 

# Generalized $(U, M)$-Derivations in Prime $\Gamma$-Rings

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**Abstract**

Let $M$ be a 2-torsion free prime $\Gamma$-ring satisfying the condition $a\alpha b\beta c = a\beta b\alpha c, \forall a, b, c \in M$ and $\alpha, \beta \in \Gamma$, $U$ be a Lie ideal of $M$ and f be a generalized $(U, M)$-derivation of $M$. Then we prove the following results:

1. If $U$ is an admissible Lie ideal of $M$, then $f(u\alpha v) = f(u)\alpha v + u\alpha d(v), \forall u, v \in U, \alpha \in \Gamma$.
2. If $u\alpha u \in U, \forall u \in U, \alpha \in \Gamma$, then $f(u\alpha m) = f(u)\alpha m + u\alpha d(m), \forall u \in U, m \in M, \alpha \in \Gamma$, where $d$ is a $(U, M)$-derivation of $M$.

**Keywords**: Derivation, Lie ideal, admissible Lie ideal, generalized $(U, M)$-derivation, $\Gamma$-ring, prime $\Gamma$-ring.

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## 1 Introduction

In [9], Herstein proved a well-known result in prime rings that every Jordan derivation is a derivation. Afterwards many Mathematicians studied extensively the derivations in prime rings. In [3], Awtar extended this result in Lie ideals. $(U, R)$-derivations in rings have been introduced by Faraj, Haetinger and Majeed [7], as a generalization of Jordan derivations on a Lie ideals of a ring. The notion of a $(U, R)$-derivation extends the concept given in [3]. In this paper [7], they proved that if $R$ is a prime ring, $\text{char}(R) \neq 2$, $U$ a square closed Lie ideal of $R$ and $d$ a $(U, R)$- derivation of $R$, then $d(ur) = d(u)r + ud(r), \forall, u \in U, r \in R$. This result is a generalization of a result in Awtar [3, Theorem in section 3].

The notion of a $\Gamma$-ring has been developed by Nobusawa [13], as a generalization of a ring. Following Barnes [4] generalized the concept of Nobusawa's $\Gamma$-ring as a more general nature. Now a days, $\Gamma$-ring theory is a showpiece of mathematical unification, bringing together several branches of the subject. It is the best research area for the Mathematicians and during 40 years, many classical ring theories have been generalized in $\Gamma$-rings by many authors.

The notions of derivation and Jordan derivation in $\Gamma$-rings have been introduced by Sapanci and Nakajima [14]. Afterwards, in the light of some significant results due to Jordan left derivation of a classical ring obtained by Jun and Kim in [11], some extensive results of left derivation and Jordan left derivation of a $\Gamma$-ring were determined by Ceven in [6]. In [8], Halder and Paul extended the results of [6] in Lie ideals.

In this article, we introduce the concept of $(U, M)$-derivation and generalized $(U, M)$-derivation, where $U$ is a Lie ideal of a $\Gamma$-ring $M$ . An example of a Lie ideal of a $\Gamma$-ring and an example of $(U, M)$-derivation and generalized $(U, M)$-derivation are given here. A result in [7, Theorem 2.8] is generalized in $\Gamma$-rings by the new concept of a $(U, M)$-derivation.

Let $M$ and $\Gamma$ be additive abelian groups. If there is a mapping
$M \times \Gamma \times M \to M$ (sending $(x, \alpha, y)$ into $x\alpha y$) such that
(i) $(x + y)\alpha z = x\alpha z + y\alpha z$,
$x(\alpha + \beta)y = x\alpha y + x\beta y$,
$x\alpha(y + z) = x\alpha y + x\alpha z$,
(ii) $(x\alpha y)\beta z = x\alpha(y\beta z)$,
for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, then $M$ is called a $\Gamma$-*ring*. This concept is more general than a ring and was introduced by Barnes [4]. A $\Gamma$-ring $M$ is called a *prime* $\Gamma$-*ring* if $\forall a, b \in M, a\Gamma M\Gamma b = 0$ implies $a = 0$ or $b = 0$ and $M$ is called *semiprime* if $a\Gamma M\Gamma a = 0$ (with $a \in M$) implies $a = 0$. A $\Gamma$-ring $M$ is *2-torsion free* if $2a = 0$ implies $a = 0, \forall a \in M$.

