Modified Homotopy Perturbation Method For Solving High-Order Integro-Differential Equation

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Abstract
In this work, a new modification of homotopy perturbation method was proposed to find analytical solution of high-order integro-differential equations. The Modification process yields the Taylor series of the exact solution. Canonical polynomials are used as basis function. The assumed approximate solution was substituted into the problem considered in which the coefficients of the homotopy perturbation parameter ***p*** were compared, and then solved, resulting to a single algebraic equation. Thus, algebraic linear system of equations were obtained by equating the coefficients of various powers of the independent variables in the equation to zero, which are then solved simultaneously using ***MAPLE 18*** software to obtain the values of the unknown constants in the equations. The values of the unknown constants were substituted back to get the initial approximation which yield the final solution. Some examples were given to illustrate the effectiveness of the method.
Keywords: *Homotopy perturbation, Integro-differential equation, Canonical polynomial, Basis function*

1. General Introduction

In every phenomenon in real life, there are many parameters and variables related to each other under the law imperious on that phenomenon. When the relation between the parameters and variables are presented in mathematical language we usually derive mathematical model of the problem, which may be equation , a differential equation, an integral equation, a system of integral equations, an integro-differential equation and etc. To solve these equations, the numerical solutions of such equations have been highly studied by many authors. In recent years, numerous works have been focusing on the development of more advanced and efficient methods for integro-differential equations such as Wavelet-Galerkin method and Tau method and semi analytical-numerical techniques such as Taylor polynomials, Adomians decomposition method and the Homotopy perturbation method (HPM).

Perturbation techniques are widely used in science and engineering to handle linear and nonlinear problems. The homotopy perturbation was first proposed by JH-He in (1998) and further developed and improved by him in (2003). This method is based on the use of traditional perturbation method and homotopy technique. In this method the solution is considered as the summation of infinite series which converges rapidly to the exact solutions. The applications of the HPM in nonlinear problems have been demonstrated by many researchers. Amongst them are S. Abbasbandy, Iterative Hes homotopy perturbation method for quadratic Riccati differential equation, Appl. Math. Comput. (2006), Babolian, Dastani, Numerical solutions of two-dimensional linear and nonlinear Volterra integral equations: homotopy perturbation method and differential transform method, (2011), Biazar, Ghazvini, Numerical solution for special non-linear Fredholm integral equation by HPM, Appl. Math. Comput. (2008).

Several authors have proposed a variety of the modified homotopy perturbation methods. Yusfoğlu (2009) proposed the improved homotopy perturbation method for solving

Fredholm type integro-differential equations. Javidi (2009) proposed the modified homotopy perturbation method for solving non-linear Fredholm integral equations. This results reveals that the (MHPM) is very effective and convenient.In another study he applied the Modified homotopy perturbation method for solving system of linear Fredholm integral equations, Javidi (2009). Golbabai et al. (2008) used the modified homotopy perturbation method for solving Fredholm integral equations. In another work, he introduced new iterative methods for nonlinear equations by (MHPM), Golbabai et al. (2007).

In this work, a new modification of homotopy perturbation method was proposed to find analytical solution of higher-order integro-differential equations. Basically the MHPM is the same as the HPM. In the HPM, in order to find the solution, the series components are to be calculated and the sum of the partial series of these components is considered as an approximation of the solution. However, in the MHPM, the method is designed in such a way that only one term of the series is calculated.

1.1 Aim and Objectives

The aim of this study is to modify the existing Homotopy perturbation method to find the solution high-order integtro-differential equations by using Canonical polynomials basis function.

The main objectives of this work are to:

- construct canonical polynomials to serve as the basis function for the proposed methods
- develop a fast and accurate algorithm for the approximation solution of high order integro-differential equations;
- generalize the proposed method for the solution of nth order integro differential equation;

1.2 Definition of Relevant Terms

Integral Equations

An Integral Equation is an equation in which the unknown function $y(x)$ appears under an integral sign. A standard integral equation is of the form:

$$y(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x,t)y(t)dt, \quad (1)$$

where $g(x)$ and $h(x)$ are the limits of integration, λ is a constant parameter, $f(x)$ is a known analytic function and $K(x,t)$ is a function of two variables x and t called the kernel or the nucleus of the integral equation, Wazwaz, (2011).

Integro-Differential Equations (IDE)

An integro-differential equations is an equation which involves both the integral and the derivatives of an unknown function. A standard integro-differential equation is of the form:

$$y^{(n)}(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x,t)y(t)dt, \quad (2)$$

where $g(x)$, $h(x)$, $f(x)$, λ and $K(x,t)$ are already explained in (1) and n is the order of the IDE. (2) is called Fredholm Integro-Differential Equation (FIDE) if both the lower and upper bounds or limits of the region of the integration are fixed numbers while it is called Voltera Integro-Differential Equation (VIDE) if the lower bound of the region of integration is a fixed number and the upper bound is a variable, Wazwaz, (2011).

2. Review of Homotopy perturbation method (HPM)

In this section, the HPM is briefly reviewed.
Consider the following equation:

$$A(u) - f(r) = 0, r \in \Omega \tag{3}$$

with boundary conditions of:

$$\mathrm{B}\left(u, \frac{\partial u}{\partial n}\right) = 0. \quad r \in \Gamma, \tag{4}$$

where $A, B, f(r)$ and Γ are a general differential operator, a boundary operator, a known analytical function and the boundary of the domain Ω, respectively. Generally speaking, the operator A can be divided into a linear part $L(u)$ and a nonlinear part $N(u)$. (3) is therefore re-written as:

$$L(u) + N(u) - f(r) = 0 \tag{5}$$

By the homotopy technique, we constructed a homotopy $v(r,p): \Omega \times [0,1] \to R$ which satisfies:

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \quad p \in [0,1], r \in \Omega, \tag{6}$$

or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0 \tag{7}$$

where $p \in [0,1]$ is an embedding parameter, while u_0 is an initial approximation of (3), which satisfies the boundary conditions. Obviously, from (6) and (7) we will have:

$$H(v,0) = L(v) - L(u_0) = 0, \tag{8}$$

$$H(v,1) = A(v) - f(r) = 0, \tag{9}$$

The changing process of p from zero to unity is just that of $v(r,p)$ from u_0 to $u(r)$. In topology, this is called deformation, while $L(v) - L(u) = 0$, and $A(v) - f(r)$ are called homotopic. According to the HPM, we can first use the embedding parameter p as a "small parameter", and assume that the solutions of (6) and (7) can be expressed as:

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \cdots \tag{10}$$

Setting $p = 1$ yields in the approximate solution of (3)

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots \tag{11}$$

The combination of the perturbation method and the homotopy method is called the HPM, which eliminates the drawbacks of the traditional perturbation methods while keeping all its advantages. The convergence of the series given by (11) has been proved in He (1999). The series (10) is convergent in most cases, and the convergence rate depends on $A(u) - f(r) = 0$, JH. He(1999).

Note that in the HPM in order to obtain an approximate solution, the components v_i for $i = 0,1,\cdots$ must be calculated. Specially for for $i \geq 3$, it needs large and sometimes complicated computations and, in the case of non linearity, the use of He's polynomials in Ghorbani (2009) is employed. To obviate this problem, the MHPM is introduced, in which v_0 is calculated in such a way that $v_1 = 0$ for $i \geq 1$. So the number of computations decreases in comparison with that in the HPM.

