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A Finite Differences Solution To The Vibrating Membrane Problem

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Abstract

A realistic approach to the solution of mechanical systems containing multiple parameters must take into account the fact that dependent variables depend not only on t, but on more space variables. The modelling of such problems leads to partial differential equations (P.D.Es), rather than Ordinary Differential Equations (O.D.Es). Here, the wave equation, a P.D E that governs the vibrating membrane problem is considered. A finite difference method (F.D.M) is provided as an alternative to the analytic methods. F.D.Ms basically involve three steps; dividing the solution into grids of nodes, approximating the given differential equation by finite difference equivalences that relate the solutions to grid points and solving the difference equations subject to the prescribed boundary and/or initial conditions. It is shown here that the error in the result is relatively negligible, and the conclusion made that the method developed can further be used to solve certain non-linear P.D.Es.

Key words; Wave Equation, Vibrating membrane, Fouling, Analytic solution, Numerical solution, Local truncation Error, Stability

1. Introduction

1.1 Studies on the vibrating membrane (and fouling)

Studies have been made on increasing the efficiency of filtration and separation systems by decreasing the rate of fouling. Low, et al [4] suggested that vibration prevented premature fouling of a membrane in a submerged membrane bioreactor application (SMBR). Ostapenko [6] developed a mathematical model in which a carrying blade was represented as a vibrating membrane, and the processable materials, as a series of strips. Cheng [2] wrote a paper on the advantages of membrane technology over water treatment by addition of toxic chemicals and biological processes.

1.2 Finite difference methods (F.D.Ms)

The finite difference method was first developed as 'The method of squares' by Thom and Apelt [10] in the 1920s, and was used to solve non-linear hydrodynamic equations. Much of the work on finite difference schemes is presented in Jain [3], Rahman [8], Morton et al [7] and Rao [9]. Jain [3] discussed more advanced finite difference methods due to Lax Friedrich, in which the term $Z_{m,l}^{n}$ in the time derivative is replaced by $\frac{1}{2}\left(Z_{m,l}^{n-1}+Z_{m,l}^{n+1}\right)$. Zhilin [12] considered the region under study to be covered by a cubic linear grid with sides parallel to the $x-$, $y-$ and $t-$ axes. The grid points $X, Y$and T were given by $x = mh_1$, $y = lh_2$ and $t = nk$, where m, l and n are the number of points in the $x-$, $y-$ and $t-$ axes respectively. Also, $h_1 = \Delta x$, $h_2 = \Delta y$, and $k = \Delta t$.

1.3 Stability of FDMs

According to Vrushali et al [11] , the numerical solution to a partial differential equation (P.D.E) is an approximation to the exact solution, and the local truncation error (LTE) is that difference which results when the exact solution is substituted into the finite difference formula. The forward time centered space scheme (FTCS) is considered here.

1.4 Fluid flow

Manyonge [5] derived the surface forces associated with the components of momentum, and Bansal [1] explained the law of conservation of momentum by using Newton's law of viscosity.

2. The model: 2-Dimensional wave equation

The wave equation governs the motion of a vibrating membrane with respective lengths in the x and y directions as L and H . We have

$$\frac{\partial^2 Z}{\partial t^2} = a^2\left(\frac{\partial^2 Z}{\partial x^2}+\frac{\partial^2 Z}{\partial y^2}\right) \tag{2.1}$$

where $Z = Z(x, y, t)$ with $Z = 0$ at the boundary, x and y are spatial co-ordinates, and a^2 is a constant with velocity dimensions, given by

$$a = \sqrt{\frac{gT}{w(L, H)}} \tag{2.2}$$

where g is the gravitational acceleration, $w(L, H)$ is the weight per unit area, and T is the tension directed along the respective tangents to the deflected curves. The set of equations governing the problem are:

$$\frac{\partial^2 Z}{\partial t^2} = a^2\left(\frac{\partial^2 Z}{\partial x^2} + \frac{\partial^2 Z}{\partial y^2}\right), \quad 0 < x < L, \quad 0 < y < H \quad t > 0 \tag{2.3}$$

$$Z(0, y, t) = Z(L, y, t) = 0 \quad 0 < y < H, \quad t > 0 \tag{2.4}$$

$$Z(x, 0, t) = Z(x, H, t) = 0 \quad 0 < x < L, \quad t > 0 \tag{2.5}$$

$$Z(x, y, 0) = F(x, y), \quad 0 < x < L, \quad 0 < y < H \tag{2.6}$$

$$\frac{\partial}{\partial t}[Z(x, y, 0)] = G(x, y), \quad 0 < x < L, \quad 0 < y < H \tag{2.7}$$

