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Mathematical Interpretation Of Electromagnetic Lightning Discharge Propagation

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ABSTRACT

An electromagnetic lightning discharge propagation is being affected by attenuation parameter γ which is defined as the product of attenuation coefficient, n and time, t. The line integral of the developed lightning discharge propagation gave rise to the energy outflow, W. Then, the energy outflow per time gave rise to the lightning discharge power, $\dot{W}$ which is expressed as $-4\epsilon_0 v_L^2 ({}^{n}/_{z}) e^{2jwt}$. At initial time, $t = 0$, dimensional investigation showed that the value, $n = \{(Nm^2/c^2)(kg.ms^{-1})\}$ is defined as the product of Coulomb's constant, K_c and Impulse, I. This value is now called Umahi's coefficient of lightning discharge.

KEYWORDS: Umahi's coefficient, and Maxwell, Coulomb's, and Newton Equations.

1 INTRODUCTION

Over the past few decades, a reasonable number of rocket flights in the ionosphere (Kelley et. al., 1985; Li et. al., 1991) have detected electric field transients (EFT) due to lightning strikes at altitudes of 80-400 km, providing the first direct clarification evidence that Very Low Frequency (VLF) lightning energy generated waves can penetrate the ionosphere (Lay et. al., 2007). In addition lightning generated whistler's wave radiation can propagate into the outer magnetosphere (Holzworth et. al. 1999). Whistlers are dispersed VLF electromagnetic signatures of lightning discharges, received after propagation (Helliwell, 1965; Collier et. al., 2006). Developments in VLF (3-30kHz) electromagnetic lightning discharge detection and Global Positioning System (GPS) timing now allow both continental and oceanic lightning detection with comparable efficiency (Lay, 2008) and accuracy. In particular, the World-Wide Lightning Location Network (WWLLN) generally accepted to monitors global lightning discharge in realtime (Dowden et al., 2002; Lay et al., 2004, Rodger et al. 2004, 2005a). VLF Lightning monitoring with WWLLN is intrinsically long range, because it takes advantage of long-range propagation paths in the Earth-Ionosphere Wave-Guide (EIWG). The wave-guide propagation paths available in VLF allow useful detection over 10^4 km (Lay, 2008). Thus, it is clearly observed that lightning discharge propagation is inline with electromagnetic wave nature. This work is to study basically physical mathematical interpretation of the electromagnetic lightning discharge propagation.

2 BASIC THEORY

2.1 Electromagnetic Wave in Free Space

In Paul and Dale (2003), Maxwell's equations impose no limit on the frequency of electromagnetic waves. It extends continuously from the long radio waves ($\sim 10^4$ Hz and 3×10^4 m) to the very high energy gamma rays observed in cosmic radiation ($\sim 10^{24}$ Hz and 3×10^{-16} m).

The simple reduced cross product part of Maxwell's equation of plane electromagnetic wave in free space is given as:

$$\nabla \times \begin{Bmatrix} E \\ H \end{Bmatrix} + j\omega \begin{Bmatrix} \mu_0 E \\ \epsilon_0 H \end{Bmatrix} = 0 \qquad (1)$$

Taking the curl of (1) we have

$$\nabla^2 \times \begin{Bmatrix} E \\ H \end{Bmatrix} + \epsilon_0 \mu_0 \omega^2 \begin{Bmatrix} E \\ H \end{Bmatrix} = 0 \qquad (2)$$

This differential equation (2) is that of an unattenuated wave travelling at the velocity, $c = (\epsilon_0 \mu_0)^{-1/2}$. The possible solution of a plane wave travelling in the x-axis is generally expressed as $e^{i(kx-\omega t).}$

The expression of c exposes the basic constants of electromagnetism as the permittivity of free space ϵ_0 inline with Coulomb's Law and the permeability of free space, μ_0 inline with magnetic force law.