2.1 Problem Considered

In this work, the following problems were considered:

Integro-differential equations

In this work, an analytical scheme based on the modified homotopy perturbation method for the following kind of integro-differential equation were present:

Fredholm Integro-differential equations the form:

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + a_{n-3}y^{(n-3)} + \cdots + a_1 y' + a_0 y + \lambda \int_{g(x)}^{h(x)} K(x,t)y(t)dt = f(x) \quad (12)$$

subject to the following conditions

$$y(a) = \alpha_1, \quad y'(a) = \alpha_2, \quad \cdots \quad y^{(n-1)}(a) = \alpha_{n-1} \quad (13)$$

where a_i,s are constant coefficients, $g(x)$ and $h(x)$ are the limit of integration, λ is a constant parameter, and $K(x,t)$ is a function of two variables x and t called the kernel of the integral equation, $f(x)$ is a known function. Where both the lower and upper bounds of the region of the integration are fixed numbers.

Volterra Integro-differential equations

Volterra Integro-differential equation of the form (12), where the lower bound of the region of integration is a fixed number and the upper bound is not, were also considered

Volterra-Fredholm integro-differential equation of the form:

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + a_{n-3}y^{(n-3)} + \cdots + a_1 y' + a_0 y + \lambda \int_0^x \int_{g(x)}^{h(x)} K(x,t)y(t)dt = f(x) \quad (14)$$

subject to the following conditions

$$y(a) = \alpha_1, \quad y'(a) = \alpha_2, \quad \cdots \quad y^{(n-1)}(a) = \alpha_{n-1} \quad (15)$$

where a_i,s are constant coefficients, $g(x)$ and $h(x)$ are the limit of integration and are fixed numbers, $f(x)$ is a known analytic function and $K(x,t)$ is a function of two variables x and t called the kernel of the integral equation.

2.2 Construction of Canonical polynomial

In this section the generalized form of canonical polynomial which served as the basis function to the approximate of all the problems considered in this work were generated .
Consider the general integro-differential equation of the form:

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + a_{n-3}y^{(n-3)} + \cdots + a_1 y' + a_0 y = \lambda \int_{g(x)}^{h(x)} K(x,t)y(t)dt + g(x) \quad (16)$$

From (16), operator L is define as :

$$Ly \equiv \left(a_n \frac{d^n}{dx^n} + a_{n-1}\frac{d^{n-1}}{dx^{n-1}} + a_{n-2}\frac{d^{n-2}}{dx^{n-2}} + a_{n-3}\frac{d^{n-3}}{dx^{n-3}} + a_1 \frac{d}{dx} + a_0\right) y \quad (17)$$

Let,

$$LQ_j(x) = x^j \quad (18)$$

So that

$$L\{LQ_j(x)\} = Lx^j \quad (19)$$

Thus,

$$L\{LQ_j(x)\} = \left(a_n \frac{d^n}{dx^n} + a_{n-1}\frac{d^{n-1}}{dx^{n-1}} + a_{n-2}\frac{d^{n-2}}{dx^{n-2}} + a_{n-3}\frac{d^{n-3}}{dx^{n-3}} + \cdots + a_1 \frac{d}{dx} + a_0\right) x^j \quad (20)$$

$$Lx^j = a_n \frac{d^n}{dx^n}x^j + a_{n-1}\frac{d^{n-1}}{dx^{n-1}}x^j + a_{n-2}\frac{d^{n-2}}{dx^{n-2}}x^j + a_{n-3}\frac{d^{n-3}}{dx^{n-3}}x^j + \cdots$$

$$+a_1 \frac{d}{dx} x^j + a_0 x^j \tag{21}$$

Thus the expansion of (21) leads to

$$Lx^j = a_n \prod_{r=0}^{n-1} (j-r) x^{(j-n)} + a_{n-1} \prod_{r=0}^{n-2} (j-r) x^{(j-n+1)} + a_{n-2} \prod_{r=0}^{n-3} (j-r) x^{(j-n+2)}$$

$$+ \quad \cdots + a_1 \prod_{r=0}^{n-n} (j-r) x^{(j-1)} + a_0 x^j \tag{22}$$

Since $LQ(x) = x^j$, (22) becomes

$$Lx^j = a_n \prod_{r=0}^{n-1} (j-r) LQ_{j-n}(x) + a_{n-1} \prod_{r=0}^{n-2} (j-r) LQ_{j-n+1}(x) + a_{n-2} \prod_{r=0}^{n-3} (j-r) LQ_{j-n+2}(x)$$

$$+ \quad \cdots + a_1 \prod_{r=0}^{n-n} (j-r) LQ_{j-1}(x) + a_0 LQ_j(x) \tag{23}$$

Thus, implies

$$Lx^j = L\{a_n \prod_{r=0}^{n-1} (j-r) Q_{j-n}(x) + a_{n-1} \prod_{r=0}^{n-2} (j-r) Q_{j-n+1}(x) + a_{n-2} \prod_{r=0}^{n-3} (j-r) Q_{j-n+2}(x) + \cdots + a_1 \prod_{r=0}^{n-n} (j-r) Q_{j-1}(x) + a_0 Q_j(x)\} \tag{24}$$

Here, it is assumed that L^{-1} exists and hence (24) becomes

$$x^j = a_n \prod_{r=0}^{n-1} (j-r) Q_{j-n}(x) + a_{n-1} \prod_{r=0}^{n-2} (j-r) Q_{j-n+1}(x) + a_{n-2} \prod_{r=0}^{n-3} (j-r) Q_{j-n+2}(x) + \cdots + a_1 \prod_{r=0}^{n-n} (j-r) Q_{j-1}(x) + a_0 Q_j(x) \tag{25}$$

Therefore the recurrence relation of (25) is

$$Q_j(x) = \frac{1}{a_0}\{x^j - a_n \prod_{r=0}^{n-1} (j-r) Q_{j-n}(x) - a_{n-1} \prod_{r=0}^{n-2} (j-r) Q_{j-n+1}(x) - a_{n-2} \prod_{r=0}^{n-3} (j-r) Q_{j-n+2}(x) - \cdots - a_1 \prod_{r=0}^{n-n} (j-r) Q_{j-1}(x)\} \tag{26}$$

For $j \geq 0$, $a_0 \neq 0$.
From (26), for case $n = 1$

$$Q_j = \frac{1}{a_0}\{x^j - a_1 j Q_{j-1}\} \tag{27}$$

Few terms of the canonical polynomials for the case $n = 1$ are given as;

$$Q_0 = \frac{1}{a_0}, Q_1 = \frac{x}{a_0} - \frac{a_1}{a_0^2}, Q_2 = \frac{x^2}{a_0} - \frac{2a_1 x}{a_0^2} + \frac{2a_1^2}{a_0^3},$$

$$Q_3 = \frac{x^3}{a_0} - \frac{3a_1 x^2}{a_0^2} + \frac{6a_1^2 x}{a_0^3} - \frac{6a_1^3}{a_0^4} \tag{28}$$

