Jordan Generalized Higher Derivations in Prime Γ-Rings

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Abstract

Let M be a 2-torsion free prime Γ-ring satisfying the condition $a\alpha b\beta c = a\beta b\alpha c, \forall a, b, c \in M$ and $\alpha, \beta \in \Gamma$. Suppose that $F = (f_i)_{i\in N_0}$ be a Jordan generalized higher derivation of M with an associated higher derivation $D = (d_i)_{i\in N_0}$ of M, then we prove that F is a generalized higher derivation of M with the associated higher derivation D.

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1 Introduction

Let M and Γ be additive abelian groups. If there is a mapping $M \times \Gamma \times M \to M$ such that $\forall x, y, z \in M, \alpha, \beta \in \Gamma$,

- $(x + y)\alpha z = x\alpha z + y\alpha z$
- $x(\alpha + \beta)y = x\alpha y + x\beta y$
- $x\alpha(y + z) = x\alpha y + x\alpha z$
- $(x\alpha y)\beta z = x\alpha(y\beta z)$

then M is called a Γ-*ring*. This concept is more general than a ring and was introduced by Barnes [4].

A Γ-ring M is *2-torsion free* if $2a = 0$ implies $a = 0, \forall a \in M$. Besides, M is called a *prime* Γ-ring if $\forall a, b \in M, a\Gamma M\Gamma b = 0$ implies $a = 0$ or $b = 0$.
And, M is called *semiprime* if $a\Gamma M\Gamma a = 0$ with $a \in M$ implies $a = 0$. Note that every prime Γ-ring is clearly a semiprime Γ-ring. We define $[a, b]_\alpha$ by $a\alpha b - b\alpha a$ which is known as a *commutator* of a and b with respect to α.

Let M be a Γ-ring. An additive mapping $d : M \to M$ is called a *derivation* if $d(a\alpha b) = d(a)\alpha b + a\alpha d(b), \forall a, b \in M$ and $\alpha \in \Gamma$ and a *Jordan derivation* if $d(a\alpha a) = d(a)\alpha a + a\alpha d(a), \forall a \in M$ and $\alpha \in \Gamma$.
An additive mapping $f : M \to M$ is called a *generalized derivation* if there exists a derivation $d : M \to M$ such that $f(a\alpha b) = f(a)\alpha b + a\alpha d(b), \forall a, b \in M$ and $\alpha \in \Gamma$. This f is known as generalized derivation with an associated derivation d.
An additive mapping $f : M \to M$ is called a *Jordan generalized derivation* if there exists a Jordan derivation $d : M \to M$ such that
$f(a\alpha a) = f(a)\alpha a + a\alpha d(a), \forall a \in M$ and $\alpha \in \Gamma$. This derivation is known as Jordan generalized derivation with an associated derivation d.

Throughout the article, we use the condition $a\alpha b\beta c = a\beta b\alpha c, \forall a, b, c \in M$ and $\alpha, \beta \in \Gamma$ and this is represented by (*).

The notion of a Γ-ring has been developed by Nobusawa [8], as a generalization of a ring. Following Barnes [1] generalized the concept of Nobusawa's Γ-ring as a more general nature.
The concepts of derivation and Jordan derivation of a Γ-ring have been introduced by M.Sapanci and A.Nakajima in [9]. It is well-known that every derivation is a Jordan derivation but the converse is not true in general.
In [9], Sapanci and Nakajima proved every Jordan derivation in a 2-torsion free completely prime Γ-ring is a derivation.
Ceven and Ozturk [2] worked on Jordan generalized derivations in Γ-rings and they proved that every Jordan generalized derivation on some Γ-rings is a generalized derivation.
A. Nakajima [7] defined the notion of generalized higher derivations and investigated some elementary relations between generalized higher derivations and higher derivations in the usual sense. They also discussed Jordan generalized higher derivations and Lie derivations on rings.
W.Cortes and C.Haetinger [4] proved that every Jordan generalized higher derivations on a ring R is a generalized higher derivation.
In [5], M.Ferrero and C.Haetinger proved that every Jordan higher derivation of a 2-torsion free semiprime ring is a higher derivation.
C.Haetinger [6] extended the above results of prime rings in Lie ideals.

By the motivations of above works, in this article,we introduce a Jordan generalized higher derivations in Γ-rings. We prove that every Jordan generalized higher derivation in a 2-torsion free prime Γ-ring with the condition $a\alpha b\beta c = a\beta b\alpha c$ $\forall a, b, c \in M$ and $\alpha, \beta \in \Gamma$,is a generalized higher derivation of M.