The solution $Z(x, y, t)$ which satisfies the above conditions is now determined. This represents the displacement of a point (x, y) on the membrane at time t from rest. Equations (2.4) and (2.5) represent the boundary conditions, while (2.6) and (2.7) represent the initial conditions.

3. Methodology

The construction of this model involves the analytic solution of the wave equation, the use of an FDM in the numerical solution and the application of the solution to a filtration system.

3.1 The Analytic solution

It can be shown that the solution $Z(x, y, t)$ of the wave equation is given by

$$Z(x, y, t) = \left(A_3 \cos\left(a\sqrt{\frac{n^2}{L^2} + \frac{m^2}{H^2}}\right)\pi t + A_4 \sin\left(a\sqrt{\frac{n^2}{L^2} + \frac{m^2}{H^2}}\right)\pi t\right)\left(\sin\frac{n\pi}{L}x\right)\left(\sin\frac{m\pi}{H}y\right) \tag{3.0}$$

Here L and H are the respective spatial dimensions in the x- and y- directions.

3.2 The Numerical Solution

It is supposed that the weight of the membrane, after it is stretched, is a known function $w(L,H)$, where $(L \times H)$ is a property (density for this case) which is proportional to the area of the membrane. The change in mass, ΔM, is given by

$$\Delta M = \frac{w(L,H)\Delta x \Delta y}{g} \tag{3.1 }$$

The acceleration produced in ΔM by these forces and by the portion of the distributed load is approximately

$$\frac{\partial^2 Z}{\partial t^2} = \frac{Tg}{w(L,H)}\left(\frac{\partial}{\partial x}\left(\frac{\partial Z}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial Z}{\partial y}\right)\right)$$

Therefore

$$\frac{\partial^2 Z}{\partial t^2} = \frac{Tg}{w(L,H)}\left(\frac{\partial^2 Z}{\partial x^2} + \frac{\partial^2 Z}{\partial y^2}\right) \tag{3.4}$$

Here, $\rho = w(L,H)$ is the density of the membrane material. The contact area $(Fig\ 1)$ is found as

$$Area = L \times H \tag{3.5}$$

3.3 The Forward Time Centered Space (FTCS) Scheme for Equation (2.1)

Equation (2.1) is re-written as

$$\frac{\partial}{\partial t}\left(\frac{\partial Z}{\partial t}\right) = a\frac{\partial}{\partial x}\left(a\frac{\partial Z}{\partial x}\right) + a\frac{\partial}{\partial y}\left(a\frac{\partial Z}{\partial y}\right) \tag{3.6}$$

Let

$$S = \frac{\partial Z}{\partial t}, \quad R = a\frac{\partial Z}{\partial x} \quad and \quad Q = a\frac{\partial Z}{\partial y} \tag{3.7}$$

Substituting (3.7) into (3.6);

$$\frac{\partial S}{\partial t} = a\frac{\partial R}{\partial x} + a\frac{\partial Q}{\partial y} \quad (3.8)$$

The FTCS scheme of (3.8) becomes

$$\frac{1}{k}\Delta_t\left(S_{m,l}^n\right) = \frac{a}{h_1}\mu\delta_x\left(R_{m,l}^n\right) + \frac{a}{h_2}\mu\delta_y\left(Q_{m,l}^n\right) \quad (3.9)$$

where k is the time step, and $h_1$and h_2 are the spatial dimensions in the $x -$ and $y -$ directions respectively. $\mu\delta_x$ and $\mu\delta_y$ are the respective averaging operators. It can then be written that

$$S_{m,l}^{n+1} = S_{m,l}^n + \frac{ak}{2h_1}\left(R_{m+1,l}^n - R_{m-1,l}^n\right) + \frac{ak}{2h_2}\left(Q_{m,l+1}^n - Q_{m,l-1}^n\right) \quad (3.10)$$

