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Action of Direct Products of Four Alternating Groups on Cartesian Product of Four Sets

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Abstract

In this paper, transitivity, primitivity, ranks and subdegrees associated with the action of direct product of four Alternating groups A_n, where n is a positive integer atleast 2 on the Cartesian product of four sets are investigated. It is shown that for $n > 2$, the action is both transitive and imprimitive. It is further shown that the rank associated with this action is a constant of 16 and the subdegrees are 1, $n-1$, $(n-1)^2$, $(n-1)^3$, $(n-1)^4$.

Keywords: Alternating Group, Transitivity, Primitivity, Ranks, Subdegrees, Direct Product, Cartesian Product, GAP

1. Introduction

The action Alternating groups on sets on sets has been explored by various authors including Nyaga (2018b,a); Gachimu et al. (2015). The results have been helpful in giving valuable information to graph theorists with applications such as; in daily life (e.g. to optimize the distance between two places, model social networks), in communication networks (such as modeling call graphs), in information networks (e.g. to model web graphs) in Chemistry (to define the natural model for molecules), in Physics (statistics on graph-theoretical properties on topology of atoms enhance quantitative study of three-dimensional structure of complicated simulated atoms in condensed matter physics) and in Computational biochemistry (in resolution of conflicts between cell samples). Unfortunately, a study investigating the action of the direct product of four Alternating groups on the Cartesian product of four sets has to the best of our knowledge not been investigated and therefore this presents a research gap.

Definition 1.1. [Product Action](Cameron et al., 2008, p.3) Let (G_1, X_1) and (G_2, X_2) be permutation groups. The direct product $G_1 \times G_2$ acts on the the Cartesian product $X_1 \times X_2$ by the rule

$$(g_1, g_2)(x_1, x_2) = (g_1x_1, g_2x_2)\ \forall\ g_1 \in G_1, g_2 \in G_2 \text{ and } x_1 \in X_1,\ x_2 \in X_2.$$

Remark 1.1. Through out this paper, the group action defined is in a similar way as in Definition 1.1 as

$$(g_1, g_2, \cdots, g_4)(x_1, x_2, \cdots, x_4) = (g_1x_1, g_2x_2, \cdots, g_4x_4)\ \forall\ g_1, g_2, \cdots, g_4 \in G \text{ and } x_i \in X$$

where $G = A_n \times A_n \times \ldots \times A_n$ and, $X_1 = \{1, 2, \cdots, n\}$, $X_2 = \{n+1, n+2, \cdots, 2n\}, \cdots, X_4 = \{n(n-1)+1, n(n-1)+2, \cdots, n^2\}$.

1.1 Definitions and Preliminary results

Definition 1.2. Let G act on X. The orbit of $x \in X$, denoted $Orb_G(x)$ is defined as the set

$$Orb_G(x) = \{gx : g \in G\}.$$

Definition 1.3. Let G act on X, and let $x \in X$. The Stabilizer of x in G, denoted G_x (sometimes $Stab_G(x)$) is set all elements in G that fix x. Thus

$$G_x = \{g \in G : gx = x\}.$$

Definition 1.4. The action of a group G on the set X is said to be transitive if for each pair of points $x, y \in X$, there exists $g \in G$ such that $gx = y$; in other words, if the action has only one orbit.

Definition 1.5. Suppose that G acts transitively on X. For each subset Y of X and each $g \in G$, let $gY = \{gy : y \in Y\} \subseteq X$. A subset Y of X is said to be a block for the action if for each $g \in G$, either $gY = Y$ or $gY \cap Y = \emptyset$; In particular, $\emptyset$, X, and all 1-element subsets of X are obviously blocks, called the trivial blocks. If these are the only blocks, then we say that G acts primitively on X. Otherwise, G acts imprimitively.

Definition 1.6. Suppose G is a group acting transitively on a set X and let G_x be the stabilizer in G of a point $x \in X$. The orbits $\triangle_0 = \{x\}, \triangle_1, \triangle_2, \cdots \triangle_{k-1}$ of G_x on X are known as suborbits of G. The rank of G in this case is k. The sizes $n_i = |\triangle_i|$ $(i = 0, 1, 2, \cdots, k-1)$ are known as the subdegrees of G. It was proved by Ivanov et al. (1983) that the rank and subdegrees of the suborbits $\triangle_i$ $(i = 0, 1, 2, \cdots, k-1)$ are independent of the choices of $x \in X$.

Definition 1.7. Let G act on the set X. The set of all elements of X fixed by $g \in G$ is called the fixed point set of g, denoted by $Fix(g)$. Thus

$$Fix(g) = \{x \in X : \ gx = x\}.$$

The character π of permutation representation of G on X is defined as

$$\pi(g) = |Fix(g)|, \ \forall\, g \in G.$$

Definition 1.8. Let $\triangle$ be an orbit of G_x on X. Define $\triangle^* = \{gx : \ g \in G, \ x \in g\triangle\}$, then $\triangle^*$ is also an orbit of G_x and is called the G_x-orbit paired with $\triangle$. Wielandt (1964) proved that if $\triangle^* = \triangle$, then $\triangle$ is called a self-paired orbit of G_x.

Theorem 1.1. [Orbit-Stabilizer Theorem]Rose (1978) Let G be a group acting on a finite set X and $x \in X$. Then $|Orb_G(x)| = |G : \text{Stab}_G(x)|$.

Lemma 1.1. [Cauchy-Frobenius Lemma]Harary (1969) Let G be a finite group acting on a set X. The number of orbits of G is given by $\frac{1}{|G|}\sum_{g\in G}|Fix(g)|$.

2. Main Results

Lemma 2.1. *The action of $A_2 \times A_2 \times A_2 \times A_2$ on $X_1 \times X_2 \times X_3 \times X_4$ is not transitive where $X_1 = \{1, 2\}, \ X_2 = \{3, 4\}, \ X_3 = \{5, 6\},$ and $X_4 = \{7, 8\}$.*

Proof. Let $G = A_2 \times A_2 \times A_2 \times A_2$. It suffices to show that $|Orb_G(1, 3, 5, 7)| \neq |X_1 \times X_2 \times X_3 \times X_4|$. Let $K = X_1 \times X_2 \times X_3 \times X_4$.
Then
$K = \{(1, 3, 5, 7), (1, 3, 5, 8), (1, 3, 6, 7), (1, 3, 6, 8), (1, 4, 5, 7), (1, 4, 5, 8), (1, 4, 6, 7), (1, 4, 6, 8), (2, 3, 5, 7), (2, 3, 5, 8), (2, 3, 6, 7), (2, 3, 6, 8), (2, 4, 5, 7), (2, 4, 5, 8), (2, 4, 6, 7), (2, 4, 6, 8)\}$. Also, since $A_2 = \{()\}$, then $G = \{()\}$. By Definition 1.3, $G_{(1,3,5,7)} = \{()\}$. Using Theorem 1.1, $|Orb_G(1, 3, 5, 7)| = \frac{1}{1} = 1 \neq | \times X_2 \times X_3 \times X_4|$.
Moreover
$Orb_G(1, 3, 5, 7) = \{(1, 3, 5, 7)\}$. Thus, the action is intransitive. □

Lemma 2.2. *The action of $A_3 \times A_3 \times A_3 \times A_3$ on $X_1 \times X_2 \times X_3 \times X_4$ is transitive where $X_1 = \{1, 2, 3\}, \ X_2 = \{4, 5, 6\}, \ X_3 = \{7, 8, 9\},$ and $X_4 = \{10, 11, 12\}$.*

Proof. Let $G = A_3 \times A_3 \times A_3 \times A_3$ and $K = X_1 \times X_2 \times X_3 \times X_4$. By using the Groups, Algorithms Programming (GAP) software, $G = \langle\{(1\,2\,3), (4\,5\,6), (7\,8\,9), (10\,11\,12)\}\rangle$ with $|G| = 81$ (G is shown in the Appendix). Also by Definition 1.3, $G_{(1,4,7,10)} = \{()\}$ so that $|G_{(1,4,7,10)}| = 1$. By Theorem 1.1, $|Orb_G(1, 4, 7, 10)| = \frac{81}{1} = 81 = 3^4 = |X_1 \times X_2 \times X_3 \times X_4$.