A plane electromagnetic wave propagating in free space is therefore transverse in nature, since it has no longitudinal components. Assuming that the wave in plane polarized having E-vector in the direction of the x-axis, its expression:

$$E_{(x,\omega,t)} = E_0(\omega)e^{j\omega(t-z/c)}i \tag{3}$$

$$H_{(y,\omega,t)} = \epsilon_0 c E_0(\omega)e^{j\omega(t-z/c)}j \tag{4}$$

Again, a plane electromagnetic wave in free space propagates in the direction of the poynting vector $S = E \times H$ as determined by

$$S_{(\boldsymbol{z},\omega,t)} = \epsilon_0 c E_0^2 e^{2j\omega(t-z/c)}k \tag{5}$$

This poynting vector S is assumed to be propagating in a homogenous and uniform isotropic media. Therefore the amplitude, E_0 is a function of the vector S. the magnitude of the amplitude has to be discussed in detail.

2.2 Amplitude Determination

Lee (1989) pointed out that effectively, all the VLF power from the first return stroke comes from the lowest 2km, which is a small fraction of a wavelength in the VLF band (10-100km), so the source of the VLF radiation is a short current element (Dowden et al., 2002). The assumption is that the phase of all the VLF Fourier components of the current is the same so that the initial (at zero range) phase is the same (ϕ_0) for all VLF fourier components of the radiated electrical field. (Dowden et al. 2002).

The vertical component of the stroke current dominates as far as VLF propagation in the EIWG is concerned (Lee, 1989), and for the lowest few hundred meters (Krider et. al., 1976). At range r from the single lightning stroke and at time t; the wave field can be expressed as

$$E_{(r,\omega,t)} = E_0(\omega)cos\big(\phi(\omega)\big) \tag{6}$$

Where $\phi(\omega) = \omega t - K(\omega)z + \phi_0$ and the vector, K is independent of frequency.

For the solution to be physically useful, $\phi(\omega)$ must not have any imaginary part if the wave function satisfy periodic boundary conditions by assuming the same value of amplitude, $E_0(\omega)$ at any distance ranging from 0 to r. Since the particle wave function is normalized, the probability density of the degree of freedom of the particle in unit (Eugen, 1961; Murugeshan and Sivaprasath, 2010). Then

$$\int_0^r \left|E_{(r,\omega,t)}\right|^2 dr = 1 \tag{7}$$

$$E_0^2(\omega)\int_0^r cos^2\big(\phi(\omega)\big)dr = 1 \tag{8}$$

Therefore

$$\frac{E_0^2(\omega)r}{2} = 1 \tag{9}$$

Equation (9) is a dimensionless parameter which contains the composition of the E_0 and all are equal to unity.

3 ELECTRIC FIELD WAVEFORM

The lightning events are classified into two main types: Cloud-to-Ground and In-Cloud. Each of these types of lightning events due generates electromagnetic wave to accompany the activity. The waveforms always clarify the propagation and impulsive characteristics of lightning activities. The impulsive contribution inline with the electromagnetic lightning propagation indicates a peak amplitude which occurs within 0-300μs for CG and 0-12 μs for IC. The changes in time scale of cloud-to-ground and in-cloud lightning electric field waveform are discussed below.

3.1 Cloud-to-Ground Lightning Discharge.

The cloud to ground (CG) electromagnetic lightning categorization is in two main types: negative (-) and positive (+). The negative charge belts in the cloud are typically lower in altitude than the positive charge belts, stroke channels lengths for (-) CGs are typically 5 -8km. It is the current flowing in this kilometer (km) length channels that allows the generation of radiation with km-scale wave lengths (3-30kHz, VLF) (Lay,E.H, 2008). An example of an electric field waveform as detected by a lightning receiver 474km from the lightning stroke are shown in figure 1.