From (3.8);

$$\frac{\partial R}{\partial t} = a\frac{\partial S}{\partial x} \quad (3.11)$$

and

$$\frac{\partial Q}{\partial t} = a\frac{\partial S}{\partial y} \quad (3.12)$$

Using the FTCS scheme on equations (3.11) and (3.12), the following equations are obtained;

$$R_{m,l}^{n+1} = R_{m,l}^n + \frac{ak}{2h_1}\left(S_{m+1,l}^n - S_{m-1,l}^n\right) \quad (3.13)$$

$$Q_{m,l}^{n+1} = Q_{m,l}^n + \frac{ak}{2h_2}\left(S_{m,l+1}^n - S_{m,l-1}^n\right) \quad (3.14)$$

The first equation in (3.7) leads to

$$S_{m,l}^n = \frac{Z_{m,l}^{n+1} - Z_{m,l}^n}{k} \quad (3.15\ a)$$

For the spatial derivatives (second and third equations in (3.7));

$$R_{m,l}^n = \frac{a}{2h_1}\left(Z_{m+1,l}^n - Z_{m-1,l}^n\right) \quad (3.15\ b)$$

$$Q_{m,l}^{n} = \frac{a}{2h_2}\left(Z_{m,l+1}^{n} - Z_{m,l-1}^{n}\right) \quad (3.15\ c)$$

In the preceding equations ($(3.15a)$ to $(3.15\ c)$), $k = \Delta t$ is the time interval and Δ_t is the forward difference operator on time. $h_1 = \Delta x$ and $h_2 = \Delta y$ are the spatial dimensions in the $x-$ and $y-$ directions respectively. Using equations (3.13) and (3.14), the respective velocity gradients can be determined. Equations (3.16), (3.17) and (3.18) are respectively obtained from equations $(3.15\ a)$, $(3.15\ b)$, and $(3.15\ c)$;

$$S_{m,l}^{n+1} = \frac{1}{k}\left(Z_{m,l}^{n+2} - Z_{m,l}^{n+1}\right) \quad (3.16)$$

$$R_{m+1,l}^{n} = \frac{a}{2h_1}\left(Z_{m+2,l}^{n} - Z_{m,l}^{n}\right) \quad \text{and} \quad R_{m-1,l}^{n} = \frac{a}{2h_1}\left(Z_{m,l}^{n} - Z_{m-2,l}^{n}\right) \quad (3.17)$$

$$Q_{m,l+1}^{n} = \frac{a}{2h_2}\left(Z_{m,l+2}^{n} - Z_{m,l}^{n}\right) \quad \text{and} \quad Q_{m,l-1}^{n} = \frac{a}{2h_2}\left(Z_{m,l}^{n} - Z_{m,l-2}^{n}\right) \quad (3.18)$$

$(3.15\ a)$, (3.16), (3.17) and (3.18) are substituted into equation (3.10) to obtain

$$Z_{m,l}^{n+2} = 2Z_{m,l}^{n+1} - Z_{m,l}^{n} + \frac{a^2k^2}{4{h_1}^2}\left(Z_{m+2,l}^{n} - 2Z_{m,l}^{n} + Z_{m-2,l}^{n}\right) + \frac{a^2k^2}{4{h_2}^2}\left(Z_{m,l+2}^{n} - 2Z_{m,l}^{n} + Z_{m,l-2}^{n}\right) \quad (3.19)$$

Equation (3.19) leads to the scheme

$$Z_{m,l}^{n+2} = 2Z_{m,l}^{n+1} - \left(1 + \frac{a^2k^2}{2{h_1}^2} + \frac{a^2k^2}{2{h_2}^2}\right)Z_{m,l}^{n} + \frac{a^2k^2}{4{h_1}^2}\left(Z_{m+2,l}^{n} + Z_{m-2,l}^{n}\right) + \frac{a^2k^2}{4{h_2}^2}\left(Z_{m,l+2}^{n} + Z_{m,l-2}^{n}\right) \quad (3.20)$$