□

Lemma 2.3. *The action of $A_4 \times A_4 \times A_4 \times A_4$ on $X_1 \times X_2 \times X_3 \times X_4$ is transitive where $X_1 = \{1, 2, 3, 4\}$, $X_2 = \{5, 6, 7, 8\}$, $X_3 = \{9, 10, 11, 12\}$, and $X_4 = \{13, 14, 15, 16\}$.*

Proof. Let $G = A_4 \times A_4 \times A_4 \times A_4$ and $K = X_1 \times X_2 \times X_3 \times X_4$. Then by using GAP software, G is a permutation group with 8 generators and $|G| = 20736$. Also, $G_{(1,5,9,13)} = \langle\{(14\ 16\ 15), (10\ 12\ 11), (6\ 8\ 7), (2\ 4\ 3)\}\rangle$ with $|G_{(1,5,9,13)}| = 81$. Using Theorem 1.1,$|Orb_G(1, 5, 9, 13)| = \frac{20736}{81} = 256 = 4^4 = |X_1 \times X_2 \times X_3 \times X_4$. Thus, the action has one orbit and hence transitive. □

Lemma 2.4. *The action of $A_5 \times A_5 \times A_5 \times A_5$ on $X_1 \times X_2 \times X_3 \times X_4$ is transitive where $X_1 = \{1, 2, 3, 4, 5\}$, $X_2 = \{6, 7, 8, 9, 10\}$, $X_3 = \{11, 12, 13, 14, 15\}$, $X_4 = \{16, 17, 18, 19, 20\}$, and $X_5 = \{21, 22, 23, 24, 25\}$.*

Proof. Let $G = A_5 \times A_5 \times A_5 \times A_5$ and $M = X_1 \times X_2 \times X_3 \times X_4$. Then by using GAP software, G is a permutation group with 8 generators and $|G| = 207360000$. Also, $G_{(1,6,11,16)}$ is a permutation group with 8 generators and $|G_{(1,6,11,16)}| = 20736$. Using Theorem 1.1,$|Orb_G(1, 6, 11, 16)| = \frac{12960000}{20736} = 625 = 5^4 = |X_1 \times X_2 \times X_3 \times X_4$. Thus, the action is transitive since it has one orbit. □

Theorem 2.1. *Let $n > 2$. The action of $A_n \times A_n \times A_n \times A_n$ on $X_1 \times X_2 \times X_3 \times X_4$ is transitive where $X_1 = \{1, 2, \cdots, n\}$, $X_2 = \{n + 1, n + 2, \cdots, 2n\}$, $X_3 = \{2n + 1, 2n + 2, \cdots, 3n\}, \cdots,$ and $X_4 = \{3n + 1, 3n + 2, \cdots, 4n\}$.*

Proof. Let $G = A_n \times A_n \times A_n \times A_n$. We show that for $n > 2$, the cardinality of $Orb_G(1, n + 1, 2n + 1, 3n + 1)$ is equal to the cardinality of $X_1 \times X_2 \times X_3 \times X_4$. Now, by Definition 1.3, $g_1, g_2, g_3, g_4 \in G$ fixes $x_1, x_2, x_3, x_4 \in X_1 \times X_2 \times X_3 \times X - 4$ if and only if $(g_1, g_2, g_3, g_4)(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4)$. By Definition 1.1, we have $g_1x_1 = x_1, g_2x_2 = x_2, g_3x_3 = x_3$ and $g_4x_4 = x_4$. Hence x_1, x_2, x_3, x_4 comes from a 1-cycle of $g_i(i = 1, 2, 3, 4)$. Therefore, $G_{(1,n+1,2n+1,3n+1)}$ is isomorphic to $A_{n-1} \times A_{n-1} \times A_{n-1} \times A_{n-1}$. Thus, $|G_{(1,n+1,2n+1),3n+1}| = (\frac{(n-1)!}{2})^4$. By Theorem 1.1, $|Orb_G(1, n+1, 2n+1, 3n+1)| = \frac{(\frac{n!}{2})^4}{(\frac{(n-1)!}{2})^4} = n \times n \times n \times n = n^4 = |X_1 \times X_2 \times X_3 \times X_4|$. Hence the action is transitive. □

Lemma 2.5. *$A_2 \times A_2 \times A_2 \times A_2$ acts on $X_1 \times X_2 \times X_3 \times X_4$ is neither primitive nor imprimitive.*

Proof. From Theorem 2.1, this action is intransitive. Thus, we can not have either primitivity nor imprimitivity. □

Lemma 2.6. *Action of of $A_3 \times A_3 \times A_3 \times A_3$ on $X_1 \times X_2 \times X_3 \times X_4$ is imprimitive.*

Proof. Let $G = A_3 \times A_3 \times A_3 \times A_3$ and $K = X_1 \times X_2 \times X_3 \times X_4$. From Theorem 2.2, this action is transitive. Let Y to be any non-trivial subset of K such that $|Y|$ divides $|K|$ by $\frac{3\times3\times3\times3}{3}$ i.e., $|Y| = 3$. For each element of Y, there exists $g = (x, y, z, w) \in G$ with 3-cycles permutations such that $g \in G$ moves an element of Y to another element not in Y. Hence $gY \cap Y = \emptyset$ implying that Y is a non-trivial block of the action. By Definition 1.5, the result follows. □

Lemma 2.7. *Action of of $A_4 \times A_4 \times A_4 \times A_4$ on $X_1 \times X_2 \times X_3 \times X_4$ is imprimitive.*

Proof. Let $G = A_4 \times A_4 \times A_4 \times A_4$ and $K = X_1 \times X_2 \times X_3 \times X_4$. From Theorem 2.3, this action is transitive. Let Y to be any non-trivial subset of K such that $|Y|$ divides $|K|$ by $\frac{4\times4\times4\times4}{4}$ i.e., $|Y| = 4$. For each element of Y, there exists $g = (x, y, z, w) \in G$ with $3, 4$-cycles permutations such that $g \in G$ moves an element of Y to another element not in Y. Hence $gY \cap Y = \emptyset$ implying that Y is a non-trivial block of the action. By Definition 1.5, the result follows. □

Lemma 2.8. *Action of of $A_5 \times A_5 \times A_5 \times A_5$ on $X_1 \times X_2 \times X_3 \times X_4$ is imprimitive.*

Proof. Let $G = A_5 \times A_5 \times A_5 \times A_5$ and $K = X_1 \times X_2 \times X_3 \times X_4$. From Theorem 2.4, this action is transitive. Let Y to be any non-trivial subset of K such that $|Y|$ divides $|K|$ by $\frac{5\times5\times5\times5}{4}$ i.e., $|Y| = 5$. For each element of Y, there exists $g = (x, y, z, w) \in G$ with $3, 4, 5$-cycles permutations such that $g \in G$ moves an element of Y to another element not in Y. Hence $gY \cap Y = \emptyset$ implying that Y is a non-trivial block of the action. By Definition 1.5, imprimitively of $A_5 \times A_5 \times A_5 \times A_5$ on $X_1 \times X_2 \times X_3 \times X_4$ follows. □

Theorem 2.2. *For $n > 2$, $A_n \times A_n \times A_n \times A_n$ acts imprimitively on $X_1 \times X_2 \times X_3 \times X_4$*

Proof. Let $G = A_n \times A_n \times A_n \times A_n$ and $K = X_1 \times X_2 \times X_3 \times X_4$. From Theorem 2.1, this action is transitive. Let Y to be any non-trivial subset of K such that $|Y|$ divides $|K|$ by $\frac{n\times n\times n\times n}{n}$ i.e., $|Y| = n$. For each element of Y, there exists $g = (x, y, z, w) \in G$ with $3, \cdots, n$-cycles permutations such that $g \in G$ moves an element of Y to another element not in Y. Hence $gY \cap Y = \emptyset$ implying that Y is a non-trivial block of the action. By Definition 1.5, imprimitively of $A_n \times A_n \times A_n \times A_n$ on $X_1 \times X_2 \times X_3 \times X_4 \times$ follows. □

Lemma 2.9. *Action of $A_3 \times A_3 \times A_3 \times A_3$ on $X_1 \times X_2 \times X_3 \times X_4$ has a rank of* 3^4.

Proof. Let $G = A_3 \times A_3 \times A_3 \times A_3$ and $K = X_1 \times X_2 \times X_3 \times X_4$. By Theorem ??, this action is transitive and $G_{()} = \langle\{()\}\rangle$. Thus, the permutations in $G_{(1,4,7,10)}$ are of the form $()$ since $G_{(1,4,7,10)}$ is identity permutation. Therefore, the number of elements of

K fixed by each $g \in G_{(1,4,7,10)}$ is 81 since identity element fixes all elements of a set. Using Theorem 1.1, the number of orbits of $G_{(1,4,7,10)}$ on K is

$$\frac{1}{G_{(1,4,7,10)}} \sum_{g_1,g_2,g_3,g_4 \in G_{(1,4,7,10)}} |Fix(g_1,g_2,g_3,g_4)| = \frac{1}{1}(1 \times 81) = 3^4.$$

Let $A = \{1,4,7,10\}$
The suborbits of of G are those which contain exactly $4, 3, 2, 1,$ and no element from A and they are;

(a) Orbits containing exactly four elements of A are;
$Orb_{G_{(1,4,7,10)}}(1,4,7,10) = \{(1,4,7,10)\} = \triangle_0.$

(b) Orbits containing exactly three elements of A are;
$Orb_{G_{(1,4,7,10)}}(1,4,7,11) = \{(1,4,7,11)\} = \triangle_1.$
$Orb_{G_{(1,4,7,10)}}(1,4,7,12) = \{(1,4,7,12)\} = \triangle_2.$
$Orb_{G_{(1,4,7,10)}}(1,4,8,10) = \{(1,4,7,10)\} = \triangle_3.$
$Orb_{G_{(1,4,7,10)}}(1,4,9,10) = \{(1,4,9,10)\} = \triangle_4.$
$Orb_{G_{(1,4,7,10)}}(1,5,7,10) = \{(1,5,7,10)\} = \triangle_5.$
$Orb_{G_{(1,4,7,10)}}(1,6,7,10) = \{(1,6,7,10)\} = \triangle_6.$
$Orb_{G_{(1,4,7,10)}}(2,4,7,10) = \{(2,4,7,10)\} = \triangle_7.$
$Orb_{G_{(1,4,7,10)}}(3,4,7,10) = \{(3,4,7,10)\} = \triangle_8.$