2 Jordan Higher Derivations :

In this section, we show that every Jordan derivation of a 2-torsion free prime Γ-ring is a derivation. For this we prepare the following Lemmas.

2.1 Lemma: Let M be a Γ-ring and let d be a Jordan derivation of M. Then $\forall a, b, c \in M$ and $\alpha, \beta \in \Gamma$, the following statements hold:
(**i**) $d(a\alpha b + b\alpha a) = d(a)\alpha b + d(b)\alpha a + a\alpha d(b) + b\alpha d(a)$.
(**ii**) $d(a\alpha b\beta a + a\beta b\alpha a) = d(a)\alpha b\beta a + d(a)\beta b\alpha a + a\alpha d(b)\beta a + a\beta d(b)\alpha a + a\alpha b\beta d(a) + a\beta b\beta d(a)$.

In particular, if M is 2-torsion free and if M satisfies the condition (*), then
(iii) $d(a\alpha b\beta a) = d(a)\alpha b\beta a + a\alpha d(b)\beta a + a\alpha b\beta d(a)$.
(iv) $d(a\alpha b\beta c + c\alpha b\beta a) = d(a)\alpha b\beta c + d(c)\alpha b\beta a + a\alpha d(b)\beta c + c\alpha d(b)\beta a + a\alpha b\beta d(c) + c\alpha b\beta d(a)$.

Proof: Compute $d((a+b)\alpha(a+b))$ and cancel the like terms from both sides to obtain (i).

Then replace $a\beta b + b\beta a$ for b in (i) to get (ii).

Using the condition (*) and since M is 2-torsion free, (iii) follows from (ii).

And finally (iv) is obtained by replacing $a + c$ for a in (iii).

2.2 Definition: Let d be a Jordan derivation of a Γ-ring M. Then $\forall a, b \in M$ and $\alpha \in \Gamma$, we define $\phi_\alpha(a, b) = d(a\alpha b) - d(a)\alpha b - a\alpha d(b)$.
Then we have $\phi_\alpha(b, a) = d(b\alpha a) - d(b)\alpha a - b\alpha d(a)$.

2.3 Lemma: Let d be a Jordan derivation of a Γ-ring M. Then $\forall a, b, c \in M$ and $\alpha, \beta \in \Gamma$,the following statements hold:
(i) $\phi_\alpha(a, b) + \phi_\alpha(b, a) = 0$;
(ii) $\phi_\alpha(a + b, c) = \phi_\alpha(a, c) + \phi_\alpha(b, c)$;
(iii) $\phi_\alpha(a, b + c) = \phi_\alpha(a, b) + \phi_\alpha(a, c)$;
(iv) $\phi_{\alpha+\beta}(a, b) = \phi_\beta(a, b) + \phi_\beta(a, b)$.

Proof: Obvious.

Remark : Note that d is a derivation of a Γ-ring M if and only if
$\phi_\alpha(a, b) = 0, \forall a, b \in M$ and $\alpha \in \Gamma$.

2.4 Lemma: Let M be a 2-torsion free Γ-ring satisfying the condition(*) and let d be a Jordan derivation of M. Then
$\phi_\alpha(a, b)\beta m\gamma[a, b]_\alpha + [a, b]_\alpha\beta m\gamma\phi_\alpha(a, b) = 0, \forall a, b, m \in M$ and $\alpha, \beta, \gamma \in \Gamma$.

Proof: For any $a, b, m \in M$ and $\alpha, \beta, \gamma \in \Gamma$,we have
$d(a\alpha b\beta m\gamma b\alpha a + b\alpha a\beta m\gamma a\alpha b) = d((a\alpha b)\beta m\gamma b\alpha a + (b\alpha a)\beta m\gamma(a\alpha b))$
$= d(a\alpha b)\beta m\gamma b\alpha a + a\alpha b\beta d(m)\gamma b\alpha a + a\alpha b)\beta m\gamma d(b\alpha a) + d(b\alpha a)\beta m\gamma a\alpha b + b\alpha a\beta d(m)\gamma a\alpha b + b\alpha a\beta m\gamma d(a\alpha b)$,
by using Lemma 2.1(iv).

