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Commuting Condition on Derivation in Prime Near Rings

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Abstract

Several results assert that the existence of a suitably-constrained derivation on a prime near-ring forces the near-ring to be a ring. Our aim in this paper is to investigate the commuting conditions on derivations in near ring. Moreover, examples proving the necessity of the primeness condition are given.

Key Words: Commuting condition, Derivation, Prime near-ring, Ring

1. Introduction

A left near-ring is a set N with two operations addition (+) and multiplication (.) such that $(N,+)$ is a group and $(N, .)$ is a semigroup satisfying the left distributive law $x . (y + z) = x . y + x . z$ for all $x, y, z \in N$. N is called Zero symmetric left near-rings satisfy $0 . x = 0$ for all $x \in N$ (recall that left distributivity yields $x . 0 = 0$). Throughout this paper we use left near ring, unless otherwise specified, we will use the word near-ring to mean zero symmetric left near-ring and denote xy instead of $x. y$. An additive mapping $d: N \rightarrow N$ is said to be a derivation if $d(xy) = xd(y) + d(x)y$ for all $x, y \in N$ or equivalently that $d(xy) = d(x)y + xd(y)$ for all $x, y \in N$. A near-ring N is said to be prime if $xNy = 0$ for $x, y \in N$ implies $x = 0$ or $y = 0$. As usual, additive commutator is denoted by $(x, y) = x - y - x -$, $[x, y] = xy + yx$ and $x \, o \, y = xy + yx$ will denote the well-known Lie and Jordan products respectively. An element x in a near-ring N is said to be 2-torsion free if $2x = o$ implies that $x = o$ for every $x \in N$. The symbol $Z(N)$ will represent the multiplicative center of N, that is $Z(N) = \{x \in N \mid xy = yx \text{ for all } y \in N\}$. Properties of commutators:- Let be a near ring, then the following properties are satisfied. Then $[x , yx] = [x , y]x$, $xo(yx) = (xoy)x$, $[x\,y, y] = [x, y]y$, $(xy \, o \, y) = (xo \, y)o \, y$, $[[x, y] , z] + [[y, z] , x] + [[z, x] , y]$, for all $x \, y \in N$. There is an increasing body of evidence that prime near-rings with derivations have ring like behavior, indeed, there are several results asserting that the existence of a suitably-constrained derivation on a prime near-ring forces the near-ring to be a ring. In this paper we continue the line of investigation regarding the study of prime near-rings with derivations. More precisely, we shall prove that a prime near-ring which admits a nonzero commuting derivation satisfying certain differential identities.

1. PRELIMINARIES RESULTS

To prove our results we start with the following definition and lemmas:

Definition 1(see [3]).A mapping $d: N \rightarrow N$ is said to be centralizing (resp. commuting) derivation on a near-ring N if $[d(x), x] \in Z(N)$ or$[d(x), x] = 0$, holds for all x in N.

Lemma 1 (see [1, Theorem 3]) .If a prime near-ring N admits a nontrivial derivation d for which

$d(N) \subseteq Z(N)$, then $(N, +)$ is abelian. Moreover, if N is 2-torsion-free, then N is a commutative ring.

Lemma 2(see [2]).Let d be an arbitrary derivation on the near-ring N, then N satisfies the following partial distributive law:

(i) $(xd(y) + d(x)y)z = xd(y)z + d(x)yz$ for all $x, y \in N$.

(ii) $\big(d(x)y + xd(y)\big)z = d(x)yz + xd(y)z$ for all $x, y \in N$.

Lemma 3.A near ring N has no non-zero nilpotent elements if and only if $a^2 = 0$ implies $a = 0, \forall a \in$ N.

2. THE MAIN RESULTS

Theorem 1(see [4]).Let N be prime near-ring. If N admits anon-zero derivation d satisfying d([x,y])=[x, y] $\forall x, y \epsilon N$. Then d is commuting (resp. centralizing derivation on N).