We shall take a special case where the spatial dimensions are equal, say $h_1 = h_2 = h$. If we let

$$\gamma = \frac{k}{h^2} \quad (3.21)$$

equation (3.20) transforms to

$$Z_{m,l}^{n+2} = 2Z_{m,l}^{n+1} - (1 + a^2\gamma^2h^2)Z_{m,l}^{n} + \frac{a^2\gamma^2h^2}{4}\left(Z_{m+2,l}^{n} + Z_{m-2,l}^{n} + Z_{m,l+2}^{n} + Z_{m,l-2}^{n}\right) \quad (3.22)$$

Equation (3.22) approximates the displacement at a point $Z_{m,l}^{n+2}$, when the values of displacement at the six points $Z_{m,l}^{n+1}$, $Z_{m,l}^{n}$, $Z_{m+2,l}^{n}$, $Z_{m-2,l}^{n}$, $Z_{m,l+2}^{n}$, $and\ Z_{m,l-2}^{n}$ have been approximated.

3.5 Case Study (flow in a river with velocity of water as 6m/s)

The case when the speed is 6 m/s is considered. The dimensions of the rectangular membrane are set to $(60 \times 80)mm^2$ and with $n \geq 1, h = 5mm$, the scheme in equations (3.22) is examined.

Notes

i. $a^2 = 6m/s$ and $h = 0.005\ m$ yields $k = 0.000833333\ s$. Outside the medium it is assumed that the displacement relative to the vibrating membrane is zero, so that for $n = -1$, equation (3.22) yields

$$Z_{m,l}^{1} = 2Z_{m,l}^{0} \qquad (3.36)$$

ii. At the initial condition $n = 0,\ \ k = 0$. Equations (3.21) yields $\beta = r = 0$. This is substituted into (3.22) to obtain

$$Z_{m,l}^{2} = 2Z_{m,l}^{1} - Z_{m,l}^{0} \qquad (3.37)$$

iii. Equations (3.36) and (3.37) lead to

$$Z_{m,l}^{2} = 3Z_{m,l}^{0} \qquad (3.38)$$

In Fig 1, there are 12 (5mm) steps $x_0, x_1, \dots, x_{12}$ on the $x - axis$ and 18 (5mm) steps $y_0, y_1, \dots, y_{15}$ on the $y - axis$. Thus, $m = 2, \dots, 10$ and $l = 2, \dots, 14$.

3.6 Discretization

By note i., equation (3.22) leads to the scheme

$$Z_{m,l}^{n+2} = 2Z_{m,l}^{n+1} - pZ_{m,l}^{n} + q\left(Z_{m+2,l}^{n} + Z_{m-2,l}^{n} + Z_{m,l+2}^{n} + Z_{m,l-2}^{n}\right) \qquad (3.39)$$

where $p = 1.1666666666667$and $q = 0.0416666666667$.

3.7 The Local Truncation Error (LTE)

The LTE,($T_{m,l}^{n}$) is found as $T_{m,l}^{n} = \left|z_{m,l}^{n} - Z_{m,l}^{n}\right|$,where $z_{m,l}^{n}$ is the analytic solution. For the values above, the following is obtained;