(c) Orbits containing exactly two elements of A are;
$Orb_{G_{(1,4,7,10)}}(1,4,8,11) = \{(1,4,8,11)\} = \triangle_9.$
$Orb_{G_{(1,4,7,10)}}(1,4,5,12) = \{(1,4,8,12)\} = \triangle_{10}.$
$Orb_{G_{(1,4,7,10)}}(1,4,9,11) = \{(1,4,9,11)\} = \triangle_{11}.$
$Orb_{G_{(1,4,7,10)}}(1,4,7,12) = \{(1,4,7,12)\} = \triangle_{12}.$
$Orb_{G_{(1,4,7,10)}}(1,5,7,11) = \{(1,5,7,11)\} = \triangle_{13}.$
$Orb_{G_{(1,4,7,10)}}(1,5,7,12) = \{(1,5,7,12)\} = \triangle_{14}.$
$Orb_{G_{(1,4,7,10)}}(1,5,8,10) = \{(1,5,8,10)\} = \triangle_{15}.$
$Orb_{G_{(1,4,7,10)}}(1,5,9,10) = \{(1,5,9,10)\} = \triangle_{16}.$
$Orb_{G_{(1,4,7,10)}}(1,6,7,11) = \{(1,6,7,11)\} = \triangle_{17}.$
$Orb_{G_{(1,4,7,10)}}(1,6,7,12) = \{(1,6,7,12)\} = \triangle_{18}.$
$Orb_{G_{(1,4,7,10)}}(1,6,8,10) = \{(1,6,8,10)\} = \triangle_{19}.$
$Orb_{G_{(1,4,7,10)}}(1,6,9,10) = \{(1,6,9,10)\} = \triangle_{20}.$
$Orb_{G_{(1,4,7,10)}}(2,4,7,11) = \{(2,4,7,11)\} = \triangle_{21}.$
$Orb_{G_{(1,4,7,10)}}(2,4,7,12) = \{(2,4,7,12)\} = \triangle_{22}.$
$Orb_{G_{(1,4,7,10)}}(2,4,8,10) = \{(2,4,8,10)\} = \triangle_{23}.$
$Orb_{G_{(1,4,7,10)}}(2,4,9,10) = \{(2,4,9,10)\} = \triangle_{24}.$
$Orb_{G_{(1,4,7,10)}}(2,5,7,10) = \{(2,5,7,10)\} = \triangle_{25}.$

$Orb_{G_{(1,4,7,10)}}(2,6,7,10) = \{(2,6,7,10)\} = \triangle_{26}$.
$Orb_{G_{(1,4,7,10)}}(3,4,7,11) = \{(3,4,7,11)\} = \triangle_{27}$.
$Orb_{G_{(1,4,7,10)}}(3,4,7,12) = \{(3,4,7,12)\} = \triangle_{28}$.
$Orb_{G_{(1,4,7,10)}}(3,4,8,10) = \{(3,4,8,10)\} = \triangle_{29}$.
$Orb_{G_{(1,4,7,10)}}(3,4,8,10) = \{(3,4,9,10)\} = \triangle_{30}$.
$Orb_{G_{(1,4,7,10)}}(3,5,7,10) = \{(3,5,7,10)\} = \triangle_{31}$.
$Orb_{G_{(1,4,7,10)}}(3,6,7,10) = \{(3,6,7,10)\} = \triangle_{32}$.

(d) Orbits containing exactly one element of A are;
$Orb_{G_{(1,4,7,10)}}(1,5,8,11) = \{(1,5,8,11)\} = \triangle_{33}$.
$Orb_{G_{(1,4,7,10)}}(1,5,8,12) = \{(1,5,8,12)\} = \triangle_{34}$.
$Orb_{G_{(1,4,7,10)}}(1,5,9,11) = \{(1,5,9,11)\} = \triangle_{35}$.
$Orb_{G_{(1,4,7,10)}}(1,5,9,12) = \{(1,5,9,12)\} = \triangle_{36}$.
$Orb_{G_{(1,4,7,10)}}(1,6,8,11) = \{(1,6,8,11)\} = \triangle_{37}$.
$Orb_{G_{(1,4,7,10)}}(1,6,8,12) = \{(1,6,8,12)\} = \triangle_{38}$.
$Orb_{G_{(1,4,7,10)}}(1,6,9,11) = \{(1,6,9,11)\} = \triangle_{39}$.
$Orb_{G_{(1,4,7,10)}}(1,6,9,12) = \{(1,6,9,12)\} = \triangle_{40}$.
$Orb_{G_{(1,4,7,10)}}(2,4,8,11) = \{(2,4,8,11)\} = \triangle_{41}$.
$Orb_{G_{(1,4,7,10)}}(2,4,8,12) = \{(2,4,8,12)\} = \triangle_{42}$.
$Orb_{G_{(1,4,7,10)}}(2,4,9,11) = \{(2,4,9,11)\} = \triangle_{43}$.
$Orb_{G_{(1,4,7,10)}}(2,4,9,12) = \{(2,4,9,12)\} = \triangle_{44}$.
$Orb_{G_{(1,4,7,10)}}(2,5,7,11) = \{(2,5,7,11)\} = \triangle_{45}$.
$Orb_{G_{(1,4,7,10)}}(2,5,7,12) = \{(2,5,7,12)\} = \triangle_{46}$.
$Orb_{G_{(1,4,7,10)}}(2,5,8,10) = \{(2,5,8,10)\} = \triangle_{47}$.
$Orb_{G_{(1,4,7,10)}}(2,5,9,10) = \{(2,5,9,10)\} = \triangle_{48}$.
$Orb_{G_{(1,4,7,10)}}(2,6,7,11) = \{(2,6,7,11)\} = \triangle_{49}$.
$Orb_{G_{(1,4,7,10)}}(2,6,7,12) = \{(2,6,7,12)\} = \triangle_{50}$.
$Orb_{G_{(1,4,7,10)}}(2,6,8,10) = \{(2,6,8,10)\} = \triangle_{51}$.
$Orb_{G_{(1,4,7,10)}}(2,6,9,10) = \{(2,6,9,10)\} = \triangle_{52}$.
$Orb_{G_{(1,4,7,10)}}(3,4,8,11) = \{(3,4,8,11)\} = \triangle_{53}$.
$Orb_{G_{(1,4,7,10)}}(3,4,8,12) = \{(3,4,8,12)\} = \triangle_{54}$.
$Orb_{G_{(1,4,7,10)}}(3,4,9,11) = \{(3,4,9,11)\} = \triangle_{55}$.
$Orb_{G_{(1,4,7,10)}}(3,4,9,12) = \{(3,4,9,12)\} = \triangle_{56}$.
$Orb_{G_{(1,4,7,10)}}(3,5,7,11) = \{(3,5,7,11)\} = \triangle_{57}$.
$Orb_{G_{(1,4,7,10)}}(3,5,7,12) = \{(3,5,7,12)\} = \triangle_{58}$.
$Orb_{G_{(1,4,7,10)}}(3,5,8,10) = \{(3,5,8,10)\} = \triangle_{59}$.
$Orb_{G_{(1,4,7,10)}}(3,5,9,10) = \{(3,5,9,10)\} = \triangle_{60}$.
$Orb_{G_{(1,4,7,10)}}(3,6,7,11) = \{(3,6,7,11)\} = \triangle_{61}$.
$Orb_{G_{(1,4,7,10)}}(3,6,7,12) = \{(3,6,7,12)\} = \triangle_{62}$.
$Orb_{G_{(1,4,7,10)}}(3,6,8,10) = \{(3,6,8,10)\} = \triangle_{63}$.

$Orb_{G_{(1,4,7,10)}}(3,6,9,10) = \{(3,6,9,10)\} = \triangle_{64}$.

(e) Orbits containing no element of A are;
$Orb_{G_{(1,4,7,10)}}(2,5,8,11) = \{(2,5,8,11)\} = \triangle_{65}$.
$Orb_{G_{(1,4,7,10)}}(2,5,8,12) = \{(2,5,8,12)\} = \triangle_{66}$.
$Orb_{G_{(1,4,7,10)}}(2,5,9,11) = \{(2,5,9,11)\} = \triangle_{67}$.
$Orb_{G_{(1,4,7,10)}}(2,5,9,12) = \{(2,5,9,12)\} = \triangle_{68}$.
$Orb_{G_{(1,4,7,10)}}(2,6,8,11) = \{(2,6,8,11)\} = \triangle_{69}$.
$Orb_{G_{(1,4,7,10)}}(2,6,8,12) = \{(2,6,8,12)\} = \triangle_{70}$.
$Orb_{G_{(1,4,7,10)}}(2,6,9,11) = \{(2,6,9,11)\} = \triangle_{71}$.
$Orb_{G_{(1,4,7,10)}}(2,6,9,12) = \{(2,6,9,12)\} = \triangle_{72}$.
$Orb_{G_{(1,4,7,10)}}(3,5,8,11) = \{(3,5,8,11)\} = \triangle_{73}$.
$Orb_{G_{(1,4,7,10)}}(3,5,8,12) = \{(3,5,8,12)\} = \triangle_{74}$.
$Orb_{G_{(1,4,7,10)}}(3,5,9,11) = \{(3,5,9,11)\} = \triangle_{75}$.
$Orb_{G_{(1,4,7,10)}}(3,5,9,12) = \{(3,5,9,12)\} = \triangle_{76}$.
$Orb_{G_{(1,4,7,10)}}(3,6,7,11) = \{(3,6,8,11)\} = \triangle_{77}$.
$Orb_{G_{(1,4,7,10)}}(3,6,7,12) = \{(3,6,8,12)\} = \triangle_{78}$.
$Orb_{G_{(1,4,7,10)}}(3,6,9,11) = \{(3,6,9,11)\} = \triangle_{79}$.
$Orb_{G_{(1,4,7,10)}}(3,6,9,12) = \{(3,6,9,12)\} = \triangle_{80}$.