Proof. We have $d([x, y]) = [x, y]$ for all $x, y \in N$

(1.1)

Replacing y by xy in equation (5.1), because of $[x, xy] = x[x, y]$, we get $x[x, y] = d(x[x, y])$ for all $x, y \in N$

Since $d(x[x, y]) = xd([x, y]) + d(x)[x, y]$,

Then according to equation (1.1) we obtain

$x[x, y] = x[x, y] + d(x[x, y])$ and therefore $d(x)[x, y] = 0$,

Hence,

$d(x)(xy - yx) = 0$ for all $x, y \in N$ (1.2)

Substituting yz for y in equation (1.2), we obtain $d(x)y(xz - zx) = 0$ and the equation (5.2) which leads to

$d(x)N(xz - zx) = \{0\}$ for all $x, z \in N$ (1.3)

Since N is prime, equation (1.3) reduces to

$d(x) = 0$ or $[x, z] = 0$ for all $x, z \in N$ (1.4)

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From equation (1.4) assume that $[x,z]=0$ for all $x,z \in N$ .Take x=z implies [x, x]=0 $\forall x \in N$, then x$\in Z(N)$ implies that d(x)$\in Z(N)$. Since z$\in Z(N)$,since $d \neq 0$ on N, then $x \in Z(N)$ implies that $d(x) \in Z(N)$ (1.5)

This shows that from equation(1.4), we obtain that [x, d(x)]=0, $\forall x$ in N…….(1.6).

In the light of equation (1.6), we have $d(N) \subseteq Z(N)$ by lemma 1.2 and using equation (1.5), we conclude that d is commuting.

Example 1(see [4]. Let R be a commutative ring which is not a zero ring and consider

$N=\left\{\begin{pmatrix}0 & 0\\ x & y\end{pmatrix}/x\,,y\in R\right\}$, if we define a derivation $d\colon N\to N$ by $d\begin{pmatrix}0 & 0\\ x & y\end{pmatrix}=\begin{pmatrix}0 & 0\\ x & 0\end{pmatrix}$, then it is straightforward to check that d is a nonzero derivation on a near ring N. On the other hand, if there is $A=\begin{pmatrix}0 & 0\\ x_1 & 0\end{pmatrix}$ and $B=\begin{pmatrix}0 & 0\\ 0 & y_1\end{pmatrix}$ for all x_1,$y_1 \in R$. We have $ANB=\{0\}$ for all $A,B\in N$, but $A\neq 0$ and $B\neq 0$, from this observations N is not prime. Moreover, d satisfies the condition $d([A,B])=[A,B]$ for all $A,B\in N$. But $A\cdot B\neq B\cdot A$, which yields that d is not commuting.

Theorem 2 (see [4]).Let N be prime near-ring. If N admits anon-zero derivation d satisfying d(x$\circ y)=x\circ y$

$\forall$x, y $\in$ N.Then d is commuting (resp. centralizing on N).

Proof: From the hypothesis, we have $d(xoy)=xy+yx\ \ \forall x,y\in N$. (2.1)
Replacing y by xy in equation (2.1), we get $\big(xo(xy)\big)=x^2y+xyx\ \forall x,y\in N$. (2.2).

Since, we have $xo(xy)=x(xoy)$,

Then equation (2.2) yields $d\big(xo(xy)\big)=d(x(xoy))$

$=xd(xoy)+d(x)xoy\ \forall x,y\in N.$

From the given we have $d(xoy)=xoy$, Hence, equation (2.2) reduces to

$d\big(xo(xy)\big)=x(xoy)+d(x)(xoy)=x^2y+xyx+d(x)(xoy), \forall x,y\in N.$ (2.3)

As we have from equations (2.3), we get

$$x^2y+xyx=x^2y+xyx+d(x)(xoy)$$

Then equation (2.3) assures that

$d(x)(xoy)=0, \forall x,y\in N$. This leads to

$d(x)xy=-d(x)yx$ for all $\forall x,y\in N$ (2.4)

Substituting yz for y in equation (2.4) we find that

$$\begin{aligned}-d(x)yzx&=d(x)xyz\\&=(-d(x)yx)z\\&=d(x)y(-x)z\ ,\forall x,y,z\in N.\end{aligned}$$

Since from equation (2.4), we get

$-d(x)yzx=d(x)y(-x)z$ and equation (5.10) becomes

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$d(x)yz(-x) = d(x)y(-x)z, \ \forall x, y, z \in N.$ (2.5)

Taking $-x$ instead of x in equation (2.5) gives $d(-x)yzx = d(-x)yxz$ for all $\forall x, y, z \in N$

So that $d(-x)y(zx - xz) = 0$, $\forall$x, y, z $\in N$.