On the other hand

$d(a\alpha(b\beta m\gamma b)\alpha a + b\alpha(a\beta m\gamma a)\alpha b)$
$= d(a\alpha(b\beta m\gamma b)\alpha a) + d(b\alpha(a\beta m\gamma a)\alpha b)$
$= d(a)\alpha b\beta m\gamma b\alpha a + a\alpha d(b\beta m\gamma b)\alpha a + a\alpha b\beta m\gamma b\alpha d(a) + d(b)\alpha a\beta m\gamma a\alpha b + b\alpha d(a\beta m\gamma a)\alpha b + b\alpha a\beta m\gamma a\alpha d(b)$,
by using 2.1 (iii).
$= d(a)\alpha b\beta m\gamma b\alpha a + a\alpha d(b)\beta m\gamma b\alpha a + a\alpha b\beta d(m)\gamma b\alpha a + a\alpha b\beta m\gamma d(b)\alpha a + a\alpha b\beta m\gamma b\alpha d(a) + d(b)\alpha a\beta m\gamma a\alpha b +$
$b\alpha d(a)\beta m\gamma a\alpha b + b\alpha a\beta d(m)\gamma a\alpha b +$
$b\alpha a\beta m\gamma d(a)\alpha b + b\alpha a\beta m\gamma a\alpha d(b)$, by using Lemma 2.1(iii)

Comparing the two relations and using the definition, we obtain

$\phi_\alpha(a,b)\beta m\gamma b\alpha a + \phi_\alpha(b,a)\beta m\gamma a\alpha b + a\alpha b\beta m\gamma\phi_\alpha(b,a) + b\alpha a\beta m\gamma\phi_\alpha(a,b) = 0$
$\phi_\alpha(a,b)\beta m\gamma[a,b]_\alpha + [a,b]_\alpha\beta m\gamma\phi_\alpha(a,b) = 0, \forall a,b,m \in M$ and $\alpha,\beta,\gamma \in \Gamma$.

2.5 Lemma: Let M be a 2-torsion free prime Γ-ring and let $a,b \in M$.
If $a\alpha m\beta b + b\alpha m\beta a = 0, \forall m \in M, \alpha,\beta \in \Gamma$,then $a = 0$ or $b = 0$.

Proof: Replacing m by $s\delta a\mu t$ in $a\alpha m\beta b + b\alpha m\beta a = 0$
we have $a\alpha s\delta a\mu t\beta b + b\alpha s\delta a\mu t\beta a = 0$
Now $b\alpha s\delta a = -a\alpha s\delta b$ and $a\mu t\beta b = -b\mu t\beta a$
Substituting these we get, $-a\alpha s\delta b\mu t\beta a - a\alpha s\delta b\mu t\beta a = 0$.
This implies that $2a\alpha s\delta b\mu t\beta a = 0$.

Since M is 2-torsion free, $a\alpha s\delta b\mu t\beta a = 0$.
Therefore $(a\alpha s\delta b)\Gamma M\Gamma a = 0$.
Since M is prime, $a\alpha s\delta b = 0$ or $a = 0$
Suppose $a\alpha s\delta b = 0$
Again applying the primeness of M, we have $a = 0$ or $b = 0$.

Now all the prepared for the proof of the main result in this section.

2.6 Theorem: Let M be a 2-torsion free prime Γ-ring satisfying the condition(*) and let d be a Jordan derivation of M. Then d is a derivation of M.

Proof: By Lemma 2.4 and Lemma 2.5, we have
$\phi_\alpha(a,b) = 0$ or $[a,b]_\alpha = 0$, Since M is prime.

If $[a,b]_\alpha = 0, \forall a,b \in M, \alpha \in \Gamma$, then $a\alpha b = b\alpha a$.
Using this in Lemma 2.1(i), we have
$2d(a\alpha b) = 2d(a)\alpha b + 2a\alpha d(b)$.
Since M is 2-torsion free, we obtain d is a derivation of M.
If $\phi_\alpha(a,b) = 0$, then d is a derivation of M.

2.7 Definitions: Let $D = (d_i)_{i\in N_0}$ be a family of additive mappings of M such that $d_0 = id_M$, where id_M is an identity mapping on M and
$\mathbf{N}_0 = \mathbf{N}\bigcup\{0\}$. Then D is said to be a *higher derivation* of M if for each $n \in \mathbf{N}_0$ and $i,j \in \mathbf{N}_0, d_n(a\alpha b) = \sum_{i+j=n} d_i(a)\alpha d_j(b), \forall a,b \in M$ and $\alpha \in \Gamma$.