Thus, rank of $A_4 \times A_3 \times A_3$ on $X_1 \times X_2 \times X_3$ is 3^4 and the subdegrees are $\underbrace{1, 1, \cdots, 1}_{81 \text{ times}}$. □

Lemma 2.10. *The group $A_4 \times A_4 \times A_4 \times A_4$ acts on $X_1 \times X_2 \times X_3 \times X_4$ with a rank of 2^4.*

Proof. Let $G = A_4 \times A_4 \times A_4 \times A_4$ and $K = X_1 \times X_2 \times X_3 \times X_4$. From Theorem 2.3, this action is transitive and $G_{(1,5,9,13)} = \langle \{(14\ 16\ 15), (10\ 12\ 11), (6\ 8\ 7), (2\ 4\ 3)\} \rangle$ with $|G_{(1,5,9,13)}| = 81$.
Also, $G_{(1,5,9,13)}$ is isomorphic to $A_3 \times A_3 \times A_3$ with $A_3 = \{(), (1\ 2\ 3), (1\ 3\ 2)\}$ having permutations of types (I) and (abc) which are 1, and 2 respectively in number. Thus the number of elements in $X_1 \times X_2 \times X_3$ fixed by each $g_1, g_2, g_2 \in G_{(1,5,9,13)}$ are given in Table 1

Table 1: Permutations in $G_{(1,5,9,13)}$ and number of fixed points

Type of ordered quadraple permutations in $G_{(1,5,9,13)}$	Number of quadraple permutations in $G_{(1,5,9,13)}$	$\lvert Fix(g_1, g_2, g_3, g_4)\rvert$
$(1, 1, 1, 1)$	1	256
$(1, 1, 1, (abc))$	2	64
$(1, 1, (abc), 1)$	2	64
$(1, 1, (abc), (abc))$	4	16
$(1, (abc), 1, 1)$	2	64
$(1, (abc), 1, (abc))$	4	16
$(1, (abc), (abc), 1)$	4	16
$(1, (abc), (abc), (abc))$	8	4
$((abc), 1, 1, 1)$	2	64
$((abc), 1, 1, (abc))$	4	16
$((abc), 1, (abc), 1)$	4	16
$((abc), 1, (abc), (abc))$	8	4
$((abc), (abc), 1, 1)$	4	16
$((abc), (abc), 1, (abc))$	8	4
$((abc), (abc), (abc), 1)$	8	4
$((abc), (abc), (abc), (abc))$	16	1

By applying Lemma 1.1, the number of orbits of $G_{(1,5,9,13)}$ acting on K is;

$$\frac{1}{|G_{(1,5,9,13)}|}\sum_{g_1,\cdots,g_4\in G_{(1,5,9,13)}}|Fix(g_1,\cdots,g_4)| = \frac{1}{81}\{(1\times 256)+(2\times 64)+(2\times 64)$$
$$+(4\times 16)+(2\times 64)+(4\times 16)$$
$$+(4\times 16)+(8\times 4)+(2\times 64)$$
$$+(4\times 16)+(4\times 16)+(8\times 4)$$
$$+(4\times 16)+(8\times 4)+(8\times 4)$$
$$+(16\times 1)\}$$
$$=\frac{1296}{81}=2^4$$

Let $A=\{1,5,9,13\}$.
The orbits of $G_{(1,5,9,13)}$ on K include the following:

(a) Orbits containing exactly four elements of A are;
$Orb_{G_{(1,5,9,13)}}(1,5,9,13)=\{(1,5,9,13)\}=\triangle_0$.

(a) Orbits containing exactly three elements of A are;
$Orb_{G_{(1,5,9,13)}}(1,5,9,14)=\{(1,5,9,14),(1,5,9,15),(1,5,9,16)\}=\triangle_1$.
$Orb_{G_{(1,5,9,13)}}(1,5,10,13)=\{(1,5,10,13),(1,5,11,13),(1,5,12,13)\}=\triangle_2$.
$Orb_{G_{(1,5,9,13)}}(1,6,9,13)=\{(1,6,9,13),(1,7,9,13),(1,8,9,13)\}=\triangle_3$.
$Orb_{G_{(1,5,9,13)}}(2,5,9,13)=\{(2,5,9,13),(3,5,9,13),(4,5,9,13)\}=\triangle_4$.

(a) Orbits containing exactly two elements of A are;
$Orb_{G_{(1,5,9,13)}}(1,5,10,14) =$

$\{(1,5,10,14),(1,5,10,15),(1,5,10,16),(1,5,12,14),$
$(1,5,11,14),(1,5,12,16),(1,5,12,15),(1,5,11,15),(1,5,11,16)\} = \triangle_5.$
$Orb_{G_{(1,5,9,13)}}(1,6,9,14) =$
$\{(1,6,9,14),(1,6,9,15),(1,6,9,16),(1,7,9,14),(1,7,9,15),$
$(1,7,9,16),(1,8,9,14),(1,8,9,15),(1,8,9,16)\} = \triangle_6.$
$Orb_{G_{(1,5,9,13)}}(1,6,10,13) =$
$\{(1,6,10,13),(1,6,11,13),(1,6,12,13),(1,7,10,13),$
$(1,7,11,13),(1,7,12,13),(1,8,10,13),(1,8,11,13),(1,8,12,13)\} = \triangle_7.$
$Orb_{G_{(1,5,9,13)}}(2,5,9,14) =$
$\{(2,5,9,14),(2,5,9,15),(2,5,9,16),(3,5,9,14),(3,5,9,15),$
$(3,5,9,16),(4,5,9,14),(4,5,9,15),(4,5,9,16)\} = \triangle_8.$
$Orb_{G_{(1,5,9,13)}}(2,5,10,13) =$
$\{(2,5,10,13),(2,5,11,13),(2,5,12,13),(3,5,10,13),$
$(3,5,11,13),(3,5,12,13),(4,5,10,13),(4,5,11,13),(4,5,12,13)\} = \triangle_9.$
$Orb_{G_{(1,5,9,13)}}(2,6,9,13) =$
$\{(2,6,9,13),(2,7,9,13),(2,8,9,13),(3,6,9,13),(3,7,9,13),$
$(3,8,9,13),(4,6,9,13),(4,7,9,13),(4,8,9,13)\} = \triangle_{10}.$

(b) Orbits containing exactly one element of A are;
$Orb_{G_{(1,5,9,13)}}(1,6,10,14) =$
$\{(1,6,10,14),(1,6,10,15),(1,6,10,16),(1,6,11,14),$
$(1,6,11,15),(1,6,11,15),(1,6,12,14),(1,6,12,15),(1,6,12,16),(1,7,10,14),$
$(1,7,10,15),(1,7,10,16),(1,7,11,14),(1,7,11,15),(1,7,11,15),(1,7,12,14),$
$(1,7,12,15),(1,7,12,16),(1,8,10,14),(1,8,10,15),(1,8,10,16),(1,8,11,14),$
$(1,8,11,15),(1,8,11,15),(1,8,12,14),(1,8,12,15),(1,8,12,16)\} = \triangle_{11}.$
$Orb_{G_{(1,5,9,13)}}(2,5,10,14) =$
$\{(2,5,10,14),(2,5,10,15),(2,5,10,16),(2,5,11,14),$
$(2,5,11,15),(2,5,11,16),(2,5,12,14),(2,5,12,15),(2,5,12,16),(3,5,10,14),$
$(3,5,10,15),(3,5,10,16),(3,5,11,14),(3,5,11,15),(3,5,11,16),(3,5,12,14),$
$(3,5,12,15),(3,5,12,16),(4,5,10,14),(4,5,10,15),(4,5,10,16),(4,5,11,14),$
$(4,5,11,15),(4,5,11,16),(4,5,12,14),(4,5,12,15),(4,5,12,16)\} = \triangle_{12}.$
$Orb_{G_{(1,5,9,13)}}(2,6,9,14) =$
$\{(2,6,9,14),(2,6,9,15),(2,6,9,16),(2,7,9,14),(2,7,9,15),$
$(2,7,9,16),(2,8,9,14),(2,8,9,15),(2,8,9,16),(3,6,9,14),(3,6,9,15),(3,6,9,16),$
$(3,7,9,14),(3,7,9,15),(3,7,9,16),(3,8,9,14),(3,8,9,15),(3,8,9,16),(4,6,9,14),$
$(4,6,9,15),(4,6,9,16),(4,7,9,14),(4,7,9,15),(4,7,9,16),(4,8,9,14),(4,8,9,15),$
$(4,8,9,16)\} = \triangle_{13}.$
$Orb_{G_{(1,5,9,13)}}(2,6,10,13) =$
$\{(2,6,10,13),(2,6,11,13),(2,6,12,13)(2,7,10,13),$
$(2,7,11,13),(2,7,12,13),(2,8,10,13),(2,8,11,13),(2,8,12,13),(3,6,10,13),$

$(3,6,11,13),(3,6,12,13)(3,7,10,13),(3,7,11,13),(3,7,12,13),(3,8,10,13),$
$(3,8,11,13),(3,8,12,13),(4,6,10,13),(4,6,11,13),(4,6,12,13)(4,7,10,13),$
$(4,7,11,13),(4,7,12,13),(4,8,10,13),(4,8,11,13),(4,8,12,13)\} = \triangle_{14}$