For the case $n = 2$

$$Q_j = \frac{1}{a_0}\{x^j - a_2 j(j-1) Q_{j-2} - a_1 j Q_{j-1}\} \tag{29}$$

Few terms of the canonical polynomial for the case $n = 2$ are given as;

$$Q_0 = \frac{1}{a_0}, Q_1 = \frac{x}{a_0} - \frac{a_1}{a_0^2}, Q_2 = \frac{x^2}{a_0} - \frac{2a_2}{a_0^2} - \frac{2a_1 x}{a_0^2} + \frac{2a_1^2}{a_0^3}, Q_3 = \frac{x^3}{a_0} - \frac{6a_2 x}{a_0^2} + \frac{12a_1 a_2}{a_0^3} -$$

$$\frac{3a_1x^2}{a_0^2}+\frac{6a_1^2x}{a_0^3}-\frac{6a_1^3}{a_0^4}, Q_4=\frac{x^4}{a_0}-\frac{12a_2x^2}{a_0^2}+\frac{24a_2^2}{a_0^3}+\frac{48a_1a_2x}{a_0^3}-\frac{72a_2a_1^2}{a_0^4}-\frac{4a_1x^3}{a_0^2}+\frac{12a_1^2x^2}{a_0^3}-\frac{24a_1^3x}{a_0^4}+\frac{24a_1x^4}{a_0^5} \quad (30)$$

For the case $n=3$,

$$Q_j=\frac{1}{a_0}\{x^j-a_3j(j-1)(j-2)Q_{j-3}-a_2j(j-1)Q_{j-2}-a_1jQ_{j-1}\} \quad (31)$$

Few terms of the canonical polynomial for the case $n=3$ is given as;

$$Q_0=\frac{1}{a_0}Q_1=\frac{x}{a_0}-\frac{a_1}{a_0^2}Q_2=\frac{x^2}{a_0}-\frac{2a_2}{a_0^2}-\frac{2a_1x}{a_0^2}+\frac{2a_1^2}{a_0^3}Q_3=\frac{x^3}{a_0}-\frac{6a_2x}{a_0^2}+\frac{12a_1a_2}{a_0^3}-\frac{3a_1x^2}{a_0^2}+\frac{6a_1^2x}{a_0^3}-\frac{6a_1^3}{a_0^4}Q_4=\frac{x^4}{a_0}-\frac{24a_3x}{a_0^2}+\frac{48a_3a_1}{a_0^3}-\frac{12a_2x^2}{a_0^2}+\frac{24a_2^2}{a_0^3}+\frac{48a_1a_2x}{a_0^3}-\frac{72a_2a_1^2}{a_0^4}-\frac{4a_1x^3}{a_0^2}+\frac{12a_1^2x^2}{a_0^3}-\frac{24a_1^3x}{a_0^4}+\frac{24a_1^4}{a_0^5}$$

(32)

For the case $n=4$

$$Q_j=\frac{1}{a_0}\{x^j-a_4j(j-1)(j-2)(j-3)Q_{j-4}-a_3j(j-1)(j-2)Q_{j-3}-a_2j(j-1)Q_{j-2}-a_1jQ_{j-1}\} \quad (33)$$

Few terms of the canonical polynomial for the case $n=4$ is given as;

$$Q_0=\frac{1}{a_0}Q_1=\frac{x}{a_0}-\frac{a_1}{a_0^2}Q_2=\frac{x^2}{a_0}-\frac{2a_2}{a_0^2}-\frac{2a_1x}{a_0^2}+\frac{2a_1^2}{a_0^3}Q_3=\frac{x^3}{a_0}-\frac{6a_2x}{a_0^2}+\frac{12a_1a_2}{a_0^3}-\frac{3a_1x^2}{a_0^2}+\frac{6a_1^2x}{a_0^3}-\frac{6a_1^3}{a_0^4}Q_4=\frac{x^4}{a_0}-\frac{24a_4}{a_0^2}-\frac{24a_3x}{a_0^2}+\frac{48a_3a_1}{a_0^3}-\frac{12a_2x^2}{a_0^2}+\frac{24a_2^2}{a_0^3}+\frac{48a_1a_2x}{a_0^3}-\frac{72a_2a_1^2}{a_0^4}-\frac{4a_1x^3}{a_0^2}+\frac{12a_1^2x^2}{a_0^3}-\frac{24a_1^3x}{a_0^4}+\frac{24a_1^4}{a_0^5} \quad (34)$$

For the case $n=8$

$$Q_j=\frac{1}{a_0}\{x^j-a_8j(j-1)(j-2)(j-3)(j-4)(j-6)(j-7)Q_{j-8}-a_7j(j-1)(j-2)(j-3)(j-4)(j-5)(j-6)Q_{j-7}-a_6j(j-1)(j-2)(j-3)(j-4)(j-5)Q_{j-6}-a_5j(j-1)(j-2)(j-3)(j-4)Q_{j-5}-a_4j(j-1)(j-2)(j-3)Q_{j-4}-a_3j(j-1)(j-2)Q_{j-3}-a_2j(j-1)Q_{j-2}-a_1jQ_{j-1}\} \quad (35)$$

Few terms of the canonical polynomial for the case $n=8$ is given as;

$$Q_0=\frac{1}{a_0}, Q_1=\frac{x}{a_0}-\frac{a_1}{a_0^2},\ Q_2=\frac{x^2}{a_0}-\frac{2a_2}{a_0^2}-\frac{2a_1x}{a_0^2}+\frac{2a_1^2}{a_0^3}, Q_3=\frac{x^3}{a_0}-\frac{6a_3}{a_0^2}-\frac{6a_2x}{a_0^2}+\frac{12a_1a_2}{a_0^3}-\frac{3a_1x^2}{a_0^2}+\frac{6a_1^2x}{a_0^3}-\frac{6a_1^3}{a_0^4},\ Q_4=\frac{x^4}{a_0}-\frac{24a_4}{a_0^2}-\frac{24a_3x}{a_0^2}+\frac{48a_3a_1}{a_0^3}-\frac{12a_2x^2}{a_0^2}+\frac{24a_2^2}{a_0^3}+\frac{48a_1a_2x}{a_0^3}-\frac{72a_2a_1^2}{a_0^4}-\frac{4a_1x^3}{a_0^2}+\frac{12a_1^2x^2}{a_0^3}-\frac{24a_1^3x}{a_0^4}+\frac{24a_1^4}{a_0^5} \quad (36)$$