And D is said to be a *Jordan higher derivation* of M if for each $n \in \mathbf{N}_0$ and $i,j \in \mathbf{N}_0, d_n(a\alpha a) = \sum_{i+j=n} d_i(a)\alpha d_j(a), \forall a \in M$ and $\alpha \in \Gamma$.

2.8 Example: Let R be an associative ring with 1.
Let us consider $M = M_{1,2}(R)$ and $\Gamma = \{\begin{pmatrix} n.1 \\ 0 \end{pmatrix} : n \in \mathbf{Z}\}$.
Then M is a Γ-ring.

Let $f_n : R \to R$ be a higher derivation for each $n \in N_0$.

For $n \in N_0$, if we define additive mappings
$d_n : M \to M$ by $d_n((a,b)) = (f_n(a), f_n(a))$.
Then an easy verifications leads to us that d_n is a higher derivation of M.

Let $P = \{(a,a) : a \in R\}$. Then P is a Γ-ring contained in M.
In fact P is a sub Γ-ring.

Now if we define $d_n((a,a)) = (f_n(a), f_n(a))$, then d_n is a Jordan higher derivation of P.

2.9 Lemma: Let M be a 2-torsion free Γ-ring satisfying the condition (*) and $D = (d_i)_{i\in N}$ is a Jordan higher derivation of M.Then for $a, b, c \in M, \alpha, \beta \in \Gamma$ and $n \in \mathbf{N}$,we have
(i) $d_n(a\alpha b + b\alpha a) = \sum_{i+j=n}[d_i(a)\alpha d_j(b) + d_i(b)\alpha d_j(b)]$
(ii) $d_n(a\alpha b\beta a) = \sum_{i+j+k=n}[d_i(a)\alpha d_j(b)\beta d_k(a)]$
(iii) $d_n(a\alpha b\beta c + c\alpha b\beta a) = \sum_{i+j+k=n}[d_i(a)\alpha d_j(b)\beta d_k(c) + d_i(c)\alpha d_j(b)\beta d_k(a)]$

Proof: The proofs of (i) and (ii) are similar to the corresponding proofs of Lemma 2.1(i) and Lemma 2.1(iii).

Replacing a by $a + c$ in (ii), we obtain
$W = d_n((a+c)\alpha b\beta(a+c))$ and compute it by using (ii).
It follows that
$W = \sum_{i+j+k=n} d_i(a)\alpha d_j(b)\beta d_k(a) + \sum_{i+j+k=n} d_i(a)\alpha d_j(b)\beta d_k(c) +$
$\sum_{i+j+k=n} d_i(c)\alpha d_j(b)\beta d_k(a) + \sum_{i+j+k=n} d_i(c)\alpha d_j(b)\beta d_k(c)$.

Also

$W = d_n(a\alpha b\beta a) + d_n(c\alpha b\beta c) + d_n(a\alpha b\beta c + c\alpha b\beta a)$
$= \sum_{i+j+k=n} d_i(a)\alpha d_j(b)\beta d_k(a) + \sum_{i+j+k=n} d_i(c)\alpha d_j(b)\beta d_k(c)$
$+ d_n(a\alpha b\beta c + c\alpha b\beta a)$.
By comparing the two results for W, we obtain (iii).

2.10 Definition: For every Jordan higher derivation $D = (d_i)_{i\in N}$ of M, we define $\phi_n^\alpha(a,b) = d_n(a\alpha b) - \sum_{i+j=n} d_i(a)\alpha d_j(b), \forall a, b \in M, \alpha \in \Gamma$ and $n \in \mathbf{N}$

Remark : Note that $\phi_n^\alpha(a,b) = 0, \forall a, b \in M, \alpha \in \Gamma$ and $n \in \mathbf{N}$ if and only if D is a higher derivation of M.

2.11 Lemma: For every $a, b, c \in M, \alpha, \beta \in \Gamma$ and $n \in \mathbf{N}$, the following is true.
(i) $\phi_n^\alpha(a,b) + \phi_n^\alpha(b,a) = 0$
(ii) $\phi_n^\alpha(a+b,c) = \phi_n^\alpha(a,c) + \phi_n^\alpha(b,c)$
(iii) $\phi_n^\alpha(a,b+c) = \phi_n^\alpha(a,b) + \phi_n^\alpha(a,c)$
(iv) $\phi_n^{\alpha+\beta}(a,b) = \phi_n^\alpha(a,b) + \phi_n^\beta(a,b)$.