(c) Orbits containing no element of A are;
$Orb_{G_{(1,8,9,13)}}(2,6,10,14) =$
$\{(2,6,10,14),(2,6,10,15),(2,6,10,16),(2,6,11,14),$
$(2,6,11,14),(2,6,11,14),(2,6,12,14),(2,6,12,15),(2,6,12,15),(2,7,10,14),$
$(2,7,10,15),(2,7,10,16),(2,7,11,14),(2,7,11,14),(2,7,11,14),(2,7,12,14),$
$(2,7,12,15),(2,7,12,15),(2,8,10,14),(2,8,10,15),(2,8,10,16),(2,8,11,14),$
$(2,8,11,14),(2,8,11,14),(2,8,12,14),(2,8,12,15),(2,8,12,15),(3,6,10,14),$
$(3,6,10,15),(3,6,10,16),(3,6,11,14),(3,6,11,14),(3,6,11,14),(3,6,12,14),$
$(3,6,12,15),(3,6,12,15),(3,7,10,14),(3,7,10,15),(3,7,10,16),(3,7,11,14),$
$(3,7,11,14),(3,7,11,14),(3,7,12,14),(3,7,12,15),(3,7,12,15),(3,8,10,14),$
$(3,8,10,15),(3,8,10,16),(3,8,11,14),(3,8,11,14),(3,8,11,14),(3,8,12,14),$
$(3,8,12,15),(3,8,12,15),(4,6,10,14),(4,6,10,15),(4,6,10,16),(4,6,11,14),$
$(4,6,11,14),(4,6,11,14),(4,6,12,14),(4,6,12,15),(4,6,12,15),(4,7,10,14),$
$(4,7,10,15),(4,7,10,16),(4,7,11,14),(4,7,11,14),(4,7,11,14),(4,7,12,14),$
$(4,7,12,15),(4,7,12,15),(4,8,10,14),(4,8,10,15),(4,8,10,16),(4,8,11,14),$
$(4,8,11,14),(4,8,11,14),(4,8,12,14),(4,8,12,15),(4,8,12,15)\} = \triangle_{15}.$

Therefore the rank of $A_4 \times A_4 \times A_4 \times A_4$ on $X_1 \times X_2 \times X_3 \times X_4$ is 2^4 and the subdegrees are $1, \underbrace{3, 3 \cdots, 3}_{4 \text{ factors}}, \underbrace{9, \cdots, 9}_{6 \text{ factors}}, \underbrace{27, \cdots, 27}_{4 \text{ factors}}, 81.$ □

Lemma 2.11. *Rank of* $A_5 \times A_5 \times A_5 \times A_5$ *acting on* $X_1 \times X_2 \times X_3 \times X_4$ *is* 2^4.

Proof. Let $G = A_5 \times A_5 \times A_5 \times A_5$ and $K = X_1 \times X_2 \times X_3 \times X_4$. By Theorem 2.4, the action of G on K is transitive with $|G| = 12960000$ and $|G_{(1,6,11,16)}| = 20736$.
We notice that $G_{(1,6,11)}$ is isomorphic to $A_4 \times A_4 \times A_4$ with
$A_4 =$
$\{(),(1\ 2)(3\ 4),(1\ 4)(2\ 3),(1\ 3)(2\ 4),(1\ 2\ 3),(1\ 3\ 2),(1\ 2\ 4),(1\ 4\ 2),(1\ 3\ 4),(1\ 4\ 3)$
$,(2\ 3\ 4),(2\ 4\ 3)\}$ having permutations of types (I), $(ab)(ab)$, and (abc) which are $1, 3,$ and 8 respectively in number. Thus the number of elements in $X_1 \times X_2 \times X_3$ fixed by each $g_1, g_2, g_3, g_4 \in G_{(1,6,11)}$ are given in Tables 2, 3, and 4.

Table 2: Permutations in $G_{(1,6,11,16)}$ and number of fixed points

Type of ordered quadraple permutations in $G_{(1,6,11,16)}$	Number of quadraple permutations in $G_{(1,6,11,16)}$	$\lvert Fix(g_1, \cdots, g_4)\rvert$
$(1, 1, 1, 1)$	1	625
$(1, 1, 1, (ab)(ab))$	3	125
$(1, 1, 1, (abc))$	8	250
$(1, 1, (ab)(ab), 1)$	3	125
$(1, 1, (ab)(ab), (ab)(ab))$	9	25
$1, (1, (ab)(ab), (abc))$	24	50
$(1, 1, (abc), 1)$	8	250
$(1, 1, (abc), (ab)(ab))$	24	50
$(1, 1, (abc), (abc))$	64	100
$(1, (ab)(ab), 1, 1)$	3	125
$(1, (ab)(ab), 1, (ab)(ab))$	9	15
$(1, (ab)(ab), 1, (abc))$	24	50
$(1, (ab)(ab), (ab)(ab), 1)$	9	25
$(1, (ab)(ab), (ab)(ab), (ab)(ab))$	27	5
$(1, (ab)(ab), (ab)(ab), (abc))$	72	10
$(1, (ab)(ab), (abc), 1)$	24	50
$(1, (ab)(ab), (abc), (ab)(ab))$	72	10
$(1, (ab)(ab), (abc), (abc))$	192	20
$(1, (abc), 1, 1)$	8	50
$(1, (abc), 1, (ab)(ab))$	24	50
$(1, (abc), 1, (abc))$	64	100
$(1, (abc), (ab)(ab), 1)$	24	50
$(1, (abc), (ab)(ab), (ab)(ab))$	72	10
$(1, (abc), (ab)(ab), (abc))$	192	20
$(1, (abc), (abc), 1)$	64	100
$(1, (abc), (abc), (ab)(ab))$	192	20
$(1, (abc), (abc), (abc))$	512	40

Table 3: Permutations in $G_{(1,6,11,16)}$ and number of fixed points

Type of ordered quadraple permutations in $G_{(1,6,11,16)}$	Number of quadraple permutations in $G_{(1,6,11,16)}$	$\|Fix(g_1, \cdots, g_4)\|$
$((ab)(ab), 1, 1, 1)$	3	125
$((ab)(ab), 1, 1, (ab)(ab))$	9	25
$((ab)(ab), 1, 1, (abc))$	24	50
$((ab)(ab), 1, (ab)(ab), 1)$	9	25
$((ab)(ab), 1, (ab)(ab), (ab)(ab))$	27	5
$(ab)(ab), (1, (ab)(ab), (abc))$	72	10
$((ab)(ab), 1, (abc), 1)$	24	50
$((ab)(ab), 1, (abc), (ab)(ab))$	72	10
$((ab)(ab), 1, (abc), (abc))$	192	20
$((ab)(ab), (ab)(ab), 1, 1)$	9	25
$((ab)(ab), (ab)(ab), 1, (ab)(ab))$	27	5
$((ab)(ab), (ab)(ab), 1, (abc))$	72	10
$((ab)(ab), (ab)(ab), (ab)(ab), 1)$	27	5
$((ab)(ab), (ab)(ab), (ab)(ab), (ab)(ab))$	81	1
$((ab)(ab), (ab)(ab), (ab)(ab), (abc))$	216	2
$((ab)(ab), (ab)(ab), (abc), 1)$	72	10
$((ab)(ab), (ab)(ab), (abc), (ab)(ab))$	216	2
$((ab)(ab), (ab)(ab), (abc), (abc))$	576	4
$((ab)(ab), (abc), 1, 1)$	24	50
$((ab)(ab), (abc), 1, (ab)(ab))$	72	10
$((ab)(ab), (abc), 1, (abc))$	192	20
$((ab)(ab), (abc), (ab)(ab), 1)$	72	10
$((ab)(ab), (abc), (ab)(ab), (ab)(ab))$	216	2
$((ab)(ab), (abc), (ab)(ab), (abc))$	576	4
$((ab)(ab), (abc), (abc), 1)$	192	20
$((ab)(ab), (abc), (abc), (ab)(ab))$	576	2
$((ab)(ab), (abc), (abc), (abc))$	1536	8

Table 4: Permutations in $G_{(1,6,11,16)}$ and number of fixed points

Type of ordered quadraple permutations in $G_{(1,6,11,16)}$	Number of quadraple permutations in $G_{(1,6,11,16)}$	$\lvert Fix(g_1, \cdots, g_4)\rvert$
$((abc), 1, 1, 1)$	8	250
$((abc), 1, 1, (ab)(ab))$	24	50
$((abc), 1, 1, (abc))$	64	100
$((abc), 1, (ab)(ab), 1)$	24	50
$((abc), 1, (ab)(ab), (ab)(ab))$	72	10
$((abc), (1, (ab)(ab), (abc))$	192	20
$((abc), 1, (abc), 1)$	64	100
$((abc), 1, (abc), (ab)(ab))$	192	20
$((abc), 1, (abc), (abc))$	512	40
$((abc), (ab)(ab), 1, 1)$	24	50
$((abc), (ab)(ab), 1, (ab)(ab))$	72	10
$((abc), (ab)(ab), 1, (abc))$	192	20
$((abc), (ab)(ab), (ab)(ab), 1)$	72	10
$((abc), (ab)(ab), (ab)(ab), (ab)(ab))$	216	2
$((abc), (ab)(ab), (ab)(ab), (abc))$	576	4
$((abc), (ab)(ab), (abc), 1)$	192	20
$((abc), (ab)(ab), (abc), (ab)(ab))$	576	4
$((abc), (ab)(ab), (abc), (abc))$	1536	8
$((abc), (abc), 1, 1)$	64	100
$((abc), (abc), 1, (ab)(ab))$	192	20
$((abc), (abc), 1, (abc))$	512	40
$((abc), (abc), (ab)(ab), 1)$	192	20
$((abc), (abc), (ab)(ab), (ab)(ab))$	576	4
$((abc), (abc), (ab)(ab), (abc))$	1536	8
$((abc), (abc), (abc), 1)$	512	40
$((abc), (abc), (abc), (ab)(ab))$	1536	8
$((abc), (abc), (abc), (abc))$	4096	16

Applying Lemma 1.1 on Tables 2, 3, and 4, gives the number of orbits of $G_{(1,6,11,16)}$ acting on K as;

$$\frac{1}{|G_{(1,6,11,16)}|} \sum_{g_1,\cdots,g_4 \in G_{(1,6,11,16)}} |Fix(g_1, \cdots, g_4)| = \frac{331776}{20736} = 2^4$$

Let $A = \{1, 6, 11, 16\}$.

