)+w^2(t)\right]dt \tag{27}$$

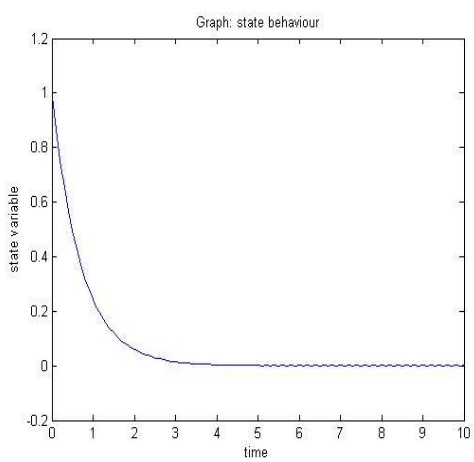
$$Subject\ to\ \ \dot{x}(t)=2x(t)+w(t),\ x(0)=1,\ \ 0\le t\le 10$$

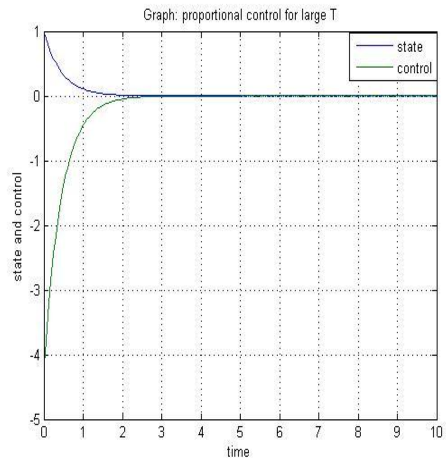
The analytical objective value from the proportional control result with the given parameters $p=1, q=1, a=2, b=1 and\ x_0=1$ is given as J_A **= 2.11803399** $for\ T=10$. The numerical objective value from the *Quasi-Newton based augmented lagrangian method using* ***MATLAB*** *subroutine is* $J_N = \mathbf{2.11816075}$. Here we take $\mu = 1000,\ \varepsilon = 10^{-5}$, $h = 0.05\ for\ \ T = 10$ as shown in the selected values of the parameters ($\mathbf{X_N}$,$\mathbf{W_N}$,$\mathbf{X_A}$,$\mathbf{W_A}$,$\mathbf{E_X}$ and $\mathbf{E_W}$) representing the state , control and errors for both the numerical and analytical results respectively as outlined in table 2 below.

Table 2: Comparison of analytical and numerical results from the newly developed scheme

t	X_N	W_N	X_A	W_A	E_X	E_W
0	1.0000	-4.2361	1.0000	-4.2361	0.0000	0.0000
0.0500	0.8942	-3.7880	0.8942	-3.7880	0.0000	0.0000
0.1000	0.7996	-3.3873	0.7996	-3.3873	0.0000	0.0000
0.1500	0.7150	-3.0290	0.7150	-3.0290	0.0000	0.0000
0.2000	0.6394	-2.7086	0.6394	-2.7086	0.0000	0.0000
0.2500	0.5718	-2.4221	0.5718	-2.4221	0.0000	0.0000
0.3000	0.5113	-2.1659	0.5113	-2.1659	0.0000	0.0000
0.3500	0.4572	-1.9367	0.4572	-1.9368	0.0000	0.0001
0.4000	0.4088	-1.7319	0.4088	-1.7319	0.0000	0.0000
0.4500	0.3656	-1.5487	0.3656	-1.5487	0.0000	0.0000
0.5000	0.3269	-1.3849	0.3269	-1.3849	0.0000	0.0000
1.0000	0.1069	-0.4527	0.1069	-0.4527	0.0000	0.0000
2.0000	0.0114	-0.0484	0.0114	-0.0484	0.0000	0.0000
3.0000	0.0012	-0.0052	0.0012	-0.0052	0.0000	0.0000
4.0000	0.0001	-0.0006	0.0001	-0.0006	0.0000	0.0000
5.0000	0.0000	-0.0001	0.0000	-0.0001	0.0000	0.0000
6.0000	0.0000	-0.0001	0.0000	0.0000	0.0000	0.0001
7.0000	0.0001	-0.0003	0.0000	0.0000	0.0001	0.0003
8.0000	0.0001	-0.0006	0.0000	0.0000	0.0001	0.0006
9.0000	0.0003	-0.0012	0.0000	0.0000	0.0003	0.0012
10.0000	0.0006	-0.0026	0.0000	0.0000	0.0006	0.0026

optimal state and control trajectories for example 6.2





6.1 Analysis of numerical results

Numerically, as the time increases ($T \to \infty$), the state decreases asymptotically to zero at a proportional rate to the control within a tolerance of $\varepsilon = \varepsilon_0 = 10^{-k}$, k *a positive integer* . Suppose at the final time T, $x(T) = x_0 e^{(a+bm)T} = x_0 e^{-rT}$, there exists a value $\varepsilon = \varepsilon_0 > 0$ for sufficiently large value of T such that

$$|x(T)| < \varepsilon \text{, then } T > \frac{\ln(\frac{x_0}{\varepsilon})}{r} \Rightarrow T > \frac{k\ln(10x_0)}{r} \; where \; r = \left[\sqrt{\frac{(pb^2 + qa^2)}{q}} \right].$$