Table 11	The LTE for the FTCS- Equation (3.39)			
	$n = 1$	$n = 2$	$n = 3$	$n = 4$
$T_{3,0}^{n}$	0	0	0	0
$T_{3,1}^{n}$	0.0000000006243	0.0000000031211	0.112088373261	0.3923093182738
$T_{3,2}^{n}$	0.0000000011536	0.000000005767	0.5606601558525	1.9623105673987
$T_{3,3}^{n}$	0.0000000015072	0.000000007535	0.7325367949023	2.5638788107913
$T_{3,4}^{n}$	0.0000000016314	0.0000000081558	0.792893196164	2.7751262175664
$T_{3,5}^{n}$	0.0000000015072	0.000000007535	0.7325367949023	2.5638788107913
$T_{3,6}^{n}$	0.0000000011536	0.000000005767	0.5606601558525	1.9623105673987
$T_{3,7}^{n}$	0.0000000006243	0.0000000031211	0.112088373261	0.3923093182738
$T_{3,8}^{n}$	0	0	0	0
$T_{3,9}^{n}$	0.0000000006243	0.0000000031211	0.112088373261	0.3923093182738
$T_{3,10}^{n}$	0.0000000011536	0.000000005767	0.5606601558525	1.9623105673987
$T_{3,11}^{n}$	0.0000000015072	0.000000007535	0.7325367949023	2.5638788107913
$T_{3,12}^{n}$	0.0000000016314	0.0000000081558	0.792893196164	2.7751262175664
$T_{3,13}^{n}$	0.0000000015072	0.000000007535	0.7325367949023	2.5638788107913
$T_{3,14}^{n}$	0.0000000011536	0.000000005767	0.5606601558525	1.9623105673987
$T_{3,15}^{n}$	0.0000000006243	0.0000000031211	0.112088373261	0.3923093182738
$T_{3,16}^{n}$	0	0	0	0

4. Conclusion

The graph in Fig 2 illustrates the size of the error at different time levels. The diagrams in Fig 3 a and Fig 3 b respectively illustrate the analytic solution and the FTCS. It is noted that the results compare, and therefore that the FTCS scheme as discussed can be used to approximate the displacement of a particle on the membrane surface with a relatively small margin of error.

5. Recommendation

The FTCS scheme is an explicit scheme and is, however, known to be conditionally stable. It is recommended here that an implicit scheme be developed to solve the problem above.

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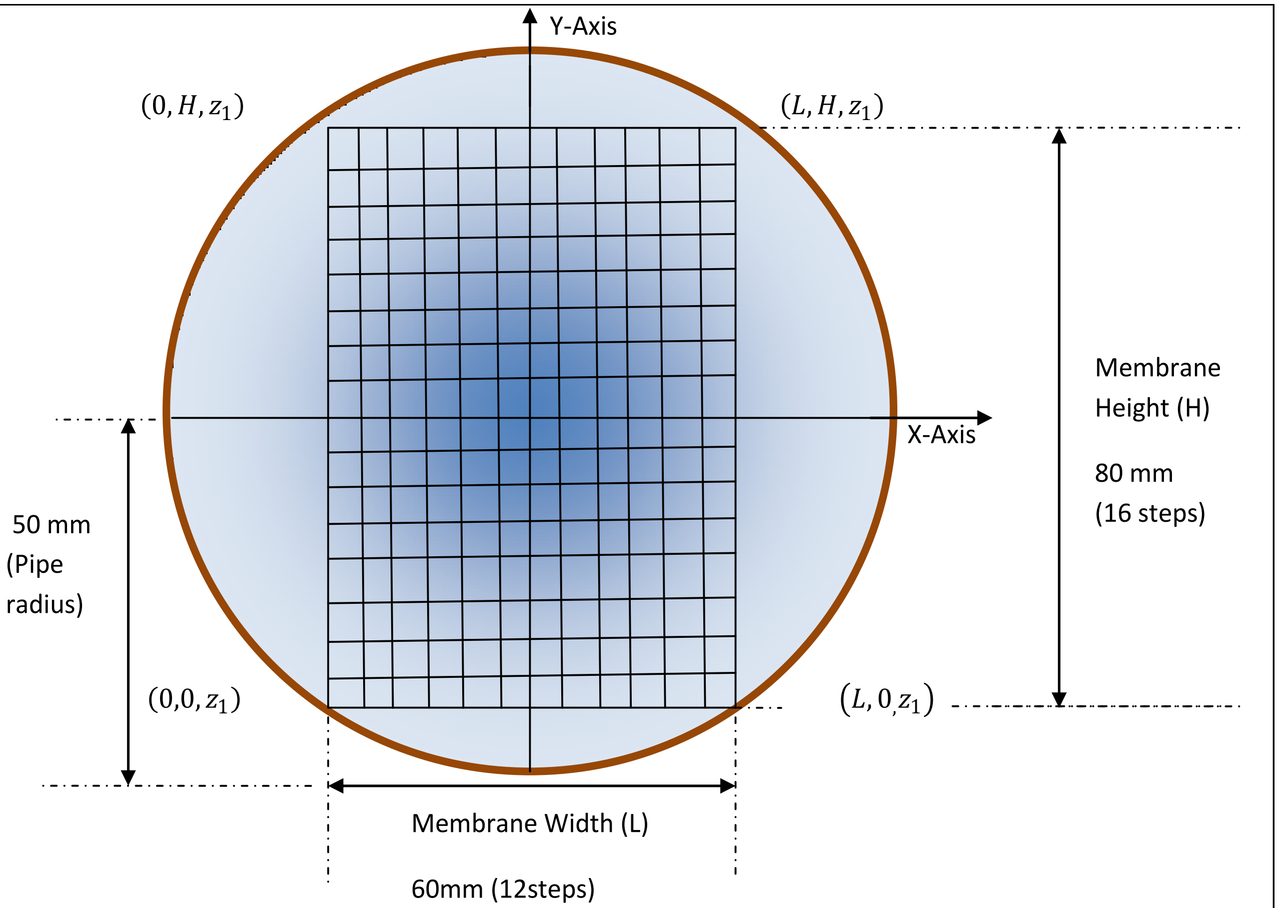