The suborbits of G are those with exactly $4, 3, 2, 1$ and no element of A and these include;

(a) Orbits containing exactly four elements of A are;
$Orb_{G_{(1,6,11,16)}}(1, 6, 11, 16) = \{(1, 6, 11, 16)\} = \triangle_0$.

(b) Orbits containing exactly three elements of A are;
$Orb_{G_{(1,6,12,16)}}(1, 6, 11, 17) = \{(1, 6, 11, 17), (1, 6, 11, 18), (1, 6, 11, 19), (1, 6, 11, 20)\} = \triangle_1$.
$Orb_{G_{(1,6,11,16)}}(1, 6, 12, 16) = \{(1, 6, 12, 16), (1, 6, 13, 16), (1, 6, 14, 16), (1, 6, 15, 16)\} = \triangle_2$.
$Orb_{G_{(1,6,11,16)}}(1, 7, 11, 16) = \{(1, 7, 11, 16), (1, 8, 11, 16), (1, 9, 11, 16),$

$(1,10,11,16)\} = \triangle_3$.
$Orb_{G_{(1,6,11,16)}}(2,6,11,16) = \{(2,6,11,16),(3,6,11,16),(4,6,11,16),$
$(5,6,11,16)\} = \triangle_4$.

(c) Orbits containing exactly two elements of A are;
$Orb_{G_{(1,6,11,16)}}(1,6,12,17) =$
$\{(1,6,12,17),(1,6,12,18),(1,6,12,19),(1,6,12,20),$
$(1,6,13,17),(1,6,13,18),(1,6,13,19),(1,6,13,20),(1,6,14,17),(1,6,14,18),$
$(1,6,14,19),(1,6,14,20),(1,6,15,17),(1,6,15,18),(1,6,15,19),(1,6,15,20)\}$
$= \triangle_5$.
$Orb_{G_{(1,6,11,16)}}(1,7,11,17) =$
$\{(1,7,11,17),(1,7,11,18),(1,7,11,19),(1,7,11,20),$
$(1,8,11,17),(1,8,11,18),(1,8,11,19),(1,8,11,20),(1,9,11,17),(1,9,11,18),$
$(1,9,11,19),(1,9,11,20),(1,10,11,17),(1,10,11,18),(1,10,11,19),(1,10,11,20)\}$
$= \triangle_6$.
$Orb_{G_{(1,6,11,16)}}(1,7,12,16) =$
$\{(1,7,12,16),(1,7,13,16),(1,7,14,16),(1,7,15,16),$
$(1,8,12,16),(1,8,13,16),(1,8,14,16),(1,8,15,16),(1,9,12,16),(1,9,13,16),$
$(1,9,14,16),(1,9,15,16),(1,10,12,16),(1,10,13,16),(1,10,14,16),(1,10,15,16)\}$
$= \triangle_7$.
$Orb_{G_{(1,6,11,16)}}(2,6,11,17) =$
$\{(2,6,11,17),(2,6,11,18),(2,6,11,19),(2,6,11,20),$
$(3,6,11,17),(3,6,11,18),(3,6,11,19),(3,6,11,20),(4,6,11,17),(4,6,11,18),$
$(4,6,11,19),(4,6,11,20),(5,6,11,17),(5,6,11,18),(5,6,11,19),(5,6,11,20)\}$
$= \triangle_8$.
$Orb_{G_{(1,6,11,16)}}(2,6,12,16) =$
$\{(2,6,12,16),(2,6,13,16),(2,6,14,16),(2,6,15,16),$
$(3,6,12,16),(3,6,13,16),(3,6,14,16),(3,6,15,16),(4,6,12,16),(4,6,13,16),$
$(4,6,14,16),(4,6,15,16),(5,6,12,16),(5,6,13,16),(5,6,14,16),(5,6,15,16)\}$
$= \triangle_9$.
$Orb_{G_{(1,6,11,16)}}(2,7,11,16) =$
$\{(2,7,11,16),(2,8,11,16),(2,9,11,16),(2,10,11,16),$
$(3,7,11,16),(3,8,11,16),(3,9,11,16),(3,10,11,16),(4,7,11,16),(4,8,11,16),$
$(4,9,11,16),(4,10,11,16),(5,7,11,16),(5,8,11,16),(5,9,11,16),(5,10,11,16)\}$
$= \triangle_{10}$.

(d) Orbits containing exactly one element of A are;
$Orb_{G_{(1,6,11,16)}}(1,7,12,17) =$
$\{(1,7,12,17),(1,7,12,18),(1,7,12,19),(1,7,12,20),$
$(1,7,13,17),(1,7,13,18),(1,7,13,19),(1,7,13,20),(1,7,14,17),(1,7,14,18),$

$(1,7,14,19),(1,7,14,20),(1,7,15,17),(1,7,15,18),(1,7,15,19),(1,7,15,20),$
$(1,7,12,17),(1,8,12,18),(1,8,12,19),(1,8,12,20),(1,8,13,17),(1,8,13,18),$
$(1,8,13,19),(1,8,13,20),(1,8,14,17),(1,8,14,18),(1,8,14,19),(1,8,14,20),$
$(1,8,15,17),(1,8,15,18),(1,8,15,19),(1,8,15,20),(1,9,12,17),(1,9,12,18),$
$(1,9,12,19),(1,9,12,20),(1,9,13,17),(1,9,13,18),(1,9,13,19),(1,9,13,20),$
$(1,9,14,17),(1,9,14,18),(1,9,14,19),(1,9,14,20),(1,9,15,17),(1,9,15,18),$
$(1,9,15,19),(1,9,15,20),(1,10,12,17),(1,10,12,18),(1,10,12,19),(1,10,12,20),$
$(1,10,13,17),(1,10,13,18),(1,10,13,19),(1,10,13,20),(1,10,14,17),(1,10,14,18),$
$(1,10,14,19),(1,10,14,20),(1,10,15,17),(1,10,15,18),(1,10,15,19),(1,10,15,20)\}$
$=\triangle_{11}.$

$Orb_{G_{(1,6,11,16)}}(2,6,12,17) =$
$\{(2,6,12,17),(2,6,12,18),(2,6,12,19),(2,6,12,20),$
$(2,6,13,17),(2,6,13,18),(2,6,13,19),(2,6,13,20),(2,6,14,17),(2,6,14,18),$
$(2,6,14,19),(2,6,14,20),(2,6,15,17),(2,6,15,18),(2,6,15,19),(2,6,15,20),$
$(3,6,12,17),(3,6,12,18),(3,6,12,19),(3,6,12,20),(3,6,13,17),(3,6,13,18),$
$(3,6,13,19),(3,6,13,20),(3,6,14,17),(3,6,14,18),(3,6,14,19),(3,6,14,20),$
$(3,6,15,17),(3,6,15,18),(3,6,15,19),(3,6,15,20),(4,6,12,17),(4,6,12,18),$
$(4,6,12,19),(4,6,12,20),(4,6,13,17),(4,6,13,18),(4,6,13,19),(4,6,13,20),$
$(4,6,14,17),(4,6,14,18),(4,6,14,19),(4,6,14,20),(4,6,15,17),(4,6,15,18),$
$(4,6,15,19),(4,6,15,20),(5,6,12,17),(5,6,12,18),(5,6,12,19),(5,6,12,20),$
$(5,6,13,17),(5,6,13,18),(5,6,13,19),(5,6,13,20),(5,6,14,17),(5,6,14,18),$
$(5,6,14,19),(5,6,14,20),(5,6,15,17),(5,6,15,18),(5,6,15,19),(5,6,15,20)\}$
$=\triangle_{12}.$

$Orb_{G_{(1,6,11,16)}}(2,7,11,17) =$
$\{(2,7,11,17),(2,7,11,18),(2,7,11,19),(2,7,11,20),$
$(2,8,11,17),(2,8,11,18),(2,8,11,19),(2,8,11,20),(2,9,11,17),(2,9,11,18),$
$(2,9,11,19),(2,9,11,20),(2,10,11,17),(2,10,11,18),(2,10,11,19),(2,10,11,20),$
$(3,7,11,17),(3,7,11,18),(3,7,11,19),(3,7,11,20),(3,8,11,17),(3,8,11,18),$
$(3,8,11,19),(3,8,11,20),(3,9,11,17),(3,9,11,18),(3,9,11,19),(3,9,11,20),$
$(3,10,11,17),(3,10,11,18),(3,10,11,19),(3,10,11,20),(4,7,11,17),(4,7,11,18),$
$(4,7,11,19),(4,7,11,20),(4,8,11,17),(4,8,11,18),(4,8,11,19),(4,8,11,20),$
$(4,9,11,17),(4,9,11,18),(4,9,11,19),(4,9,11,20),(4,10,11,17),(4,10,11,18),$
$(4,10,11,19),(4,10,11,20),(5,7,11,17),(5,7,11,18),(5,7,11,19),(5,7,11,20),$
$(5,8,11,17),(5,8,11,18),(5,8,11,19),(5,8,11,20),(5,9,11,17),(5,9,11,18),$
$(5,9,11,19),(5,9,11,20),(5,10,11,17),(5,10,11,18),(5,10,11,19),(5,10,11,20)\}$
$=\triangle_{13}.$