The proof is obvious from the definition 3.4 and Lemma 3.3(i).

2.12 Lemma: Let M be a 2-torsion free Γ-ring satisfying the condition (*) and let $D = (d_i)_{i\in N}$ be a Jordan higher derivation of M. Let $n \in N$ be and assume that $a, b \in M, \alpha, \beta, \gamma \in \Gamma$. If $\phi_n^\alpha(a,b) = 0$ for every $m < n$, then

$\phi_n^\alpha(a,b)\beta w\gamma[a,b]_\alpha + [a,b]_\alpha\beta w\gamma\phi_n^\alpha(a,b) = 0$, for every $w \in M$.

Proof: Consider $G = d_n(a\alpha b\beta w\gamma b\alpha a + b\alpha a\beta w\gamma a\alpha b)$.
First we compute $G = d_n(a\alpha(b\beta w\gamma b)\alpha a) + d_n(b\alpha(a\beta w\gamma a)\alpha b)$.
Using Lemma 3.3(ii), we have
$G = \sum_{i+j+k+h+l=n} d_i(a)\alpha d_j(b)\beta d_k(w)\gamma d_h(b)\alpha d_l(a) + d_i(b)\alpha d_j(a)\beta d_k(w)\gamma d_h(a)\alpha d_l(b))$

On the other hand

$G = d_n((a\alpha b)\beta w\gamma(b\alpha a) + (b\alpha a)\beta w\gamma(a\alpha b))$.
Using Lemma 3.3(iii), we obtain
$G = \sum_{r+s+t=n}(d_r(a\alpha b)\beta d_s(w)\gamma d_t(b\alpha a) + d_r(b\alpha a)\beta d_s(w)\gamma d_t(a\alpha b))$.

By the inductive assumption we can put
$d_n(x\alpha y)$ for $\sum_{i+j=r} d_i(x)\alpha d_j(y)$, when $r < n$, for $x = a, b$ and $y = b, a$.
Thus, we obtain by computation
$G = \sum_{i+j+k+h+l=n}(d_i(a)\alpha d_j(b)\beta d_k(w)\gamma d_h(b)\alpha d_l(a)) -$
$\sum_{r+s+t=n}(d_r(a\alpha b)\beta d_s(w)\gamma d_t(b\alpha a)) = -(\phi_n^\alpha(a,b)\beta w\gamma b\alpha a + a\alpha b\beta w\gamma\phi_n^\alpha(b,a))$.

Thus comparing both expression of G, we obtain

$\phi_n^\alpha(a,b)\beta w\gamma b\alpha a + a\alpha b\beta w\gamma\phi_n^\alpha(b,a) + \phi_n^\alpha(b,a)\beta w\gamma a\alpha b + b\alpha a\beta w\gamma\phi_n^\alpha(a,b) = 0$.
By Lemma 3.5(i), we have
$\phi_n^\alpha(a,b)\beta w\gamma b\alpha a - a\alpha b\beta w\gamma\phi_n^\alpha(a,b) - \phi_n^\alpha(a,b)\beta w\gamma a\alpha b + b\alpha a\beta w\gamma\phi_n^\alpha(a,b) = 0$.
This implies that
$\phi_n^\alpha(a,b)\beta w\gamma[a,b]_\alpha + [a,b]_\alpha\beta w\gamma\phi_n^\alpha(a,b) = 0$, for every $w \in M$.
Which is the required result.

Now we prove the main result.

2.13 Theorem: Let M be a 2-torsion free prime Γ-ring satisfying the condition(*). Then every Jordan higher derivation of M is a higher derivation of M.

Proof: By definition $\phi_0^\alpha(a,b) = 0, \forall a, b \in M$, and $\alpha \in \Gamma$.
Also by theorem 2.6, $\phi_1^\alpha(a,b) = 0, \forall a, b \in M$, and $\alpha \in \Gamma$.
Now we proved by induction.

Suppose that $\phi_m^\alpha(a,b) = 0$, This implies that
$d_m(a\alpha b) = \sum_{i+j=m} d_i(a)\alpha d_j(b), \forall a, b \in M$ and $\alpha \in \Gamma$ and $m < n$
Taking $a, b \in M$, by Lemma 3.6, we get
$\phi_n^\alpha(a,b)\beta w\gamma[a,b]_\alpha + [a,b]_\alpha\beta w\gamma\phi_n^\alpha(a,b) = 0$, for every $w \in M$ and $\alpha, \beta, \gamma \in \Gamma$.
Since M is prime, by Lemma 2.5, $\phi_n^\alpha(a,b) = 0$, or $[a,b] = 0$.