For any $x, y \in M$ and $\alpha \in \Gamma$, we induce a new product , the *Lie product* by $[x, y]_\alpha = x\alpha y - y\alpha x$.
An additive subgroup $U \subset M$ is said to be a *Lie ideal* of $M$ if whenever $u \in U, m \in M$ and $\alpha \in \Gamma$, then $[u, m]_\alpha \in U$.
In the main results of this article we assume that the Lie ideal U verifies $u\alpha u \in U, \forall u \in U$. A Lie ideal of this type is called a *square closed Lie ideal*.
Furthermore, if the Lie ideal $U$ is square closed and $U$ is not contained in $Z(M)$,where $Z(M)$denotes the center of $M$,then $U$ is called an *admissible Lie ideal* of $M$.

Let $M$ be a $\Gamma$-ring. An additive mapping $d : M \to M$ is called a *derivation* if $d(a\alpha b) = d(a)\alpha b + a\alpha d(b), \forall a, b \in M$ and $\alpha \in \Gamma$.
An additive mapping $d : M \to M$ is called a *Jordan derivation* if
$d(a\alpha a) = d(a)\alpha a + a\alpha d(a), \forall a \in M$ and $\alpha \in \Gamma$.

Throughout the article, we use the condition $a\alpha b\beta c = a\beta b\alpha c, \forall a, b, c \in M$ and $\alpha, \beta \in \Gamma$ and this is represented by (*).
We make the basic commutator identities:
$[x\alpha y, z]_\beta = [x, z]_\beta\alpha y + x[\alpha, \beta]_z y + x\alpha[y, z]_\beta$.
and $[x, y\alpha z]_\beta = [x, y]_\beta\alpha z + y[\alpha, \beta]_x z + y\alpha[x, z]_\beta$, $\forall a, b, c \in M$ and $\forall \alpha, \beta \in \Gamma$.
According to the condition (*), the above two identities reduces to
$[x\alpha y, z]_\beta = [x, z]_\beta\alpha y + x\alpha[y, z]_\beta$.
and $[x, y\alpha z]_\beta = [x, y]_\beta\alpha z + y\alpha[x, z]_\beta, \forall a, b, c \in M$ and $\forall \alpha, \beta \in \Gamma$.

# 2 Generalized $(U, M)$-Derivation

**2.1 Definition:** Let $M$ be a $\Gamma$-ring and $U$ be a Lie ideal of $M$. An additive mapping $d : M \to M$ is said to be a $(U, M)$- *derivation* of $M$ if $d(u\alpha m + s\alpha u) = d(u)\alpha m + u\alpha d(m) + d(s)\alpha u + s\alpha d(u), \forall u \in U, m, s \in M$ and $\alpha \in \Gamma$.

**2.2 Definition:** Let $M$ be a $\Gamma$-ring and $U$ be a Lie ideal of $M$. An additive mapping $f : M \to M$ is said to be a *generalized* $(U, M)$- *derivation* of $M$ if there exists a $(U, M)$-derivation $d$ of $M$ such that $f(u\alpha m + s\alpha u) = f(u)\alpha m + u\alpha d(m) + f(s)\alpha u + s\alpha d(u), \forall u \in U, m, s \in M$ and $\alpha \in \Gamma$.
The existence of a Lie ideal of a $\Gamma$-ring, $(U, M)$-derivation and a generalized $(U, M)$-derivation are confirmed by the following examples:

**2.3 Example:** Let $R$ be a commutative ring with characteristic 2 having unity element 1. Let $M = M_{2,2}(R)$ and
$\Gamma = \left\{ \begin{pmatrix} n_1.1 & n_3.1 \\ n_2.1 & n_4.1 \end{pmatrix} : n_i \in (Z - 2Z), i = 1, 2, 3, 4; n_1 = n_4, n_2 = n_3 \right\}$.
Then M is a $\Gamma$-ring.
Let $U = \left\{ \begin{pmatrix} x & y \\ y & x \end{pmatrix} : x, y \in R \right\}$.
Then $U$ is a Lie ideal of $M$.