Therefore we get $d(-x)N[z, x] = 0$, $\forall x, z \in N$. (2.6)

By primness, equation (2.6) assures that for each $x \in N$, either $x \in Z(N)$ or $d(x) = 0$,

Accordingly,

$d(x) = 0$ or $[x, z] = 0$, $\forall x, z \in N$. (2.7)

From equation (2.7) it follows that for each $x \in N$, we have

$d(x) = 0$ or $x \in Z(N)$, $\forall x \in N$. (2.8)

But $x \in Z(N)$ also implies that $d(x) \in Z(N)$ and equation (2.8) forces

$d(x) \in Z(N)$, $\forall x \in N$. (2.9)

In the light of equation (2.9), it follows that $d(N) \subseteq Z(N)$ we conclude that d is a commuting.

Example 2 (see [4]) .Let S be any ring. Next, let us consider the ring $N = \left\{\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} / x, y, z \in S\right\}$.

Define a map $d: N \to N$ such that

$d\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} = \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ for all $x, y \in S$. Then d is a nonzero derivation on N. If we take $A = \begin{pmatrix} 0 & x_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 & y_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ with $A \neq 0$ and $B \neq 0$, then $ANB = \{0\}$ proving that N is not prime. Moreover, it can be easily seen that d is a derivation on N and satisfies $d(AoB) = AoB$ for all $A, B \in N$. In the case of primness hypothesis not satisfied and d is not commuting.

Theorem 3 . Let N be a 2-torsion free prime near-ring. If N admits a non-zero derivation d satisfying

d([x, y])=xoy , for all $x, y \in N$. Then d is centralizer on N.

Proof: We have d([x, y])=xoy, for all $x, y \in N$.Replacing y by x, we obtain $2x^2 = 0$, for all $x, y \in N$. Since N is 2-torsion free, we get $x^2 = 0$ for all x ∈ N. Replacing x by d(x) with using Lemma1.3 , we get d(x)=0,for all $x \in N$…….(3.1).Then from equation (3.1), we obtain d(x)∈ $Z(N)$, for all x ∈ N.

Theorem 4 .Let N be a 2-torsion free near-ring. If N admits a non-zero derivation d is satisfying

d$(xoy) = [x, y]$ $\forall x, y \in N$. Then d(N^2) is centralizer on N.

Proof: Assume that for any x∈ N, then $x^2 \in$ Z(N),where N is 2-torsion free with characteristic different from two. We have 2d(x)=0, $\forall x \in N$. This implies that d(x)∈ N, $\forall x \in N$ ...…(4.1). By the definition of derivation on N, we have d(x^2)=xd(x)+d(x)x=2d(x)x=d(x)(2x), $\forall x \in N$ ...…(4.2). Since, d$\neq 0$ is not left zero divisors in N. It follows that from equation (4.2), 2d(x)x-d(x)(2x)=0, $\forall x \in N$…..(4.3). Then equation (4.3) reduces

2[d(x), x]=0, $\forall x \in N$……(4.4). Since N is 2-torsion free, we get [d(x), x]=0, $\forall x \in N$ … … . (4.5).

By substituting yx in x, then we have [d(x), yx]=0, $\forall x, y \in N$ ….…(4.6). Since from equation (4.6), we obtain that [d(x), y]x=0, $\forall x, y \in N$ … … (4.7). By replacing x by zx in equation (4.7), we get [d(x),y]zx=0,

$\forall x, y, z \in N$ .....(4.8). For any z$\in N$, then we have [d(x), y]Nx=0, $\forall x, y \in N$ ….…(4.9). Since N is prime near-ring, then either [d(x),y]=0 or x=0, $\forall x, y \in N$ .....(4.10). Thus x$\in N$, then d(x)$\neq 0$, where as

d(x)$\in Z(N)$. Then from equation (4.10), we have[d(x),y]=0, $\forall x, y \in N$….(4.11). Therefore, for any x$\in N$ implies that d(x)$\in Z(N), \forall x \in N$.Since d(x)$\in Z(N)$ shows that d(N)$\subseteq Z(N)$, by lemma 1.2. Hence from equation (4.11), we conclude that d is centralizing in N.

Conclusion

In this paper, we study the prime near-rings with derivations. We prove that a prime near-ring which admits a nonzero derivation satisfying certain differential identities is a commuting on derivations d. For future research, more general constraints on the derivation would be interesting. In addition, can the hypotheses of Theorem 4 be weakened such that the identities hold in some nonzero semi-group ideal *U* of *N*?

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