Suppose that $\phi_n^\alpha(a,b) = 0$, Then by the remark after the definition 3.4 , D is a higher derivation of M.

If $[a,b]_\alpha = 0$, then $a\alpha b = b\alpha a$.
Using this in Lemma 3.3(i), we obtain

$2d_n(a\alpha b) = 2\sum_{i+j=n} d_i(a)\alpha d_j(b)$.
Since M is 2-torsion free
we have $d_n(a\alpha b) = \sum_{i+j=n} d_i(a)\alpha d_j(b), \forall a, b \in M$ and $\alpha \in \Gamma$.
This completes the proof.

3 Jordan Generalized Higher Derivations

In this section, we introduce Jordan generalized higher derivations in Γ-rings. We begin with the following definitions.

3.1 Definitions: Let $F = (f_i)_{i \in N_0}$ be a family of additive mappings of a Γ-ring M such that $f_0 = id_M$, where id_M is an identity mapping on M and $\mathbf{N}_0 = \mathbf{N} \bigcup \{0\}$.

Then F is said to be a *generalized higher derivation* of M if there exists a higher derivation $D = (d_i)_{i \in N_0}$ of M such that for each $n \in \mathbf{N}_0$ and $i, j \in \mathbf{N}_0$, $f_n(a\alpha b) = \sum_{i+j=n} f_i(a)\alpha d_j(b), \forall a, b \in M$ and $\alpha \in \Gamma$.

And, F is said to be a *Jordan generalized higher derivation* of M if there exists a higher derivation $D = (d_i)_{i \in N_0}$ of M such that for each $n \in \mathbf{N}_0$ and $i, j \in \mathbf{N}_0$, $f_n(a\alpha a) = \sum_{i+j=n} f_i(a)\alpha d_j(a), \forall a \in M$ and $\alpha \in \Gamma$.

3.2 Example: Let R be an associative ring with 1 and let $F = (f_i)_{i \in N}$ be a generalized higher derivation on R. Then there exists a higher derivation $D = (d_i)_{i \in N}$ of R such that $f_n(xy) = \sum_{i+j=n} f_i(x)d_j(y), \forall x, y \in M$.
Now let us consider $M = M_{1,2}(R)$ and $\Gamma = \{\begin{pmatrix} n.1 \\ 0 \end{pmatrix} : n \in \mathbf{Z}\}$.
Then M is a Γ-ring.
If we define the mapping $K = (k_i)_{i \in \mathbf{N}}$ of M by
$k_n((x, y)) = (d_n(x), d_n(y))$, then K is a derivation of M.

Let $G = (g_i)_{i \in \mathbf{N}}$ be a additive mapping of M defined by
$g_n((x, y)) = (f_n(x), f_n(y))$. Then it is clear that G is a generalized higher derivation on M with the associated derivation K.

Let us define $N = \{(x, x) : x \in R\}$ of M.
Then N is a Γ-ring contained in M.

We define the mapping $G : N \to N$ by $g_n((x, x)) = (f_n(x), f_n(x))$ and $k_n((x, x)) = (d_n(x), d_n(x))$, then we have seen that G is a Jordan generalized higher derivation on N with the associated generalized higher derivation K.

We need the following Lemmas for proving the main results.

3.3 Lemma: Let M be a 2-torsion free Γ-ring satisfying the condition (*) and let $F = (f_i)_{i \in N}$ be a Jordan generalized higher derivation of M with the associated higher derivation $D = (d_i)_{i \in N}$ of M. Then for each fixed $n \in N$ and for every $a, b, c \in M$ and $\alpha, \beta \in \Gamma$ the following statements hold:

(i) $f_n(a\alpha b+b\alpha a)=\sum_{i+j=n}[f_i(a)\alpha d_j(b)+f_i(b)\alpha d_j(b)]$;
(ii) $f_n(a\alpha b\beta a)=\sum_{i+j+k=n}[f_i(a)\alpha d_j(b)\beta d_k(a)]$;
(iii) $f_n(a\alpha b\beta c+c\alpha b\beta a)=\sum_{i+j+k=n}[f_i(a)\alpha d_j(b)\beta d_k(c)+f_i(c)\alpha d_j(b)\beta d_k(a)]$;

Proof: Since $F=(f_i)_{i\in N_0}$ is a Jordan generalized higher derivation of M, we have $f_n(a\alpha a)=\sum_{i+j=n} f_i(a)d_j(a)$......(1)
Replacing $a+b$ for a in (1) and calculating, we obtain (i).