Let us define a mapping $f : M \to M$ by
$f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & -d \end{pmatrix}, \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M$
Then there exists a $(U, M)$-derivation, $d$ of $M$ which is defined by
$d\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}, \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M$
Then $f$ is a generalized $(U, M)$-derivation of $M$.

**2.4 Lemma:** Let $M$ be a 2-torsion free $\Gamma$-ring satisfying the condition (*). $U$ be a Lie ideal of $M$ and $f$ be a generalized $(U, M)$-derivation of $M$.Then
(i) $f(u\alpha m\beta u) = f(u)\alpha m\beta u + u\alpha d(m)\beta u + u\alpha m\beta d(u), \forall u \in U, m \in M$ and $\alpha, \beta \in \Gamma$.
(ii) $f(u\alpha m\beta v + v\alpha m\beta u) = f(u)\alpha m\beta v + u\alpha d(m)\beta v + u\alpha m\beta d(v) + f(v)\alpha m\beta u + v\alpha d(m)\beta u + v\alpha m\beta d(u), \forall u, v \in U, m \in M$ and $\alpha, \beta \in \Gamma$.

**Proof:** By the definition of generalized $(U, M)$-derivation of $M$,we have
$f(u\alpha m + s\alpha u) = f(u)\alpha m + u\alpha d(m) + f(s)\alpha u + s\alpha d(u), \forall u \in U, m, s \in M$ and $\alpha \in \Gamma$.
Replacing $m$ and $s$ by $(2u)\beta m + m\beta(2u)$ and let
$w = u\alpha((2u)\beta m + m\beta(2u)) + ((2u)\beta m + m\beta(2u))\alpha u$.
Then by using (*)
$f(w) = 2(f(u)\alpha(u\beta m + m\beta u) + u\alpha d(u\beta m + m\beta u) + f(u\beta m + m\beta u)\alpha u + (u\beta m + m\beta u)\alpha d(u))$
$= 2(f(u)\alpha u\beta m + f(u)\alpha m\beta u + u\alpha d(u)\beta m + u\alpha u\beta d(m) + u\alpha d(m)\beta u + u\alpha m\beta d(u) + f(u)\beta m\alpha u + u\beta d(m)\alpha u + f(m)\beta u\alpha u + m\beta d(u)\alpha u + u\beta m\alpha d(u) + m\beta u\alpha d(u))$

$= 2(f(u)\alpha u\beta m + f(u)\alpha m\beta u + u\alpha d(u)\beta m + u\alpha u\beta d(m) + u\alpha d(m)\beta u + u\alpha m\beta d(u) + f(u)\alpha m\beta u + u\alpha d(m)\beta u + f(m)\alpha u\beta u + m\alpha d(u)\beta u + u\alpha m\beta d(u) + m\alpha u\beta d(u))$......(1).

On the other hand

$f(w) = f((2u\alpha u)\beta m + m\beta(2u\alpha u)) + 2f(u\beta m\alpha u) + 2f(u\alpha m\beta u)$
$= 2(f(u)\alpha u\beta m + u\alpha d(u)\beta m + u\alpha u\beta d(m) + f(m)\beta u\alpha u + m\beta d(u)\alpha u + m\beta u\alpha d(u) + 4f(u\alpha m\beta u)$
$= 2(f(u)\alpha u\beta m + u\alpha d(u)\beta m + u\alpha u\beta d(m) + m\alpha d(u)\beta u + m\alpha u\beta d(u) + f(m)\alpha u\beta u) + 4f(u\alpha m\beta u)$......(2)

By comparing (1) and (2) and since $M$ is 2-torsion free, we obtain
$f(u\alpha m\beta u) = f(u)\alpha m\beta u + u\alpha d(m)\beta u + u\alpha m\beta d(u)$......(3),
$\forall u \in U, m \in M$ and $\alpha, \beta \in \Gamma$.
If we linearize (3) on u, then (ii) is obtained.