$Orb_{G_{(1,6,11,16)}}(2,7,12,16) =$
$\{(2,7,12,16),(2,7,13,16),(2,7,14,16),(2,7,15,16),(2,8,12,16),$
$(2,8,13,16),(2,8,14,16),(2,8,15,16),(2,9,12,16),(2,9,13,16),(2,9,14,16),$

$(2,9,15,16),(2,10,12,16),(2,10,13,16),(2,10,14,16),(2,10,15,16),$
$(3,7,12,16),(3,7,13,16),(3,7,14,16),(3,7,15,16),(3,8,12,16),(3,8,13,16),$
$(3,8,14,16),(3,8,15,16),(3,9,12,16),(3,9,13,16),(3,9,14,16),(3,9,15,16),$
$(3,10,12,16),(3,10,13,16),(3,10,14,16),(3,10,15,16),(4,7,12,16),(4,7,13,16),$
$(4,7,14,16),(4,7,15,16),(4,8,12,16),(4,8,13,16),(4,8,14,16),(4,8,15,16),$
$(4,9,12,16),(4,9,13,16),(4,9,14,16),(4,9,15,16),(4,10,12,16),(4,10,13,16),$
$(4,10,14,16),(4,10,15,16),(5,7,12,16),(5,7,13,16),(5,7,14,16),(5,7,15,16),$
$(5,8,12,16),(5,8,13,16),(5,8,14,16),(5,8,15,16),(5,9,12,16),(5,9,13,16),$
$(5,9,14,16),(5,9,15,16),(5,10,12,16),(5,10,13,16),(5,10,14,16),(5,10,15,16)\}$
$= \triangle_{14}.$

(e) Orbits containing no element of A are;
$Orb_{G_{(1,6,11,16)}}(2,7,12,17) =$
$\{(2,7,12,17),(2,7,12,18),(2,7,12,19),(2,7,12,20),$
$(2,7,13,17),(2,7,13,18),(2,7,13,19),(2,7,13,20),(2,7,14,17),(2,7,14,18),$
$(2,7,14,19),(2,7,14,20),(2,7,15,17),(2,7,15,18),(2,7,15,19),(2,7,15,20),$
$(2,7,12,17),(2,8,12,18),(2,8,12,19),(2,8,12,20),(2,8,13,17),(2,8,13,18),$
$(2,8,13,19),(2,8,13,20),(2,8,14,17),(2,8,14,18),(2,8,14,19),(2,8,14,20),$
$(2,8,15,17),(2,8,15,18),(2,8,15,19),(2,8,15,20),(2,9,12,17),(2,9,12,18),$
$(2,9,12,19),(2,9,12,20),(2,9,13,17),(2,9,13,18),(2,9,13,19),(2,9,13,20),$
$(2,9,14,17),(2,9,14,18),(2,9,14,19),(2,9,14,20),(2,9,15,17),(2,9,15,18),$
$(2,9,15,19),(2,9,15,20),(2,10,12,17),(2,10,12,18),(2,10,12,19),(2,10,12,20),$
$(2,10,13,17),(2,10,13,18),(2,10,13,19),(2,10,13,20),(2,10,14,17),(2,10,14,18),$
$(2,10,14,19),(2,10,14,20),(2,10,15,17),(2,10,15,18),(2,10,15,19),(2,10,15,20),$
$(3,7,12,17),(3,7,12,18),(3,7,12,19),(3,7,12,20),(3,7,13,17),(3,7,13,18),$
$(3,7,13,19),(3,7,13,20),(3,7,14,17),(3,7,14,18),(3,7,14,19),(3,7,14,20),$
$(3,7,15,17),(3,7,15,18),(3,7,15,19),(3,7,15,20),(3,7,12,17),(3,8,12,18),$
$(3,8,12,19),(3,8,12,20),(3,8,13,17),(3,8,13,18),(3,8,13,19),(3,8,13,20),$
$(3,8,14,17),(3,8,14,18),(3,8,14,19),(3,8,14,20),(3,8,15,17),(3,8,15,18),$
$(3,8,15,19),(3,8,15,20),(3,9,12,17),(3,9,12,18),(3,9,12,19),(3,9,12,20),$
$(3,9,13,17),(3,9,13,18),(3,9,13,19),(3,9,13,20),(3,9,14,17),(3,9,14,18),$
$(3,9,14,19),(3,9,14,20),(3,9,15,17),(3,9,15,18),(3,9,15,19),(3,9,15,20),$
$(3,10,12,17),(3,10,12,18),(3,10,12,19),(3,10,12,20),(3,10,13,17),(3,10,13,18),$
$(3,10,13,19),(3,10,13,20),(3,10,14,17),(3,10,14,18),(3,10,14,19),(3,10,14,20),$
$(3,10,15,17),(3,10,15,18),(3,10,15,19),(3,10,15,20),(4,7,12,17),(4,7,12,18),$
$(4,7,12,19),(4,7,12,20),(4,7,13,17),(4,7,13,18),(4,7,13,19),(4,7,13,20),$
$(4,7,14,17),(4,7,14,18),(4,7,14,19),(4,7,14,20),(4,7,15,17),(4,7,15,18),$
$(4,7,15,19),(4,7,15,20),(4,7,12,17),(4,8,12,18),(4,8,12,19),(4,8,12,20),$
$(4,8,13,17),(4,8,13,18),(4,8,13,19),(4,8,13,20),(4,8,14,17),(4,8,14,18),$
$(4,8,14,19),(4,8,14,20),(4,8,15,17),(4,8,15,18),(4,8,15,19),(4,8,15,20),$

$$
\begin{aligned}
&(4,9,12,17),(4,9,12,18),(4,9,12,19),(4,9,12,20),(4,9,13,17),(4,9,13,18),\\
&(4,9,13,19),(4,9,13,20),(4,9,14,17),(4,9,14,18),(4,9,14,19),(4,9,14,20),\\
&(4,9,15,17),(4,9,15,18),(4,9,15,19),(4,9,15,20),(4,10,12,17),(4,10,12,18),\\
&(4,10,12,19),(4,10,12,20),(4,10,13,17),(4,10,13,18),(4,10,13,19),(4,10,13,20),\\
&(4,10,14,17),(4,10,14,18),(4,10,14,19),(4,10,14,20),(4,10,15,17),(4,10,15,18),\\
&(4,10,15,19),(4,10,15,20),(5,7,12,17),(5,7,12,18),(5,7,12,19),(5,7,12,20),\\
&(5,7,13,17),(5,7,13,18),(5,7,13,19),(5,7,13,20),(5,7,14,17),(5,7,14,18),\\
&(5,7,14,19),(5,7,14,20),(5,7,15,17),(5,7,15,18),(5,7,15,19),(5,7,15,20),\\
&(5,7,12,17),(5,8,12,18),(5,8,12,19),(5,8,12,20),(5,8,13,17),(5,8,13,18),\\
&(5,8,13,19),(5,8,13,20),(5,8,14,17),(5,8,14,18),(5,8,14,19),(5,8,14,20),\\
&(5,8,15,17),(5,8,15,18),(5,8,15,19),(5,8,15,20),(5,9,12,17),(5,9,12,18),\\
&(5,9,12,19),(5,9,12,20),(5,9,13,17),(5,9,13,18),(5,9,13,19),(5,9,13,20),\\
&(5,9,14,17),(5,9,14,18),(5,9,14,19),(5,9,14,20),(5,9,15,17),(5,9,15,18),\\
&(5,9,15,19),(5,9,15,20),(5,10,12,17),(5,10,12,18),(5,10,12,19),(5,10,12,20),\\
&(5,10,13,17),(5,10,13,18),(5,10,13,19),(5,10,13,20),(5,10,14,17),(5,10,14,18),\\
&(5,10,14,19),(5,10,14,20),(5,10,15,17),(5,10,15,18),(5,10,15,19),(5,10,15,20)\}\\
&=\triangle_{15}.
\end{aligned}
$$

Thus the rank of $A_5 \times A_5 \times A_5 \times A_5$ acting on $X_1 \times X_2 \times X_3 \times X_4$ is 2^4 with subdegrees $1, \underbrace{4, \cdots, 4}_{4 \text{ factors}}, \underbrace{16, \cdots, 16}_{6 \text{ factors}}, \underbrace{64, \cdots, 64}_{4 \text{ factors}}, 256.$ □

Theorem 2.3. *For $n > 3$, the rank of $A_n \times A_n \times A_n \times A_n$ acting on $X_1 \times X_2 \times X_3 \times X_4$ is 2^4 where $X_1 = \{1, 2, \cdots, n\}$, $X_2 = \{n+1, n+2, \cdots, 2n\}$, $X_3 = \{2n+1, 2n+2, \cdots, 3n\}$, $X_4 = \{3n+1, 3n+2, \cdots, 4n\}$.*

Proof. Let $G = A_n \times A_n \times A_n \times A_n$ and $K = X_1 \times X_2 \times X_3 \times X_4$ and $B = \{1, n+1, 2n+1, 3n+1\}$.
The suborbits of G include those with exactly $4, 3, 2, 1$ and no element from B. The number of suborbits of G are given in Table 5.