2.3 Description and Implementation of the proposed method

Consider the following integro - differential equation

$$y^{(n)}+f(x)y(x)+\int_0^x k(x,t)y(t)dt=g(x), \quad (37)$$

with initial conditions

$$y(0)=\beta_0, y'(0)=\beta_1,\cdots,y^{(n)}(0)=\beta_n \quad (38)$$

Solving (37) by MHPM we construct the following convex homotopy

$$(1-p)(V^{(n)}(x)-y_0(x))+p(V^{(n)(x)}+f(x)V(x)+\int_0^x k(x,t)y(t)dt-g(x))=0 \quad (39)$$

or equivalently

$$V^{(n)}(x)=y_0(x)-p(y_0(x)+f(x)V(x)+\int_0^x k(x,t)y(t)dt-g(x)) \quad (40)$$

Applying the inverse operator, $L^{-1}=\int_0^x\int_0^\eta\cdots\int_0^\tau(\cdot)d\xi\cdots d\mu d\eta$
to both sides of (40) we obtain

$$\begin{aligned}V(x)&=V(0)+xV'(0)+\frac{x^2}{2!}V''(0)+\cdots+\frac{x^n}{n!}V^{(n)}(0)\\&+\int_0^x\int_0^\eta\cdots\int_0^\tau(y_0(\xi)-p(y_0(\xi)+f(\xi)V(\xi)\\&+\int_0^x k(\xi,t)V(t)dt-g(x)\Big)\Big)d\xi\cdots d\mu d\eta,\end{aligned} \quad (41)$$

where $V(0)=\beta_0, V'(0)=\beta_1, V''(0)=\beta_2,\cdots,V^{(n)}(0)=\beta_n$.
Suppose the solution of (41) have the following form

$$V(x)=V_0(x)+pV_1(x)+p^2V_2(x)+\cdots,$$

(42)

where $V_i(x), i=1,2,3,\cdots$ are functions which are to be determined.
Now suppose that the initial approximation to the solutions $y_0(x)$ is of the form

$$y_0(x)=\sum_{j=0}^{\infty}\alpha_jP_j(x) \quad (43)$$

where α_j are unknown coefficient, $P_0(x),P_1(x),P_2(x),\cdots$ are specific function depending on the problem.
Substituting (43) into (42) and equating the corresponding coefficients of p with the same power leads to

$$\begin{aligned}\sum_{n=0}^{\infty}p^nV_n(x)&=V(0)+xV'(0)+\frac{x^2}{2!}V''(0)+\cdots+\frac{x^n}{n!}\mathrm{V}^{(n)}(0)\\&+\int_0^x\int_0^\eta\cdots\int_0^\tau(y_0(\xi)-p(y_0(\xi)+f(\xi)\sum_{n=0}^{\infty}p^nV_n(\xi)\\&+\int_0^x k(\xi,t)\sum_{n=0}^{\infty}p^nV_n(t)dt-g(x)\Big)\Big)d\xi\cdots d\mu d\eta,\end{aligned} \quad (44)$$

$$\begin{aligned}&p^0{:}\,V_0(x)=V(0)+xV'(0)+\frac{x^2}{2!}V''(0)+\cdots+\frac{x^n}{n!}V^{(n)}(0)\\&+\int_0^x\int_0^\eta\cdots\int_0^\tau(y_0(\xi)d\xi\cdots d\mu d\eta,\\&p^1{:}\,V_1(x)=-\int_0^x\int_0^\eta\cdots\int_0^\tau\big(y_0(\xi)+f(\xi)V_0(\xi)+\int_0^x k(\xi,t)V_0(t)dt-\\&g(x)\big)d\xi\cdots d\mu d\eta,\\&p^2{:}\,V_2(x)=-\int_0^x\int_0^\eta\cdots\int_0^\tau\big(f(\xi)V_1(\xi)+\int_0^x k(\xi,t)V_1(\mathrm{t})dt\big)d\xi\cdots d\mu d\eta,\\&p^3{:}\,V_3(x)=-\int_0^x\int_0^\eta\cdots\int_0^\tau\big(f(\xi)V_2(\xi)+\int_0^x k(\xi,t)V_2(t)dt\big)d\xi\cdots d\mu d\eta,\\&p^j{:}\,V_j(x)=-\int_0^x\int_0^\eta\cdots\int_0^\tau\left(f(\xi)V_{j-1}(\xi)+\int_0^x k(\xi,t)V_{j-1}(t)dt\right)d\xi\cdots d\mu d\eta,\\&\vdots\end{aligned} \quad (45)$$

Now if these equations are solved in a way that $V_1(x)=0$, then (63) resulted to $V_2(x)=V_3(x)=\cdots=0$, therefore the solution is obtained by using

$$y(x)=V_0(x)=\beta_0+\beta_1x+\beta_2\frac{x^2}{2!}+\beta_3\frac{x^3}{3!}+\beta_2\frac{x^n}{n!}+\int_0^x\int_0^\eta\cdots\int_0^\tau y_0\xi d\xi\cdots d\mu d\eta \quad (46)$$

It is worthwhile to note that if $V_1(x)$ is analytic at $x=x_0$, then their Taylor series

$$V_1(x)=\sum_{n=0}^{\infty}(x-x_0)^n, \quad (47)$$

is used in (46) where $\beta_0,\beta_1,\beta_2,\cdots$ are known coefficients and $\alpha_0,\alpha_1\alpha_2,\alpha_3,\cdots,$ are unknown to be determined.

3 Demonstation of proposed method on some problems

In order to demonstrate the efficiency and applicability of the new method proposed, the following examples were considered.

Numerical Example 1

Consider the following integro-differential equation

$$y^{iv}(x) - y(x) = x(1 + e^x) + 3e^x - \int_0^x y(t)dt, \tag{48}$$

with the initial conditions

$$y(0) = 1, y'(0) = 1, y''(0) = 2, y'''(0) = 3,$$

The exact solution is $y(x) = 1 + e^x$.

For solving this equation by MHPM we construct the following convex homotopy

$$(1 - p)(V^{iv}(x) - y_0(x)) + p\left(V^{iv}(x) - V(x) - (x + xe^x + 3e^x) + \int_0^x V(t)dt\right) = 0 \tag{49}$$

we therefore simplify (49) to have

$$V^{iv}(x) = y_0(x) - py_0(x) + pV(x) + p(x + xe^x + 3e^x) - p\int_0^x V(t)dt \tag{50}$$

By integrating (50) we have

$$V(x) = V(0) + xV'(0) + \frac{x^2}{2}V''(0) + \frac{x^3}{6}V'''(0) + \int_0^x \int_0^\tau \int_0^\eta \int_0^\phi y_0(\xi)d\xi d\phi d\eta d\tau$$
$$-p\int_0^x \int_0^\tau \int_0^\eta \int_0^\phi y_0(\xi)d\xi d\phi d\eta d\tau + p\int_0^x \int_0^\tau \int_0^\eta \int_0^\phi V(\xi)d\xi d\phi d\eta d\tau$$
$$+p\int_0^x \int_0^\tau \int_0^\eta \int_0^\phi (\xi + \xi e^\xi + 3e^\xi)d\xi d\phi d\eta d\tau$$
$$-p\int_0^x \int_0^\tau \int_0^\eta \int_0^\phi \int_0^\xi V(t)dtd\xi d\phi d\eta d\tau \tag{51}$$

where

$$V(0) = 1, V'(0) = 1, V''(0) = 2, V'''(0) = 3,$$

Now suppose that the solution of (51) is in the following form

$$V(x) = V_0(x) + pV_1(x) + p^2V_2(x) + p^3V_3(x) + \cdots \tag{52}$$

Substituting (52) into (51) to have

$$\sum_{n=0}^{\infty} p^nV_n(x) = 1 + x + x^2 + \frac{x^3}{2} + \int_0^x \int_0^\tau \int_0^\eta \int_0^\phi y_0(\xi)d\xi d\phi d\eta d\tau$$
$$-p\int_0^x \int_0^\tau \int_0^\eta \int_0^\phi y_0(\xi)d\xi d\phi d\eta d\tau$$
$$+p\int_0^x \int_0^\tau \int_0^\eta \int_0^\phi \sum_{n=0}^{\infty} p^nV_n(\xi)d\xi d\phi d\eta d\tau$$
$$+p\int_0^x \int_0^\tau \int_0^\eta \int_0^\phi (\xi + \xi e^\xi + 3e^\xi)d\xi d\phi d\eta d\tau$$
$$-p\int_0^x \int_0^\tau \int_0^\eta \int_0^\phi \int_0^\xi \sum_{n=0}^{\infty} p^nV_n(t)dtd\xi d\phi d\eta d\tau \tag{53}$$