**2.5 Definition:** Let $d$ be a $(U, M)$-derivation of $M$, then we define $\Phi_\alpha(u, m) = d(u\alpha m) - d(u)\alpha m - u\alpha d(m)$
$\forall u \in U, m \in M$ and $\alpha \in \Gamma$.

**2.6 Lemma:** Let $d$ be a $(U, M)$-derivation of $M$, then
(i)$\Phi_\alpha(u, m) = -\Phi_\alpha(m, u)$, $\forall u \in U, m \in M$ and $\alpha \in \Gamma$.
(ii) $\Phi_\alpha(u + v, m) = \Phi_\alpha(u, m) + \Phi_\alpha(v, m), \forall u, v \in U, m \in M$ and $\alpha \in \Gamma$.
(iii) $\Phi_\alpha(u, m + n) = \Phi_\alpha(u, m) + \Phi_\alpha(u, n), \forall u \in U, m, n \in M$ and $\alpha \in \Gamma$.
(iv) $\Phi_{\alpha+\beta}(u, m) = \Phi_\alpha(u, m) + \Phi_\beta(u, m), \forall u \in U, m \in M$ and $\alpha, \beta \in \Gamma$.
The proofs are obvious by using the definition 2.5

**2.7 Definition:** If $f$ is a generalized $(U, M)$-derivation of $M$ and $d$ is a $(U, M)$-derivation of $M$, then we define $\Psi_\alpha(u, m) = f(u\alpha m) - f(u)\alpha m - u\alpha d(m), \forall u \in U, m \in M$ and $\alpha \in \Gamma$.

**2.8 Lemma:** Let $f$ be a generalized $(U, M)$-derivation of $M$, then
(i)$\Psi_\alpha(u, m) = -\Psi_\alpha(m, u)$, $\forall u \in U, m \in M$ and $\alpha \in \Gamma$.
(ii) $\Psi_\alpha(u + v, m) = \Psi_\alpha(u, m) + \Psi_\alpha(v, m), \forall u, v \in U, m \in M$ and $\alpha \in \Gamma$. (iii) $\Psi_\alpha(u, m + n) = \Psi_\alpha(u, m) + \Psi_\alpha(u, n), \forall u \in U, m, n \in M$ and $\alpha \in \Gamma$.
(iv) $\Psi_{\alpha+\beta}(u, m) = \Psi_\alpha(u, m) + \Psi_\beta(u, m), \forall u \in U, m \in M$ and $\alpha, \beta \in \Gamma$.
The proofs are obvious by using the definition 2.7

**2.9 Lemma:** Let $M$ be a 2-torsion free prime $\Gamma$-ring satisfying the condition (*), $U$ an admissible Lie ideal of $M$ and $f$ a generalized $(U, M)$- derivation of $M$ then $\Psi_\alpha(u, v)\beta w\gamma[u, v]_\alpha = 0, \forall u, v, w \in U$ and $\alpha, \beta, \gamma \in \Gamma$.

**Proof:** Let $x = 4(u\alpha v\beta w\gamma v\alpha u + v\alpha u\beta w\gamma u\alpha v)$.
Then by using Lemma 2.4(ii), we have
$f(x) = f((2u\alpha v)\beta w\gamma(2v\alpha u) + (2v\alpha u)\beta w\gamma(2u\alpha v))$
$= f(2u\alpha v)\beta w\gamma 2v\alpha u + 2u\alpha v\beta d(w)\gamma 2v\alpha u + 2u\alpha v\beta w\gamma d(2v\alpha u) + f(2v\alpha u)\beta w\gamma 2u\alpha v + 2v\alpha u\beta d(w)\gamma 2u\alpha v + 2v\alpha u\beta w\gamma d(2u\alpha v)$