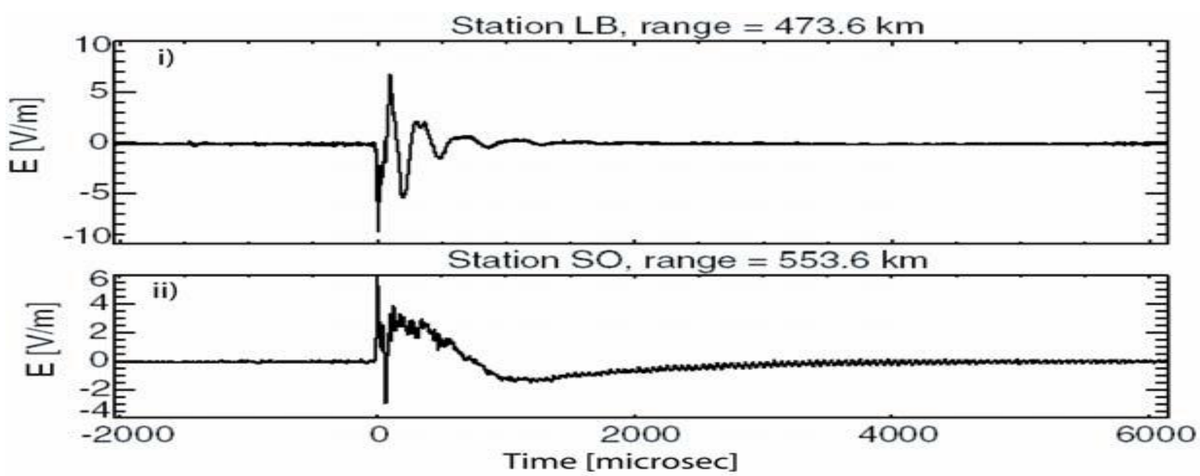

Figure 1: Example of electric field changes due to lightning: (i) a negative CG stroke and (ii) a positive CG stroke.(Smith et al., 2002).

3.2 In-cloud Lightning Discharge

A newly discovered type of in-cloud (IC) lightning: Narrow Bipolar Event (NBE), transfers electric charge between cloud charge layers but is usually strong (peak currents ~30-40kA) and significantly very fast (~25μs) when compared to common IC events (LeVine, 1980; Willett, 1989; Smith et al., 1999; Lay,E.H, 2008). An example of NBE electric field waveform is shown in figure 2.

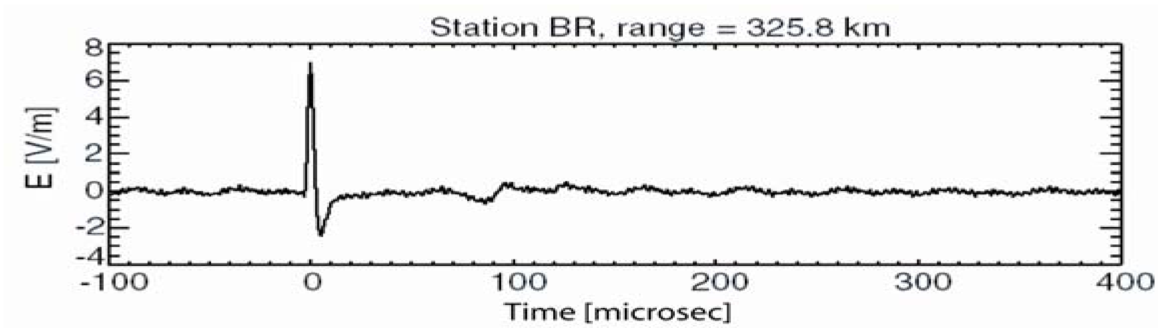

Figure 2: Example of electric field change due to NBE lightning stroke.(Smith et al., 2002).

4 MATHEMATICAL ANALYSIS

The observed cloud to ground electromagnetic lightning discharge has a fluctuating power and impulsive nature when propagating through atmosphere. The electromagnetic wave in line with lightning discharge has a significant factor that attenuates its power out flow. Partial deferential and integration processes have to be applied to Maxwell Equation for this investigation.

4.1 Lightning Discharge Propagation Method

Lightning discharges are powerful impulsive electrostatic sources of electromagnetic energy over a wide bandwidth (up to optical) (Rodger et al., 2004), with significant radiated electromagnetic power from a few hertz to several hundred megahertz (Magono, 1980), and the bulk of the energy radiated in the frequency bands <30 kHz (Pierce, 1977).

Lightning is easily detected and measured at VLF, at ranges of several thousand kilometers (km). Propagation over such ranges in the EIWG disperses the initial sharp pulse of the lightning stroke into a wave train lasting a millisecond or more.