Equating the coefficient of p in (53) with the same power leads to

$$p^0: V_0(x) = 1 + x + x^2 + \frac{x^3}{2} + \int_0^x \int_0^\tau \int_0^\eta \int_0^\phi y_0(\xi)d\xi d\phi d\eta d\tau \tag{54}$$

$$p^1: V_1(x) = -\int_0^x \int_0^\tau \int_0^\eta \int_0^\phi y_0(\xi)d\xi d\phi d\eta d\tau$$
$$+\int_0^x \int_0^\tau \int_0^\eta \int_0^\phi \sum_{n=0}^{\infty} V_0(\xi)d\xi d\phi d\eta d\tau$$
$$+\int_0^x \int_0^\tau \int_0^\eta \int_0^\phi (\xi + \xi e^\xi + 3e^\xi)d\xi d\phi d\eta d\tau \tag{55}$$
$$-\int_0^x \int_0^\tau \int_0^\eta \int_0^\phi \int_0^\xi V_0(t)d\xi d\phi d\eta d\tau$$

$$p^j: V_j(\xi) = \int_0^x \int_0^\tau \int_0^\eta \int_0^\phi V_{j-1}(\xi) - \int_0^x \int_0^\tau \int_0^\eta \int_0^\phi \int_0^\xi V_{j-1}(t)dtd\xi d\phi d\eta d\tau$$

Now assume that

$$y_0(x) = \sum_{n=0}^{\infty} \alpha_nQ_n(x),$$

where $Q_k(x)$ is the Canonical polynomials generated in (34) for the case of $n = 4$. We have

$$y_0(x) = \alpha_0 Q_0(x) + \alpha_1 Q_1(x) + \alpha_2 Q_2(x) + \alpha_3 Q_3(x) + \alpha_4 Q_4(x) + \cdots \quad (56)$$

Now comparing (48) with (16) we have $a_0 = -1$, $a_1 = a_2 = a_3 = 0$ and $a_4 = 1$.

Therefore

$$\begin{cases} Q_0(x) = -1 \\ Q_1(x) = -x \\ Q_2(x) = -x^2 \\ Q_3(x) = -x^3 \\ Q_4(x) = -x^4 - 24 \end{cases} \quad (57)$$

Then,

$$y_0(\mathrm{x}) = -\alpha_0 - \alpha_1 x - \alpha_2 x^2 - \alpha_3 x^3 - \alpha_4(x^4 + 24) \quad (58)$$

Substituting (58) into (54) and solve results in

$$V_0(x) = 1 + x + x^2 + \frac{1}{2}x^3 - \frac{\alpha_4 x^8}{1680} - \frac{\alpha_3 x^7}{840} - \frac{\alpha_2 x^6}{360} - \frac{\alpha_1 x^5}{120} + \frac{1}{4}\left(-\frac{1}{6}\alpha_0 - 4\,\alpha_4\right)x^4 \quad (59)$$

Solving for $V_1(x)$ such that $V_0(x) = 0$ in (54), we have the Taylor series as

$$V_1(x) = \left(\frac{1}{24}\alpha_0 + \alpha_4 + \frac{1}{6}\right)x^4 + \left(\frac{1}{24} + \frac{\alpha_1}{120}\right)x^5 + \left(\frac{1}{120} + \frac{\alpha_2}{360}\right)x^6 + \left(\frac{1}{720} + \frac{\alpha_3}{840}\right)x^7 + \left(-\frac{\alpha_0}{40320} + \frac{1}{10080}\right)x^8 + \cdots = 0 \quad (60)$$

So, to find α_0,α_1, α_2, α_3 and α_4 in (60), we equate to zero, the coefficients of various power of x to have

$$\begin{cases} \left(\frac{1}{24}\alpha_0 + \alpha_4 + \frac{1}{6}\right) = 0 \\ \left(\frac{1}{24} + \frac{\alpha_1}{120}\right) = 0 \\ \left(\frac{1}{120} + \frac{\alpha_2}{360}\right) = 0 \\ \left(\frac{1}{720} + \frac{\alpha_3}{840}\right) = 0 \\ \left(-\frac{\alpha_0}{40320} + \frac{1}{10080}\right) = 0 \end{cases} \quad (61)$$

Solving the above simultaneous equations using $MAPLE18$, we obtain

$$\alpha_0 = 4, \alpha_1 = -5, \alpha_2 = -3, \alpha_3 = -\frac{7}{6}, \alpha_4 = -\frac{1}{3}, \quad (62)$$

we substitute (62), into $V_0(x)$ in (59).

Therefore the exact solution of integro-differential equation (48) can be expressed as

$$\begin{aligned} y(x) = V_0(x) &= 1 + x + x^2 + \frac{1}{2}x^3 - \frac{\alpha_4 x^8}{1680} - \frac{\alpha_3 x^7}{840} - \frac{\alpha_2 x^6}{360} - \frac{\alpha_1 x^5}{120} + \frac{1}{4}\left(-\frac{1}{6}\alpha_0 - 4\,\alpha_4\right)x^4 \\ &= 1 + x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5 + \frac{x^6}{120} + \frac{x^7}{720} + \frac{x^8}{5040} \\ &= 1 + x\left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7 + \cdots\right) \\ &= \left(1 + x\sum_{n=0}^{\infty}\frac{x^n}{n!}\right) \\ &= 1 + xe^x \end{aligned}$$

Numerical Example 2

Consider the Fredholm integro-differential equation

$$y^8(x) - y(x) = -8e^x + x^2 + \int_0^1 x^2 y'(t)dt, \quad (63)$$

with the initial conditions

$$y(0) = 1, y'(0) = 0, y''(0) = -1, y'''(0) = -2,$$
$$y^{iv}(0) = -3, y^{v}(0) = -4, y^{vi}(0) = -5, y^{vii}(0) = -6,$$

The exact solution is $y(x) = (1-x)e^{x}$.