On the other hand, by using Lemma 2.4(i),we have
$f(x) = f(u\alpha(4v\beta w\gamma v)\alpha u + v\alpha(4u\beta w\gamma u)\alpha v)$
$= f(u)\alpha 4v\beta w\gamma v\alpha u + u\alpha d(4v\beta w\gamma v)\alpha u + u\alpha 4v\beta w\gamma v\alpha d(u) +$
$f(v)\alpha 4u\beta w\gamma u\alpha v + v\alpha d(4u\beta w\gamma u)\alpha v + v\alpha 4u\beta w\gamma u\alpha d(v)$
$= 4f(u)\alpha v\beta w\gamma v\alpha u + 4u\alpha d(v)\beta w\gamma v\alpha u + 4u\alpha v\beta d(w)\gamma v\alpha u +$
$4u\alpha v\beta w\gamma d(v)\alpha u + 4u\alpha v\beta w\gamma v\alpha d(u) + 4f(v)\alpha u\beta w\gamma u\alpha v +$
$4v\alpha d(u)\beta w\gamma u\alpha v + 4v\alpha u\beta d(w)\gamma u\alpha v + 4v\alpha u\beta w\gamma d(u)\alpha v +$
$4v\alpha u\beta w\gamma u\alpha d(v)$.

Comparing the right side of $f(x)$ and using the 2-torsion freeness of $M$
$f(u\alpha v)\beta w\gamma v\alpha u+u\alpha v\beta w\gamma d(v\alpha u)+f(v\alpha u)\beta w\gamma u\alpha v+v\alpha u\beta w\gamma d(u\alpha v) = f(u)\alpha v\beta w\gamma v\alpha u+u\alpha d(v)\beta w\gamma v\alpha u+u\alpha v\beta w\gamma d(v)\alpha u+u\alpha v\beta w\gamma v\alpha d(u)+f(v)\alpha u\beta w\gamma u\alpha v+v\alpha d(u)\beta w\gamma u\alpha v+v\alpha u\beta w\gamma d(u)\alpha v+v\alpha u\beta w\gamma u\alpha d(v)$.
Therefore
$(f(u\alpha v) - f(u)\alpha v - u\alpha d(v))\beta w\gamma v\alpha u + (f(v\alpha u) - f(v)\alpha u - v\alpha d(u))\beta w\gamma u\alpha v + u\alpha v\beta w\gamma(d(v\alpha u) - d(v)\alpha u - v\alpha d(u)) + v\alpha u\beta w\gamma(d(u\alpha v) - d(u)\alpha v - u\alpha d(v)) = 0$
By using the definitions 2.5 and 2.7, we obtain
$\Psi_\alpha(u,v)\beta w\gamma v\alpha u + \Psi_\alpha(v,u)\beta w\gamma u\alpha v$
$+ u\alpha v\beta w\gamma\Phi_\alpha(v,u) + v\alpha u\beta w\gamma\Phi_\alpha(u,v) = 0$

Now using Lemma 2.6(i)and 2.8(i), we have
$\Psi_\alpha(u,v)\beta w\gamma[u,v]_\alpha + [u,v]_\alpha\beta w\gamma\Phi_\alpha(u,v) = 0, \forall u,v,w \in U, \alpha,\beta,\gamma \in \Gamma$
Since $d$ is a $(U,M)$-derivation, we have $\Phi_\alpha(u,v) = 0, \forall u,v \in U$ and $\alpha \in \Gamma$. Using this we obtain the desired result.

**2.10 Lemma:** Let $U$ be a Lie ideal of a 2-torsion free prime $\Gamma$-ring $M$ and $U$ is not contained in $Z(M)$. Then there exists an ideal $I$ of $M$ such that $[I,M]_\Gamma \subseteq U$ but $[I,M]_\Gamma$ is not contained in $Z(M)$.