The amplitude of the received wave train ("sferic") rises slowly (a few hundred microseconds) from the noise floor (Dowden et al., 2002). Whistlers are dispersed VLF electromagnetic signatures of lightning discharges, received after propagation (Helliwell, 1965; Collier et al., 2006).

The plane electromagnetic wave propagate as a result of $E \times H$ in (5) is assumed to be inline with lightning discharge waveform E_L and speed of lightning, v_L: Since the amplitude varies, an attenuation parameter, γ is introduced and expressed as

$$E_{(z,\omega,t)} = v_L \epsilon_0 \{E_0(1-\gamma)\}^2 e^{2j\omega(t-z/v_L)} K \qquad (10)$$

Where $\gamma = nt$ which is the product of attenuation coefficient, n and time, t and v_L is the speed of lightning discharge.

We can now calculate the out flux of energy, W flowing along the discharge pathway (assuming the pathway in enclosed in a cylinder of distance, z), by evaluating the line integral of (10) as follows:

$$W_{(z,\omega,t)} = \epsilon_0 v_L E_0^2 \int_0^z \left\{1 - 2n\left(t - \frac{z}{v_L}\right) + n^2\left(t - \frac{z}{v_L}\right)\right\} e^{2j\omega\left(t-\frac{z}{v_L}\right)} dz \qquad (11)$$

Let $a = \epsilon_0 v_L E_0^2$, and assume the parameter n to be small then n2 should be very small and could be neglected. Thus ,the total energy out flux:

$$W = \frac{-a}{j\omega}\left\{\frac{v_L}{2} + nv_L t + zn + \frac{2v_L}{2\omega}\right\} e^{2j\omega\left(t-\frac{z}{v_L}\right)} \qquad (12)$$

Finally, the VLF band (3-30 KHz) contains the highest power spectral density of lightning radiation (Malan, 1963, Pierce 1977). Therefore, the lightning discharge power, W is expressed as the energy per time,

$$\dot{W} = -\frac{anv_L}{j\omega}.2\omega.e^{2j\omega t} = \{-2anv_L\}e^{2j\omega t} \qquad (13)$$

Substituting $E_0^2(\omega) = \frac{2}{z}$ (of equation (9)) $\alpha = \epsilon_0 v_L E_0^2 = \frac{2\epsilon_0 v_L}{z}$ equation (22) becomes

$$\dot{W} = -4\epsilon_0 v_L^2 \left(\frac{n}{z}\right) e^{2j\omega t} \qquad (14)$$

The expression of equation (14) describes the electromagnetic lightning propagation.

5 CONCLUSION

The expression of the power in the lightning discharge is shown in equations (14) which have a propagating nature. One greater question is the value of n inline with equation (14). Let assume $t = 0$, then

$$\dot{W} = -4\epsilon_0 v_L^2 \left(\frac{n}{z}\right) \qquad (15)$$

Equation (15) can be investigated dimensionally. Re-arrange equation (15) to have

$$n = \frac{z\dot{W}}{\epsilon_0 v_L^2} \qquad (16)$$

Where z is measured in metres, power ($\dot{W}$) in kgm^2s^{-3}, ϵ_0 in $C^2.s^2.kg^{-1}.m^{-3}$ and speed of lightning, $v_L = ms^{-1}$. By substitution we have

$$n = \left(\frac{Nm^2}{C^2}\right)(kg.ms^{-1}) = (Coulomb'sConstant\ (K_c) \times Impulse\ (I)) \qquad (17)$$

Therefore, the attenuation coefficient n is directly proportional to the product of Coulomb's constant, K_c, Impulse, I, which is called ***Umahi's Coefficient of Lightning discharge.***

Comparing equations (15) and (17), the power in electromagnetic lightning discharge at time,t is given by

$$P = -P_0 e^{2j\omega t} \tag{18}$$

Where

$$P_0 = \frac{4\epsilon_0 v_L^2 K_c I}{z} \tag{19}$$

Equation (19) is the varying amplitude of the electromagnetic lightning discharge which is being affected by *Umahi's coefficiency* .$K_c I$. Which are within the lightning discharge location z. This location z is longitude and latitude dependent. The negative sign accompanying the electromagnetic discharge (as expressed in equation 18) indicates that the electromagnetic wave is retarding towards the earth surface.

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