For solving this equation by MHPM we construct the following convex homotopy

$$(1-p)(V^8(x) - y_0(x)) + p\left(V^8(x) - V(x) - (x^2 - 8e^x) - \int_0^1 x^2 V'(t)dt\right) = 0 \quad (64)$$

We therefore simplify (64) to have

$$V^8(x) = y_0(x) - py_0(x) + pV(x) + p(x^2 - 8e^x) + p\int_0^1 x^2 V'(t)dt \quad (65)$$

By integrating (65) we have

$$V(x) = V(0) + xV'(0) + \frac{x^2}{2}V''(0) + \frac{x^3}{6}V'''(0) + \frac{x^4}{24}V^{iv}(0) + \frac{x^5}{120}V^{v}(0) + \frac{x^6}{720}V^{vi}(0) + \frac{x^7}{5040}V^{vii}(0) + \int_0^x\int_0^\tau\int_0^\eta\int_0^\phi\int_0^\omega\int_0^\sigma\int_0^\psi\int_0^\rho y_0(\xi)d\xi d\rho d\phi d\sigma d\omega d\phi d\eta d\tau$$
$$-p\int_0^x\int_0^\tau\int_0^\eta\int_0^\phi\int_0^\omega\int_0^\sigma\int_0^\psi\int_0^\rho y_0(\xi)d\xi d\rho d\phi d\sigma d\omega d\phi d\eta d\tau$$
$$+p\int_0^x\int_0^\tau\int_0^\eta\int_0^\phi\int_0^\omega\int_0^\sigma\int_0^\psi\int_0^\rho V(\xi)d\xi d\rho d\phi d\sigma d\omega d\phi d\eta d\tau \quad (66)$$
$$+p\int_0^x\int_0^\tau\int_0^\eta\int_0^\phi\int_0^\omega\int_0^\sigma\int_0^\psi\int_0^\rho (\xi^2 - 8e^\xi)d\xi d\rho d\phi d\sigma d\omega d\phi d\eta d\tau$$
$$+p\int_0^x\int_0^\tau\int_0^\eta\int_0^\phi\int_0^\omega\int_0^\sigma\int_0^\psi\int_0^\rho\int_0^\xi \xi^2 V'(t)dt d\xi d\rho d\phi d\sigma d\omega d\phi d\eta d\tau$$

where

$$V(0) = 1, V'(0) = 0, V''(0) = -1, V'''(0) = -2,$$
$$V^{iv}(0) = -3, V^{v}(0) = -4, V^{vi}(0) = -5, V^{vii}(0) = -6,$$

Now suppose that the solution of (66) is in the following form

$$V(x) = V_0(x) + pV_1(x) + p^2V_2(x) + p^3V_3(x) + \cdots \quad (67)$$

Substituting (67) into (66) to have

$$\sum_{n=0}^{\infty} p^n V_n(x) = V(0) + xV'(0) + \frac{x^2}{2}V''(0) + \frac{x^3}{6}V'''(0) + \frac{x^4}{24}V^{iv}(0) + \frac{x^5}{120}V^{v}(0) + \frac{x^6}{720}V^{vi}(0) + \frac{x^7}{5040}V^{vii}(0)$$
$$+\int_0^x\int_0^\tau\int_0^\eta\int_0^\phi\int_0^\omega\int_0^\sigma\int_0^\psi\int_0^\rho y_0(\xi)d\xi d\rho d\phi d\sigma d\omega d\phi d\eta d\tau$$
$$-p\int_0^x\int_0^\tau\int_0^\eta\int_0^\phi\int_0^\omega\int_0^\sigma\int_0^\psi\int_0^\rho y_0(x)d\xi d\rho d\phi d\sigma d\omega d\phi d\eta d\tau$$
$$+p\int_0^x\int_0^\tau\int_0^\eta\int_0^\phi\int_0^\omega\int_0^\sigma\int_0^\psi\int_0^\rho \sum_{n=0}^{\infty} p^n V_n(\xi)d\xi d\rho d\phi d\sigma d\omega d\phi d\eta d\tau$$
$$+p\int_0^x\int_0^\tau\int_0^\eta\int_0^\phi\int_0^\omega\int_0^\sigma\int_0^\psi\int_0^\rho (\xi^2 - 8e^\xi)d\xi d\rho d\phi d\sigma d\omega d\phi d\eta d\tau$$

$$+p\int_0^x\int_0^\tau\int_0^\eta\int_0^\phi\int_0^\omega\int_0^\sigma\int_0^\psi\int_0^\rho\left(\int_0^\xi \sum_{n=0}^{\infty} \xi^2 p^n V'_n(t)dt\right)d\xi d\rho d\phi d\sigma d\omega d\phi d\eta d\tau \quad (68)$$

Equating the coefficient of p in (68) with the same power of x leads to

$$p^0: V_0(x) = V(0) + xV'(0) + \frac{x^2}{2}V''(0) + \frac{x^3}{6}V'''(0) + \frac{x^4}{24}V^{iv}(0) + \frac{x^5}{120}V^{v}(0) + \frac{x^6}{720}V^{vi}(0) + \frac{x^7}{5040}V^{vii}(0) + \int_0^x\int_0^\tau\int_0^\eta\int_0^\phi\int_0^\omega\int_0^\sigma\int_0^\psi\int_0^\rho y_0(\xi)d\xi d\rho d\phi d\sigma d\omega d\phi dr$$

$$p^1: V_1(x) = -\int_0^x\int_0^\tau\int_0^\eta\int_0^\phi\int_0^\omega\int_0^\sigma\int_0^\psi\int_0^\rho y_0(\xi)d\xi d\rho d\phi d\sigma d\omega d\phi d\eta d\tau$$
$$+\int_0^x\int_0^\tau\int_0^\eta\int_0^\phi\int_0^\omega\int_0^\sigma\int_0^\psi\int_0^\rho V_0(\xi)d\xi d\rho d\phi d\sigma d\omega d\phi d\eta d\tau$$

$$+\int_0^x\int_0^\tau\int_0^\eta\int_0^\phi\int_0^\omega\int_0^\sigma\int_0^\psi\int_0^\rho(\xi^2-8e^\xi)d\xi d\rho d\phi d\sigma d\omega d\phi d\eta d\tau$$
$$+\int_0^x\int_0^\tau\int_0^\eta\int_0^\phi\int_0^\omega\int_0^\sigma\int_0^\psi\int_0^\rho\left(\int_0^\xi \xi^2V'_0(t)dt\right)d\xi d\rho d\phi d\sigma d\omega d\phi d\eta d\tau \quad (70)$$

$$p^j: V_j(x)=\int_0^x\int_0^\tau\int_0^\eta\int_0^\phi\int_0^\omega\int_0^\sigma\int_0^\psi\int_0^\rho V_{j-1}(\xi)d\xi d\rho d\phi d\sigma d\omega d\phi d\eta d\tau$$
$$+\int_0^x\int_0^\tau\int_0^\eta\int_0^\phi\int_0^\omega\int_0^\sigma\int_0^\psi\int_0^\rho\left(\int_0^\xi x^2V'_{j-1}(t)dt\right)d\xi d\rho d\phi d\sigma d\omega d\phi d\eta d\tau$$

Now assume that

$$y_0(x)=\sum_{n=0}^{\infty}\alpha_nQ_n(x),$$

where $Q_k(x)$ is the Canonical polynomials generated in (36) for the case of $n=8$. We have

$$y_0(x)=\alpha_0Q_0(x)+\alpha_1Q_1(x)+\alpha_2Q_2(x)+\alpha_3Q_3(x)+\alpha_4Q_4(x)+\alpha_5Q_5(x)+$$
$$\alpha_6Q_6(x)+\alpha_7Q_7(x)+\alpha_8Q_8(x)+\cdots \quad (71)$$

Now comparing (63) with (16) we have $a_0=-1$, $a_1=a_2=a_3=\cdots=a_7=0$ and $a_8=1$. Therefore

$$\left(\begin{array}{l} Q_0(x)=-1\\ Q_1(x)=-x\\ Q_2(x)=-x^2\\ Q_3(x)=-x^3\\ Q_4(x)=-x^4\\ Q_5(x)=-x^5\\ Q_6(x)=-x^6\\ Q_7(x)=-x^7\\ Q_8(x)=-x^8-40320 \end{array}\right. \quad (72)$$