As $t \to T$, there exists a time interval $t \in [T_k, T] \subset [0, T]$ for which both the state and the control variables oscillate within the neighbourhood of $|x(T)| = \varepsilon$ about $x(T) = 0$ [i.e. $|x(t)| < x(\varepsilon)$ *and* $|w(t)| < w(\varepsilon)$ *for* $T \to \infty$ *and* ε *very small*]. This then implies that for smooth feedback optimal control law (as in the *Riccati*) for which a constant proportionality function is guaranteed, it is required that the *tolerance* $\varepsilon = \varepsilon_0 > 0$ *be reduced for sufficiently large value of the final time* T. We then found out that by the discretization of the continuous optimal control for linear system with a quadratic objective function, we obtain a proportional feedback control, with the controller gaining the solution of a Riccati equation and a constant for an infinite control time. This then gives a more accurate result within the tolerance limit of the objective value (performance index) as the value of the upper integral limit increases.

6. Error and Convergence Analysis

Suppose $\{z_k\} \subset \mathbb{R}^n$ represents the sequence of solution approaching a limit z^*, then the error $e(z_k) = e_k$ such that $e(z_k) = e_k = |z_k - z^*| \geq 0$ for $\forall z_k \subset \mathbb{R}^n$ and $e(z^*) \neq 0$.

Assuming that the convergence ratio is represented by β, then

$$\beta = \lim_{k\to\infty} \frac{e_{k+1}}{e_k} = \lim_{k\to\infty} \frac{\|z_{k+1} - z^*\|}{\|z_k - z^*\|} \quad for \ e_k \neq 0 \ \forall k \tag{28}$$

For, $0 < \beta < 1$ and $\beta = 1$ implies quadratic, suoer-linear and sub-linear convergence respectively

However, the convergence ratio profile of the earlier example (6.1) using the newly developed algorithm (discretization scheme) expressed in terms of the penalty parameter (μ) used in the developed algorithm is shown in table 3 below.

Table 3: Convergence ratio profile

penalty parameter (μ)	**Objective value (r)**	**convergence ratio (β)**
1.0×10^2	0.384021	**0.1934**
1.0×10^3	0.371689	**0.1335**
1.0×10^4	0.371094	**0.1237**
1.0×10^5	0.370990	**0.0440**

The result on the table shows that the convergence ratio (β) hovers round the average figure of $\beta = \mathbf{0.12365}$ for increasing values of the penalty parameter which makes the convergence linear, though close to being super-Linear because of its proximity to zero. This convergence is satisfactory for optimization algorithms since the convergence is not close to one.

7. Conclusion

This research paper has demonstrated that the Quasi-Newton Algorithm constructed via the Augmented lagrangian multiplier method can generate the state and control variables that optimize the objective function with an optimal feedback (control) law whose feedback gain is a constant estimate of the analytically known result from the rigorous indirect method of the Riccati or Hamilton-Jacobi Bellman (HJB) with very high level of precision.

REFERENCES

1. Adekunle, A.I and Olotu. O. (2012). *An Algorithm for optimal control of Delay Differential Equation.* The Pacific Journal of Science and Technology, Vol. 13, Number 1, Pp. 228-237.
2. Betts J.T. (2001). Practical Methods for Optimal Control Problem Using Nonlinear Programming. SIAM, Philadelphia.
3. Olotu, O**. (**2010). Imbedding the Multiplier in a Discretized Optimal Control Problem with real Coefficients via the Penalty and Multiplier Methods. Journal of Mathematics and Statistics, 6(1), 23-27 **USA.**
4. Fiacco A. V. and McCormick G. P. (1968). *Nonlinear Programming: Sequential Unconstrained* Minimization Techniques. Wiley, New York.
5. Gollman L. D. and Maurer H, (2008). Optimal Control Problems with delays in state and control variables subject to mixed control – state constraints, Optimal control application method published by John Wiley, Interscience, USA.
6. Hestenes M.R. (1969). *Multiplier and gradient methods*. J.O.T.A. 4. 303 – 320.
7. Ibiejugba M.A and Onumanyi (1984). *A control operator and some of its Applications.* Journal of Mathematics Analysis and Application Vol. 103, 31-47.
8. Morton M.D 1969, *Optimization by variational methods.* , McGraw-Hill Publishers, New York.
9. Olotu O. and Akeremale O.C. (2012). *Augmented lagrangian method for discretized optimal control problems*, Journal of the Nigerian Association of mathematical physics, vol.2 Pp.185-192
10. Olotu O. and Olorunsola S.A.(2009). Convergence profile of a discretized scheme for constrained problem via the penalty –multiplier method. Journal of the Nigerian Association of Mathematical Physics, Vol.14 (1), 341-348.
11. Poljak, B.T., (1971). *The convergence rate of the penalty function method*. Z.V.ychist. Mat, Fiz, pp3-11.
12. Powell, M.J.D. (1969). *A method for nonlinear constraints in minimization problems in Optimization*, pp. 283 – 298. Academic Press, New York.