**Proof:** Since $M$ is 2-torsion free and $U$ is not contained in $Z(M)$, it follows from the result in [1] that $[U,U]_\Gamma \neq 0$ and $[I,M]_\Gamma \subseteq U$,where $I = I\Gamma[U,U]_\Gamma\Gamma M \neq 0$ is an ideal of $M$ generated by $[U,U]_\Gamma$.

Now $U$ is not contained in $Z(M)$ implies that $[I,M]_\Gamma$ is not contained in $Z(M)$; for if $[I,M]_\Gamma \subseteq Z(M)$, then $[I,[I,M]_\Gamma]_\Gamma = 0$, which implies that $I \subseteq Z(M)$ and hence $I \neq 0$ is an ideal of $M$, so $M = Z(M)$.

**2.11 Lemma:** Let $U$ be a Lie ideal of a 2-torsion free prime $\Gamma$-ring $M$ satisfying the condition (*) and $U$ is not contained in $Z(M)$. If $a,b \in M$ (resp.$b \in U$ and $a \in M$) such that $a\alpha U\beta b = 0, \forall\alpha,\beta \in \Gamma$, then $a = 0$ or $b = 0$.

**Proof:** By Lemma 2.10, there exists an ideal $I$ of $M$ such that $[I,M]_\Gamma \subseteq U$ and $[I,M]_\Gamma$ is not contained in $Z(M)$. Now take $u \in U, c \in I, m \in M$ and $\alpha,\beta,\gamma \in \Gamma$, we have $[c\alpha a\beta u,m]_\Gamma \in [I,M]_\Gamma \subseteq U$ and so

$0 = a\delta[c\alpha a\beta u,m]_\gamma\mu b, \forall\delta,\mu \in \Gamma.$
$= a\delta[c\alpha a,m]_\gamma\beta u\mu b + a\delta c\alpha a\beta[u,m]_\gamma\mu b$, by using (*)
$= a\delta[c\alpha a,m]_\gamma\beta u\mu b$ since $a\beta[u,m]_\gamma\mu b \in a\beta U\mu b = 0$
$= a\delta(c\alpha a\gamma m - m\gamma c\alpha a)\beta u\mu b$
$= a\delta c\alpha a\gamma m\beta u\mu b - a\delta m\gamma c\alpha a\beta u\mu b$
$= a\delta c\alpha a\gamma m\beta u\mu b$, by using assumption $a\beta u\mu b = 0$

Thus $a\delta I\alpha a\gamma M\beta U\mu b = 0$. If $a \neq 0$, then by the primeness of $M, U\mu b = 0$.
Now if $u \in U$ and $m \in M$, then $[u,m]_\alpha \in U, \forall\alpha \in \Gamma$.
Hence $[u,m]_\alpha\beta b = 0, \forall\beta \in \Gamma$. Since $m\alpha u\beta b = 0, u\alpha m\beta b = 0$.
Since $U \neq 0$, we must have $b = 0$.
In the similar manner, it can be shown that if $b \neq 0$, then $a = 0$.

**2.12 Lemma:** Let $M$ be a 2-torsion free prime $\Gamma$-ring and $U$ an admissible Lie ideal of $M$. Let $G_1, G_2, ..., G_n$ be additive groups, $S : G_1 \times G_2 \times ... \times G_n \to M$ and $T : G_1 \times G_2 \times ... \times G_n \to M$ be mappings which are additive in each argument. If $S_\alpha(a_1, ..., a_n)\beta x\gamma T_\alpha(a_1, ..., a_n) = 0$, for every $x \in U, a_i \in G, i = 1, 2, ..., n, \alpha, \beta, \gamma \in \Gamma$, then $S_\alpha(a_1, ..., a_n)\beta x\gamma T_\delta(b_1, ..., b_n) = 0$