Then,

$$y_0(x)=-\alpha_0-\alpha_1x-\alpha_2x^2-\alpha_3x^3-\alpha_4-\alpha_5x^5-\alpha_6x^6-\alpha_7x^7-\alpha_8(x^8+$$
$$0320) \quad (73)$$

Substituting (73) into (69),then imposing the conditions and solve results in

$$V_0(x)=1-\frac{1}{2}x^2-\frac{1}{3}x^3-\frac{1}{8}x^4-\frac{1}{30}x^5-\frac{x^6}{144}-\frac{x^7}{840}$$

$$+\frac{1}{8}\left(-\frac{\alpha_0}{5040}-8\,\alpha_8\right)x^8-\frac{\alpha_1x^9}{362880}-\frac{\alpha_2x^{10}}{1814400}-\frac{\alpha_3x^{11}}{6652800}-\frac{\alpha_4x^{12}}{19958400}-\frac{\alpha_5x^{13}}{51891840}$$
$$-\frac{\alpha_6x^{14}}{121080960}-\frac{\alpha_7x^{15}}{259459200}-\frac{\alpha_8x^{16}}{518918400}+\cdots \quad (74)$$

Solving for $V_1(x)$ such that $V_1(x)=0$ in (70), we have the Taylor series as

$$V_1(x)=\left(\frac{\alpha_0}{40320}+\alpha_8-\frac{1}{5760}\right)x^8+\left(-\frac{1}{45360}+\frac{\alpha_1}{362880}\right)x^9+\left(-\frac{\alpha_0}{73156608000}\right.$$
$$-\frac{\alpha_1}{658409472000}-\frac{22679}{9144576000}+\frac{1814399\,\alpha_2}{3292047360000}-\frac{\alpha_3}{12070840320000}$$
$$-\frac{\alpha_4}{36212520960000}-\frac{\alpha_5}{94152554496000}-\frac{\alpha_6}{219689293824000}-\frac{518918401\,\alpha_8}{941525544960000}$$
$$\left.-\frac{\alpha_7}{470762772480000}\right)x^{10}$$
$$+\left(-\frac{1}{3991680}+\frac{\alpha_3}{6652800}\right)x^{11}+\left(-\frac{1}{43545600}+\frac{\alpha_4}{19958400}\right)x^{12} \quad (75)$$
$$+\left(-\frac{1}{518918400}+\frac{\alpha_5}{51891840}\right)x^{13}+\left(-\frac{1}{6706022400}+\frac{\alpha_6}{121080960}\right)x^{14}$$

$$+\left(-\frac{1}{93405312000}+\frac{\alpha_7}{259459200}\right)x^{15}+\left(-\frac{\alpha_0}{20922789888000}-\frac{1}{2615348736000}\right)x^{16}+\cdots$$
$$=0$$

So, to find the value of $\alpha's$ in (75), we equate to zero, the coefficients of various power of x to have

$$\left\{\begin{array}{l}
\left(\frac{\alpha_0}{40320}+\alpha_8-\frac{1}{5760}\right)=0\\
\left(-\frac{1}{45360}+\frac{\alpha_1}{362880}\right)=0\\
\left(-\frac{\alpha_0}{73156608000}-\frac{\alpha_1}{658409472000}-\frac{22679}{9144576000}+\frac{1814399\,\alpha_2}{3292047360000}-\frac{\alpha_3}{12070840320000}-\frac{\alpha_4}{36212520960000}-\frac{\alpha_5}{94152554496000}-\frac{\alpha_6}{219689293824000}-\frac{518918401\,\alpha_8}{941525544960000}-\frac{\alpha_7}{470762772480000}\right)=0\\
\left(-\frac{1}{3991680}+\frac{\alpha_3}{6652800}\right)=0\\
\left(-\frac{1}{43545600}+\frac{\alpha_4}{19958400}\right)=0\\
\left(-\frac{1}{518918400}+\frac{\alpha_5}{51891840}\right)=0\\
\left(-\frac{1}{6706022400}+\frac{\alpha_6}{121080960}\right)=0\\
\left(-\frac{1}{518918400}+\frac{\alpha_5}{51891840}\right)x^{13}+\left(-\frac{1}{6706022400}+\frac{\alpha_6}{121080960}\right)x^{14}=0\\
\left(-\frac{1}{93405312000}+\frac{\alpha_7}{259459200}\right)=0\\
\left(-\frac{\alpha_0}{20922789888000}-\frac{1}{2615348736000}\right)=0
\end{array}\right. \tag{76}$$

Solving the above simultaneous equations using $MAPLE18$, we obtain

$$\alpha_0=-8,\ \alpha_1=8,\ \alpha_2=\frac{9}{2},\ \alpha_3=\frac{5}{3},\alpha_4=\frac{11}{24},\ \alpha_5=\frac{1}{10},\ \alpha_6=\frac{13}{720},\ \alpha_7=\frac{1}{360},\ \alpha_8=\frac{1}{2688} \tag{77}$$

We substituted (77) into $V_0(x)$ in (74).

Therefore the exact solution of integro-differential equation (63) can be expressed as

$$y(x)=V_0(x)=1-\frac{1}{2}x^2-\frac{1}{3}x^3-\frac{1}{8}x^4-\frac{1}{30}x^5-\frac{x^6}{144}-\frac{x^7}{840}+\frac{1}{8}\left(-\frac{\alpha_0}{5040}-8\,\alpha_8\right)x^8-\frac{\alpha_1x^9}{362880}-\frac{\alpha_2x^{10}}{1814400}-\frac{\alpha_3x^{11}}{6652800}-\frac{\alpha_4x^{12}}{19958400}-\frac{\alpha_5x^{13}}{51891840}-\frac{\alpha_6x^{14}}{121080960}-\frac{\alpha_7x^{15}}{259459200}-\frac{\alpha_8x^{16}}{518918400}-\cdots$$

$$=1-\frac{1}{2}x^2-\frac{1}{3}x^3-\frac{1}{8}x^4-\frac{1}{30}x^5-\frac{x^6}{144}-\frac{x^7}{840}-\frac{x^8}{5760}-\frac{x^9}{45360}-\frac{x^{10}}{403200}-\frac{x^{11}}{3991680}-\frac{x^{12}}{43545600}-\frac{x^{13}}{518918400}-\frac{x^{14}}{6706022400}-\frac{x^{15}}{93405312000}-\frac{x^{16}}{1394852659200}-\cdots \tag{78}$$

which is the Taylor series of the exact solution

Numerical Example 3

Consider the mixed Volterra-Fredholm integro-differential equation

$$u''(x) = -\frac{20}{3} + \frac{2}{3}x^3 + \int_0^x \int_{-1}^1 (rt^2 - r^2t)u(t)dtdr, \quad (79)$$

with the initial conditions

$u(0) = 2, u'(0) = 3,$

The exact solution is given by $2 + 3x - \frac{10}{3}x^2$.