Then replace $a\beta b+b\beta a$ for b in (i) and using the condition (*), we obtain (ii), since M is 2-torsion free.

For (iii), we replace a by $a+c$ in (ii), we obtain for $w=(a+c)\alpha b\beta(a+c)$,
$f_n(w)=\sum_{i+j+k=n} f_i(a)\alpha d_j(b)\beta d_k(c)+\sum_{i+j+k=n} f_i(c)\alpha d_j(b)\beta d_k(a)+$
$\sum_{i+j+k=n} f_i(a)\alpha d_j(b)\beta d_k(a)+\sum_{i+j+k=n} f_i(a)\alpha d_j(b)\beta d_k(c)$.

On the other hand, using (ii)

$f_n(w)=f_n(a\alpha b\beta c+c\alpha b\beta a)+$
$\sum_{i+j+k=n}(f_i(a)\alpha d_j(b)\beta d_k(a)+f_i(c)\alpha d_j(b)\beta d_k(c)$.

Comparing the above two expressions for $f_n(w)$, we obtain (iii).

3.4 Definition: For every Jordan generalized higher derivation $F=(f_i)_{i\in\mathbf{N}}$ of M, we define
$\psi_n^\alpha(a,b)=f_n(a\alpha b)-\sum_{i+j=n} f_i(a)\alpha d_j(b), \forall a,b\in M, \alpha\in\Gamma$ and $n\in\mathbf{N}$

Remark : It is clear that F is a generalized higher derivation of M if and only if $\psi_n^\alpha(a,b)=0, \forall a,b\in M, \alpha\in\Gamma$ and $n\in\mathbf{N}$.

3.5 Lemma: For every $a,b,c\in M, \alpha,\beta\in\Gamma$ and $n\in\mathbf{N}$, the following statements hold:
(i) $\psi_n^\alpha(a,b)+\psi_n^\alpha(b,a)=0$
(ii) $\psi_n^\alpha(a+b,c)=\psi_n^\alpha(a,c)+\psi_n^\alpha(b,c)$
(iii) $\psi_n^\alpha(a,b+c)=\psi_n^\alpha(a,b)+\psi_n^\alpha(a,c)$
(iv) $\psi_n^{\alpha+\beta}(a,b)=\psi_n^\alpha(a,b)+\psi_n^\beta(a,b)$.

3.6 Lemma: Let M be a 2-torsion free Γ-ring satisfying the condition (*) and let $F=(f_i)_{i\in N}$ be a Jordan generalized higher derivation of M with the associated higher derivation $D=(d_i)_{i\in N}$ of M. Assume that $a,b\in M, \alpha,\beta,\gamma\in\Gamma$ and $n\in\mathbf{N}$. If $\psi_m^\alpha(a,b)=0$ for every $m<n$ with $\phi_m^\alpha(a,b)=0$ for every $m<n$, then $\psi_n^\alpha(a,b)\beta w\gamma[a,b]_\alpha+[a,b]_\alpha\beta w\gamma\phi_n^\alpha(a,b)=0$, for every $w\in M$.

Proof: Consider $G=f_n(a\alpha b\beta w\gamma b\alpha a+b\alpha a\beta w\gamma a\alpha b)$.
First we compute $G=f_n(a\alpha(b\beta w\gamma b)\alpha a)+f_n(b\alpha(a\beta w\gamma a)\alpha b)$.

Using Lemma 3.3(ii), we obtain
$G=\sum_{i+j+k+h+l=n} f_i(a)\alpha d_j(b)\beta d_k(w)\gamma d_h(b)\alpha d_l(a)+f_i(b)\alpha d_j(a)\beta d_k(w)\gamma d_h(a)\alpha d_l(b))$

On the other hand

$G=f_n((a\alpha b)\beta w\gamma(b\alpha a)+(b\alpha a)\beta w\gamma(a\alpha b))$.
Using Lemma 3.3(iii), we obtain

$G = \sum_{r+s+t=n}(f_r(a\alpha b)\beta d_s(w)\gamma d_t(b\alpha a) + f_r(b\alpha a)\beta d_s(w)\gamma d_t(a\alpha b))$.