Table 5: Rank of action of $A_n \times A_n \times A_n \times A_n$ on $X_1 \times X_2 \times X_3 \times X_4$

Suborbit	**Number of suborbits**
Orbit containing exactly 4 elements of B	$\binom{4}{4}$
Orbit containing exactly 3 elements of B	$\binom{4}{3}$
Orbits containing exactly 2 elements of B	$\binom{4}{2}$
Orbits containing exactly 1 element of B	$\binom{4}{1}$
Orbits containing no elements of B	$\binom{4}{0}$

From Table 5, the rank of action of $A_n \times A_n \times A_n \times A_n$ on $X_1 \times X_2 \times X_3 \times X_4$ is

$$\begin{aligned} R(4) &= \binom{4}{4} + \binom{4}{3} + \binom{4}{2} + \binom{4}{1} + \binom{4}{0} \\ &= 1 + 4 + 6 + 4 + 1 \\ &= 16 \\ &= 2^4 \end{aligned}$$

These 2^4 suborbits of the action include;

(a) Orbits containing exactly four elements of B are;
$Orb_{G_{(1,n+1,2n+1,3n+1)}}(1, n+1, 2n+1, 3n+1) = \{(1, n+1, 2n+1, 3n+1)\} = \triangle_0$.

(b) Orbits containing exactly three elements of B are;
$Orb_{G_{(1,n+1,2n+1,3n+1)}}(1, n+1, 2n+1, 3n+2) = \{(1, n+1, 2n+1, 3n+2), (1, n+1, 2n+1, 3n+3), \cdots, (1, n+1, 2n+1, 4n)\} = \triangle_1$.
$Orb_{G_{(1,n+1,2n+1,3n+1)}}(1, n+1, 2n+2, 3n+1) = \{(1, n+1, 2n+2, 3n+1), (1, n+1, 2n+3, 3n+1), \cdots, (1, n+1, 3n, 3n+1)\} = \triangle_2$.
$Orb_{G_{(1,n+1,2n+1,3n+1)}}(1, n+2, 2n+1, 3n+1) = \{(1, n+2, 2n+1, 3n+1), (1, n+3, 2n+1, 3n+1), \cdots, (1, 2n, 2n+1, 3n+1)\} = \triangle_3$.
$Orb_{G_{(1,n+1,2n+1,3n+1)}}(2, n+1, 2n+1, 3n+1) = \{(2, n+1, 2n+1, 3n+1), (3, n+1, 2n+1, 3n+1), \cdots, (n, n+1, 2n+1, 3n+1)\} = \triangle_4$.

(c) Orbits containing exactly two elements of B are;
$Orb_{G_{(1,n+1,2n+1,3n+1)}}(1, n+1, 2n+2, 3n+2) = \{(1, n+1, 2n+2, 3n+2), (1, n+1, 2n+3, 3n+2), \cdots, (1, n+1, 3n, 3n+2), \cdots, (1, n+1, 2n+2, 3n+3), (1, n+1, 2n+2, 3n+4), \cdots, (1, n+1, 2n+2, 4n), (1, n+1, 3n, 4n)\} = \triangle_5$.
$Orb_{G_{(1,n+1,2n+1,3n+1)}}(1, n+2, 2n+2, 3n+1) = \{(1, n+2, 2n+2, 3n+1), (1, n+3, 2n+2, 3n+1), \cdots, (1, 2n, 2n+1, 3n+1), \cdots, (1, n+2, 2n+3, 3n+1), (1, n+2, 2n+4, 3n+1), \cdots, (1, n+2, 3n, 3n+1), (1, 2n, 3n, 3n+1)\} = \triangle_6$.
$Orb_{G_{(1,n+1,2n+1,3n+1)}}(1, n+2, 2n+1, 3n+2) = \{(1, n+2, 2n+1, 3n+2), (1, n+3, 2n+1, 3n+2), \cdots, (1, 2n, 2n+1, 3n+2), \cdots, (1, n+2, 2n+1, 3n+3), (1, n+2, 2n+1, 3n+4), \cdots, (1, n+2, 2n+1, 4n), (1, 2n, 2n+1, 4n)\} = \triangle_7$.
$Orb_{G_{(1,n+1,2n+1,3n+1)}}(2, n+2, 2n+1, 3n+1) = \{(2, n+2, 2n+1, 3n+1), (3, n+2, 2n+1, 3n+1), \cdots, (n, n+2, 2n+1, 3n+1), \cdots, (2, n+3, 2n+1, 3n+1), (2, n+4, 2n+1, 3n+1), \cdots, (2, 2n, 2n+1, 3n+1), (n, 2n, 2n+1, 3n+1)\} = \triangle_8$.
$Orb_{G_{(1,n+1,2n+1,3n+1)}}(2, n+1, 2n+2, 3n+1) = \{(2, n+1, 2n+2, 3n+1), (3, n+1, 2n+2, 3n+1), \cdots, (n, n+1, 2n+2, 3n+1), \cdots, (2, n+1, 2n+3, 3n+1), (2, n+1, 2n+4, 3n+1), \cdots, (2, n+1, 3n, 3n+1), (n, n+1, 3n, 3n+1)\} = \triangle_9$.
$Orb_{G_{(1,n+1,2n+1,3n+1)}}(2, n+1, 2n+1, 3n+2) = \{(2, n+1, 2n+1, 3n+2), (3, n+$

$1, 2n+1, 3n+2), \cdots, (n, n+1, 2n+1, 3n+2), \cdots, (2, n+1, 2n+1, 3n+3), (2, n+1, 2n+1, 3n+4), \cdots, (2, n+1, 2n+1, 4n), (n, n+1, n+1, 4n)\} = \triangle_{10}$.

(d) Orbits containing exactly one element of B are;

$Orb_{G_{(1,n+1,2n+1,3n+1)}}(1, n+2, 2n+2, 3n+2) = \{(1, n+2, 2n+2, 3n+2), (1, n+3, 2n+2, 3n+2), \cdots, (1, 2n, 2n+2, 3n+2), (1, n+2, 2n+3, 3n+2), (1, n+2, 2n+4, 3n+2), \cdots, (1, n+2, 3n, 3n+2), (1, n+2, 2n+2, 3n+3), (1, n+2, 2n+2, 3n+3), \cdots, (1, n+2, 2n+2, 4n), (1, 2n, 3n, 4n)\} = \triangle_{11}$.

$Orb_{G_{(1,n+1,2n+1,3n+1)}}(2, n+1, 2n+2, 3n+2) = \{(2, n+1, 2n+2, 3n+2), (3, n+1, 2n+2, 3n+2), \cdots, (n, n+1, 2n+2, 3n+2), (2, n+1, 2n+3, 3n+2), (1, n+2, 2n+4, 3n+2), \cdots, (2, n+1, 3n, 3n+2), (2, n+1, 2n+2, 3n+3), (2, n+1, 2n+2, 3n+4), \cdots, (2, n+1, 2n+2, 4n), (n, n+1, 3n, 4n)\} = \triangle_{12}$.

$Orb_{G_{(1,n+1,2n+1,3n+1)}}(2, n+2, 2n+1, 3n+2) = \{(2, n+2, 2n+1, 3n+2), (3, n+2, 2n+1, 3n+2), \cdots, (n, n+2, 2n+1, 3n+2), (2, n+3, 2n+1, 3n+2), (2, n+4, 2n+1, 3n+2), \cdots, (2, 2n, 2n+1, 3n+2), (2, n+2, 2n+1, 3n+3), (2, n+2, 2n+1, 3n+4), \cdots, (2, n+2, 2n+1, 4n), (n, 2n, 2n+1, 4n)\} = \triangle_{13}$.

$Orb_{G_{(1,n+1,2n+1,3n+1)}}(2, n+2, 2n+2, 3n+1) = \{(2, n+2, 2n+2, 3n+1), (3, n+2, 2n+2, 3n+1), \cdots, (n, n+2, 2n+2, 3n+1), (2, n+3, 2n+2, 3n+1), (2, n+4, 2n+2, 3n+1), \cdots, (2, 2n, 2n+2, 3n+2), (2, n+2, 2n+3, 3n+1), (2, n+2, 2n+4, 3n+1), \cdots, (2, n+2, 3n, 3n+1), (n, 2n, 3n, 3n+1)\} = \triangle_{14}$.

(e) Orbits containing exactly one element of B are;

$Orb_{G_{(1,n+1,2n+1,3n+1)}}(2, n+2, 2n+2, 3n+2) = \{(2, n+2, 2n+2, 3n+2), (3, n+2, 2n+2, 3n+2), \cdots, (n, n+2, 2n+2, 3n+2), (2, n+3, 2n+2, 3n+2), (2, n+4, 2n+2, 3n+2), \cdots, (2, 2n, 2n+2, 3n+2), (2, n+2, 2n+3, 3n+2), (2, n+2, 2n+4, 3n+2), \cdots, (2, n+2, 3n, 3n+2), (2, n+2, 2n+2, 3n+3), (2, n+2, 2n+2, 3n+4), \cdots, (2, n+2, 2n+2, 4n), (n, 2n, 3n)\} = \triangle_{15}$.

Thus, for $n > 3$ the action of $A_n \times A_n \times A_n$ on $X_1 \times X_2 \times X_3 \times X_4$ has rank 2^4 with respective subdegrees $1, \underbrace{n-1, \cdots, n-1}_{4 \text{ factors}}, \underbrace{(n-1)^2, \cdots, (n-1)^2}_{6 \text{ factors}}, \underbrace{(n-1)^3, \cdots, (n-1)^3)}_{4 \text{ factors}}, (n-1)^4$. □

3. Conclusion

For $n \geq 3$, the action of $A_n \times A_n \times A_n \times A_n \times A_n$ on $X_1 \times X_2 \times X_3 \times X_4 \times X_5$ is transitive and imprimitive but for $n \geq 4$, the associated rank is 2^4 and subdegrees are 1, $n-1$, $(n-1)^2$, $(n-1)^3$, $(n-1)^4$.

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