**Proof:** It suffices to prove the case $n = 1$.
The general proof is obtained by induction on $n$.
If $S_\alpha(a)\beta x\gamma T_\alpha(a) = 0$, for every $u \in U, a \in G_1$, we get
$(T_\alpha(a)\beta x\gamma S_\alpha(a))\mu y\nu(T_\alpha(a)\beta x\gamma S_\alpha(a)) = 0$, for all $x, y \in U$ and $\mu, \nu \in \Gamma$.
Then by Lemma 2.11, $T_\alpha(a)\beta x\gamma S_\alpha(a) = 0$, for every $x \in U, a \in G_1$ and $\beta, \gamma \in \Gamma$.
Now linearizing $T_\alpha(a)\beta x\gamma S_\alpha(a) = 0$ we obtain
$S_\alpha(a)\beta x\gamma T_\alpha(b) + S_\alpha(b)\beta x\gamma T_\alpha(a) = 0$, for every $x \in U, a, b \in G_1$.
Hence $(S_\alpha(a)\beta x\gamma T_\alpha(b))\mu y\nu((S_\alpha(a)\beta x\gamma T_\alpha(b))$
$= -S_\alpha(a)\beta x\gamma T_\alpha(b)\mu y\nu S_\alpha(b)\beta x\gamma T_\alpha(a) = 0, \forall x, y \in U$.
By Lemma 2.11, $S_\alpha(a)\beta x\gamma T_\alpha(b) = 0$
Similarly we can prove that $T_\alpha(b)\beta x\gamma S_\alpha(a) = 0, \forall a, b \in G_1$ and $\alpha, \beta, \gamma \in \Gamma$.
Putting $\alpha + \delta$ for $\alpha$ in the equation $S_\alpha(a)\beta x\gamma T_\alpha(b) = 0$ and using Lemma 2.6(iv), we have
$S_\alpha(a)\beta x\gamma T_\delta(b) + S_\delta(a)\beta x\gamma T_\alpha(b) = 0$.
Therefore, we have $(S_\alpha(a)\beta x\gamma T_\delta(b))\mu y\nu(S_\alpha(a)\beta x\gamma T_\delta(b))$
$= -S_\alpha(a)\beta x\gamma T_\delta(b)\mu y\nu(S_\delta(a)\beta x\gamma T_\alpha(b) = 0$
Hence by Lemma 2.11, $S_\alpha(a)\beta x\gamma T_\delta(b) = 0$.

**2.13 Theorem:** Let $M$ be a 2-torsion free prime $\Gamma$-ring satisfying the condition (*), $U$ be an admissible Lie ideal of $M$ and $f$ be a generalized $(U, M)$- derivation of $M$, then $\Psi_\alpha(u, v) = 0, \forall u, v \in U$ and $\alpha \in \Gamma$.

**Proof:** By Lemma 2.9, we have $\Psi_\alpha(u, v)\beta w\gamma[u, v]_\alpha = 0, \forall u, v, w \in U$ and $\alpha, \beta, \gamma \in \Gamma$.
Using the Lemma 2.12 in the above relation, we obtain
$\Psi_\alpha(u, v)\beta w\gamma[x, y]_\delta = 0, \forall u, v, w, x, y \in U$ and $\alpha, \beta, \gamma, \delta \in \Gamma$.
Since $U$ is not contained in $Z(M)$, $[x, y]_\delta \neq 0$
So by Lemma 2.11, we get $\Psi_\alpha(u, v) = 0, \forall u, v \in U$ and $\alpha \in \Gamma$.

**Remark:** If we replace $U$ by a square closed Lie ideal in the theorem 2.13, then the theorem is also true.
Now we are in position to prove the main result.

**2.14 Theorem:** Let $M$ be a 2-torsion free prime $\Gamma$-ring satisfying the condition (*), $U$ a square closed Lie ideal of $M$ and $f$ be a generalized $(U, M)$- derivation of $M$, then $f(u\alpha m) = f(u)\alpha m + u\alpha d(m), \forall u \in U\ m \in M$ and $\alpha \in \Gamma$.