By the inductive assumption we can put
$f_n(x\alpha y)$ for $\sum_{i+j=r} f_i(x)\alpha d_j(y)$, when $r < n$.
For $x = a, b$ and $y = b, a$, we obtain by computation

$G = \sum_{i+j+k+h+l=n}(f_i(a)\alpha d_j(b)\beta d_k(w)\gamma d_h(b)\alpha d_l(a)) -$
$\sum_{r+s+t=n}(f_r(a\alpha b)\beta d_s(w)\gamma d_t(b\alpha a)) = -(\psi_n^\alpha(a,b)\beta w\gamma b\alpha a + (a\alpha b)\beta w\gamma\phi_n^\alpha(b,a))$.

Thus comparing both expression of G, we obtain

$\psi_n^\alpha(a,b)\beta w\gamma b\alpha a + a\alpha b\beta w\gamma\phi_n^\alpha(b,a) + \psi_n^\alpha(b,a)\beta w\gamma a\alpha b + b\alpha a\beta w\gamma\phi_n^\alpha(a,b) = 0$.

By Lemma 3.5(i), we have
$\psi_n^\alpha(a,b)\beta w\gamma b\alpha a - a\alpha b\beta w\gamma\phi_n^\alpha(a,b) - \psi_n^\alpha(a,b)\beta w\gamma a\alpha b + b\alpha a\beta w\gamma\phi_n^\alpha(a,b) = 0$.
This implies that
$\psi_n^\alpha(a,b)\beta w\gamma[a,b]_\alpha + [a,b]_\alpha\beta w\gamma\phi_n^\alpha(a,b) = 0$, for every $w \in M$.
Which is the required result.

3.7 Lemma: Let M be a 2-torsion free prime Γ-ring satisfying the condition (*) and let $F = (f_i)_{i\in N}$ be a Jordan generalized higher derivation of M with the associated higher derivation $D = (d_i)_{i\in N}$ of M, having the same condition as in Lemma 3.6. Then $\psi_n^\alpha(a,b)\beta w\gamma[a,b]_\alpha = 0$.

Proof: By Theorem 2.13, we have $\phi_n^\alpha(a,b) = 0$ and thus the required result is obtained.

3.8 Lemma: Let M be a 2-torsion free semiprime Γ-ring satisfying the condition (*),then $\psi_n^\alpha(a,b)\beta w\gamma[c,d]_\alpha = 0, \forall a,b,c,d \in M, \alpha,\beta,\gamma \in \Gamma$.

The proof is similar to the proof of Lemma 2.19 in [3].

3.9 Lemma: Let M be a 2-torsion free semiprime Γ-ring satisfying the condition (*),then $\psi_n^\alpha(a,b)\beta w\gamma[c,d]_\delta = 0, \forall a,b,c,d \in M$ and $\alpha,\beta,\gamma,\delta \in \Gamma$.

The proof is similar to the proof of Lemma 2.20 in [3].

Now we have the position to prove the main result in this paper.

3.10 Theorem: Let M be a 2-torsion free prime Γ-ring satisfying the condition(*). Then every Jordan generalized higher derivation of M is a generalized higher derivation of M.

Proof: By definition $\psi_0^\alpha(a,b) = 0, \forall a,b \in M$, and $\alpha \in \Gamma$.
Also by theorem 2.4 in [2], we have $\psi_1^\alpha(a,b) = 0, \forall a,b \in M$, and $\alpha \in \Gamma$.

Now we proceed by induction.

Suppose that $\psi_m^\alpha(a,b) = 0$, This implies that
$d_m(a\alpha b) = \sum_{i+j=m} f_i(a)\alpha d_j(b), \forall a,b \in M$ and $\alpha \in \Gamma$ and $m < n$.

Taking $a, b, c, d \in M$ and $\alpha, \beta, \gamma, \delta \in \Gamma$. By Lemma 3.9, we get
$\psi_n^\alpha(a,b)\beta w\gamma[c,d]_\delta = 0$, for every $w \in M$.
Since M is prime, $\psi_n^\alpha(a,b) = 0$, or $[a,b]_\delta = 0$.

If $\psi_n^\alpha(a,b) = 0$, Then by the remark after the definition 3.4,
D is a generalized higher derivation of M.

If $[c,d]_\delta = 0$, then $c\delta d = d\delta c$, for every $c, d \in M$ and $\delta \in \Gamma$.
Using this in Lemma 3.3(i), we again have the required result.

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