Fig1: The spatially discretized rectangular membrane

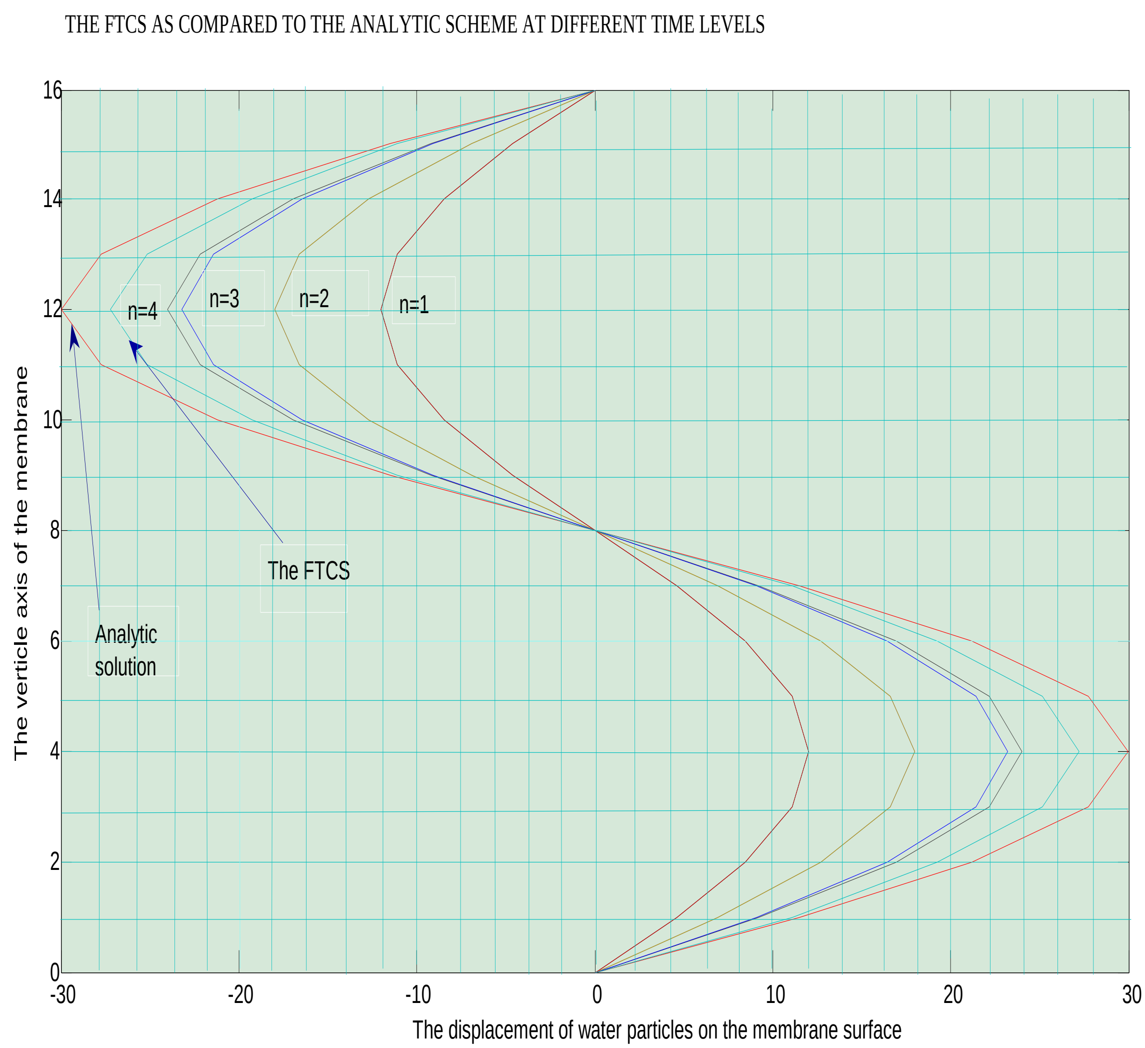

Fig 2: Graph showing the behaviour of the FTCS as compared to the analytic scheme.