**Proof:** From Theorem 2.13 and the Remark after the Theorem 2.13, we have $\Psi_\alpha(u, v) = 0, \forall u, v \in U$ and $\alpha \in \Gamma$......(A)
Replace $v$ by $u\beta m - m\beta u$ in (A), we get
$\Psi_\alpha(u, u\beta m - m\beta u) = 0$.
Since $u\beta m - m\beta u \in U, \forall u \in U$, $m \in M$ and $\alpha, \beta \in \Gamma$.
Therefore $0 = \Psi_\alpha(u, u\beta m - m\beta u)$
$= f(u\alpha(u\beta m - m\beta u)) - f(u)\alpha(u\beta m - m\beta u) - u\alpha d(u\beta m - m\beta u)$
$= f(u\alpha u\beta m) - f(u\alpha m\beta u) - f(u)\alpha u\beta m + f(u)\alpha m\beta u - u\alpha d(u)\beta m - u\alpha u\beta d(m) + u\alpha d(m)\beta u +$

$u\alpha m\beta d(m)$
$= f(u\alpha u\beta m) - f(u)\alpha m\beta u - u\alpha d(m)\beta u - u\alpha m\beta d(u) - f(u)\alpha u\beta m + f(u)\alpha m\beta u - u\alpha d(u)\beta m - u\alpha u\beta d(m) + u\alpha d(m)\beta u + u\alpha m\beta d(m)$
$= f(u\alpha u\beta m) - f(u)\alpha u\beta m - u\alpha d(u)\beta m - u\alpha u\beta d(m)$

This implies that
$f(u\alpha u\beta m) = f(u)\alpha u\beta m - u\alpha d(u)\beta m - u\alpha u\beta d(m)$......(B)
Now let $x = u\alpha u\beta m + u\beta m\alpha u$.
Then by the definition of generalized $(U, M)$-derivation, we have
$f(x) = f(u)\alpha u\beta m + u\alpha d(u\beta m) + f(u\beta m)\alpha u + u\beta m\alpha d(u)$
$= f(u)\alpha u\beta m + u\alpha d(u)\beta m + u\alpha u\beta d(m) + f(u\beta m)\alpha u + u\beta m\alpha d(u)$......(C)

On the other hand
$f(x) = f(u\alpha u\beta m) + f(u\beta m\alpha u)$
$= f(u)\alpha u\beta m + u\alpha d(u)\beta m + u\alpha u\beta d(m) + f(u)\beta m\alpha u + u\beta d(m)\alpha u + u\beta m\alpha d(u)$......(D)
Comparing (C) and (D)
$(f(u\beta m) - f(u)\beta m - u\beta d(m))\alpha u = 0$
This gives
$\Psi_\beta(u, m)\alpha u = 0, \forall u \in U, m \in M$ and $\alpha, \beta \in \Gamma$......(E)
Linearize (E) on $u$ and using equation (E), we get
$\Psi_\beta(u, m)\alpha v + \Psi_\beta(v, m)\alpha u = 0$......(F)
Replace $v$ by $v\gamma v$ in equation (F), we obtain
$\Psi_\beta(u, m)\alpha v\gamma v + \Psi_\beta(v\gamma v, m)\alpha u = 0$.
Since $\Psi_\beta(v\gamma v, m) = 0, \forall v \in U, m \in M$ and $\beta, \gamma \in \Gamma$
This is seen in the equation (B) for $v\gamma v$ in place of $u\alpha u$.
Therefore, we have
$\Psi_\beta(u, m)\alpha v\gamma v = 0, \forall u, v \in U, m \in M$ and $\alpha, \beta, \gamma \in \Gamma$......(G)

If $U$ is noncentral, then replace $v$ by $u + v$ in equation (G) to obtain
$\Psi_\beta(u, m)\alpha(u + v)\gamma(u + v) = 0$
This implies that $\Psi_\beta(u, m)\alpha v\gamma u = 0, \forall u, v \in U, m \in M$ and $\alpha, \beta, \gamma \in \Gamma$.
By Lemma 2.11, $\Psi_\beta(u, m) = 0$, since $U$ is noncentral.

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