For solving this equation by MHPM we construct the following convex homotopy

$$(1-p)(V''(x) - u_0(x)) + p\begin{pmatrix} V''(x) + \left(\frac{20}{3} - \frac{2}{3}x^3\right) \\ -\int_0^x \int_{-1}^1 (rt^2 - r^2t)V(t)dtdr \end{pmatrix} = 0 \quad (80)$$

We therefore proceed as before to obtain

$$V(x) = V(0) + xV'(0) + \int_0^x \int_0^\tau u_0(\xi)d\xi d\tau - p\int_0^x \int_0^\tau u_0(\xi)d\xi d\tau$$
$$-p\int_0^x \int_0^\tau \left(\frac{20}{3} - \frac{2}{3}\xi^3\right)d\xi d\tau + p\int_0^x \int_0^\tau \left(\int_0^\xi \int_{-1}^1 (rt^2 - r^2t)V(t)dtdr\right)d\xi d\tau \quad (81)$$

where $V(0) = 2$ and $V'(0) = 3$.

Now suppose that the solution of (81) is in the following form

$$V(x) = V_0(x) + pV_1(x) + p^2V_2(x) + p^3V_3(x) + \cdots \quad (82)$$

Therefore (81) becomes

$$\sum_{n=0}^\infty p^n V_n(x) = 2 + 3x + \int_0^x \int_0^\tau u_0(\xi)d\xi d\tau - p\int_0^x \int_0^\tau u_0(\xi)d\xi d\tau -$$
$$p\int_0^x \int_0^\tau \left(\frac{20}{3} - \frac{2}{3}\xi^3\right)d\xi d\tau + p\int_0^x \int_0^\tau \left(\int_0^\xi \int_{-1}^1 (rt^2 - r^2t)\sum_{n=0}^\infty p^n V_n(t)dtdr\right)d\xi d\tau \quad (83)$$

Now, equating the coefficient of various power of p leads to

$$p^0: V_0(x) = 2 + 3x + \int_0^x \int_0^\tau u_0(\xi)d\xi d\tau \quad (84)$$
$$p^1: V_1(x) = -\int_0^x \int_0^\tau u_0(\xi)d\xi d\tau - \int_0^x \int_0^\tau \left(\frac{20}{3} - \frac{2}{3}\xi^3\right)d\xi d\tau$$
$$+\int_0^x \int_0^\tau \left(\int_0^\xi \int_{-1}^1 (rt^2 - r^2t)\sum_{n=0}^\infty V_0(t)dtdr\right)d\xi d\tau \quad (85)$$
$$p^n: V_n(x) = \int_0^x \int_0^\tau \left(\int_0^\xi \int_{-1}^1 (rt^2 - r^2t)\sum_{n=0}^\infty V_{n-1}(t)dtdr\right)d\xi d\tau$$

Now assume that

$$y_0(x) = \sum_{n=0}^\infty \alpha_n Q_n(x),$$

where $Q_k(x)$ is the Canonical polynomials generated in (36) . We have

$$y_0(x) = \alpha_0 Q_0(x) + \alpha_1 Q_1(x) + \alpha_2 Q_2(x) + \alpha_3 Q_3(x) + \alpha_4 Q_4(x) + \cdots \quad (86)$$

Now comparing (79) with (16) we therefore re-structure (79) into the form (16) to have

$$u''(x) + u(x) = u(x) - \frac{20}{3} + \frac{2}{3}x^3 + \int_0^x \int_{-1}^1 (rt^2 - r^2t)u(t)dtdr, \quad (87)$$

then, we have

$a_2 = 1$, $a_1 = 0$ and $a_0 = 1$.

also

$$\begin{cases} Q_0(x) = 1 \\ Q_1(x) = x \\ Q_2(x) = x^2 - 2 \\ Q_3(x) = x^3 - 6x \\ Q_4(x) = x^4 - 12x^2 + 24 \end{cases} \quad (88)$$

therefore,

$$u_0(\xi) = \alpha_0 + \alpha_1\xi + \alpha_2(\xi^2 - 2) + \alpha_3(\xi^3 - 6\,\xi) + \alpha_4(\xi^4 - 12\,\xi^2 + 24) \quad (89)$$

Substituting (89) into (84) and solve results in

$$V_0(x) = 2 + 3\,x + \frac{1}{30}\alpha_4 x^6 + \frac{1}{20}\alpha_3 x^5 + \frac{1}{4}\left(\frac{1}{3}\alpha_2 - 4\,\alpha_4\right)x^4 + \frac{1}{3}\left(\frac{1}{2}\alpha_1 - 3\,\alpha_3\right)x^3 + \frac{1}{2}(\alpha_0 - 2\,\alpha_2 + 24\,\alpha_4)x^2 \quad (90)$$

Solving for $V_1(x)$ such that $V_1(x) = 0$ we have the

$$V_1(x) = \left(-\frac{1}{2}\alpha_0 + \alpha_2 - 12\,\alpha_4 - \frac{10}{3}\right)x^2 + \left(-\frac{1}{6}\alpha_1 + \alpha_3\right)x^3 + \left(-\frac{499\,\alpha_2}{5040} + \frac{26953\,\alpha_4}{22680} + \frac{\alpha_0}{120} + \frac{1}{18}\right)x^4 + \left(-\frac{61\,\alpha_3}{1400} - \frac{\alpha_1}{900}\right)x^5 - \frac{1}{30}\alpha_4 x^6 + \cdots = 0 \quad (91)$$

So, to find the value of $\alpha's$ in (91), we equate to zero, the coefficients of various power of x to have

$$\begin{cases} \left(-\frac{1}{2}\alpha_0 + \alpha_2 - 12\,\alpha_4 - \frac{10}{3}\right) = 0 \\ \left(-\frac{1}{6}\alpha_1 + \alpha_3\right) = 0 \\ \left(-\frac{499\,\alpha_2}{5040} + \frac{26953\,\alpha_4}{22680} + \frac{\alpha_0}{120} + \frac{1}{18}\right) = 0 \\ \left(-\frac{61\,\alpha_3}{1400} - \frac{\alpha_1}{900}\right) = 0 \\ -\frac{1}{30}\alpha_4 = 0 \end{cases} \quad (92)$$

Solving the above simultaneous equations using $MAPLE18$, we obtain

$$\alpha_0 = -\frac{20}{3},\ \alpha_1 = 0,\ \alpha_2 = 0,\ \alpha_3 = 0, \alpha_4 = 0, \cdots \quad (93)$$

We substituted (93) into $V_0(x)$ in (90).
Therefore the exact solution of integro-differential equation (79) can be expressed as

$$u(x) = V_0(x) = 2 + 3\,x + \frac{1}{30}\alpha_4 x^6 + \frac{1}{20}\alpha_3 x^5 + \frac{1}{4}\left(\frac{1}{3}\alpha_2 - 4\,\alpha_4\right)x^4 + \frac{1}{3}\left(\frac{1}{2}\alpha_1 - 3\,\alpha_3\right)x^3 + \frac{1}{2}(\alpha_0 - 2\,\alpha_2 + 24\,\alpha_4)x^2 \quad (94)$$

$$= 2 + 3\,x - \frac{10}{3}x^2 \quad (95)$$

4. Conclusion

The MHPM has been presented and illustrated to solve high-order IDEs. The advantage of this method is that only once iteration is required to get the exact solution of a given problem. Canonical polynomial is used as the basis function for the solution of example 1, 2 and 3. Both methods performed creditably well for the examples considered. Example1 and 2 shows that high-order IDEs with initial conditions can be solved without linearization or discretization.

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