Figure 3 a: THE 3-D ANALYTIC SOLUTION AT $t = 0.0025$

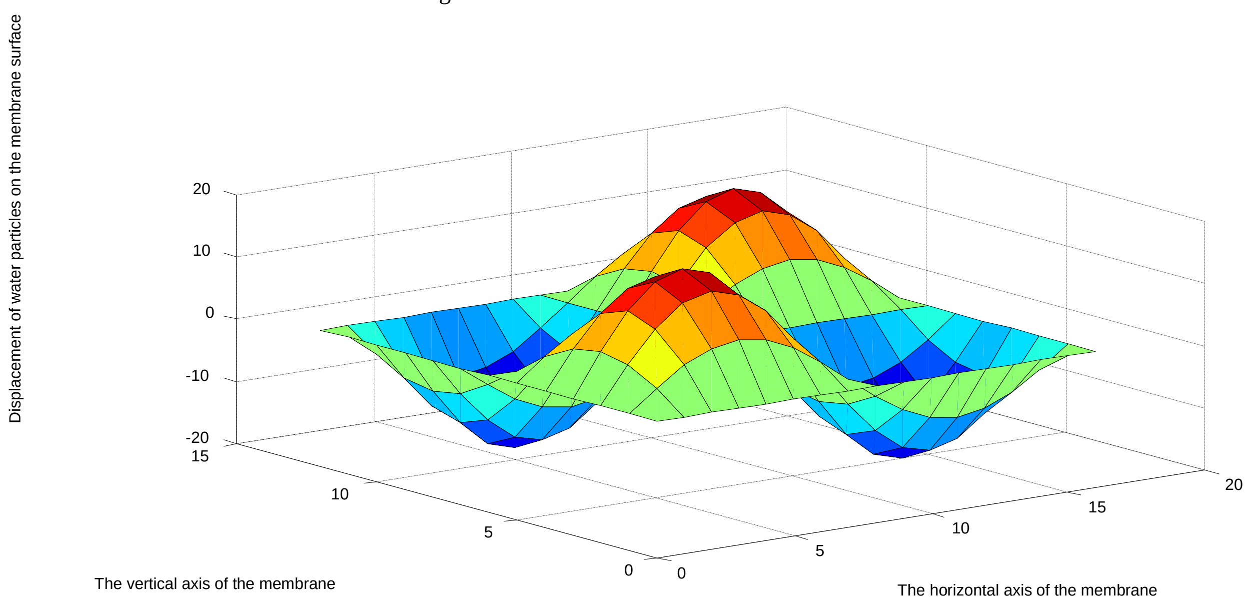

Figure 3 b: THE 3-D SOLUTION OF THE FTCS SCHEME AT $t = 0.0025$