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A HYBRID NUMERICAL METHOD WITH GREATER EFFICIENCY FOR SOLVING INITIAL VALUE PROBLEMS

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ABSTRACT

The main focus of this paper is to develop hybrid numerical method with greater efficiency for getting the solution of initial value problems (IVPs) in ordinary differential equations (ODEs) by merging the slopes used in Modified Euler's method (MEM), Improved Euler's method and a 2^{nd} stage 2^{nd} order contra harmonic mean method. Developed method has tested and analyzed for the stability, consistency and accuracy and observed that the developed method is more stable, consistent and accurate as compared to modified Euler's method (MEM), Improved Euler's method(IEM), modified improved modified Euler's method (MIME) and a 2nd stage 2^{nd} order contra harmonic mean method (C_0M).

Keywords: A hybrid numerical method, Ordinary differential equations, Initial value problems, Stability and consistency

1. INTRODUCTION

Many researchers are busy for solving different physical problems of real world. They have generated different theories and laws relating to these physical problems and nature. Mostly, these laws occur in the differential equations' form and these differential equations are categories in ordinary differential equations (ODEs) and partial differential equations (PDEs) [1-2]. If any change occurs in any physical problem corresponding to any single parameter, there arise ordinary differential equations (ODEs) and if any change occurs in any physical problem corresponding to more than two parameters then, partial differential equations arise there. The differential equations play a major role in the history of Mathematics to solve scientific problems. In science and engineering, there are a lot of physical problems which occur in the differential equations' form [3]. The differential equations are also widely used in field of Physics, Chemistry, Biology and economics etc. [3]. There are various analytical methods are available for getting the solution of differential equations. But at some situations; analytical methods do not able to give the solution of some complicated or complex differential equations. For getting the solution of complex differential equation, numerical methods are employed [4-5]. Numerical methods are very important tools for getting the solution of complex problems very quickly with the help of computer programming.

Many researchers had developed numerous numerical techniques for getting the solution of ordinary differential equations (ODEs) which are in the type of initial value problems (IVPs). These techniques are generated by many researchers from different point of views. Some of them, tried to improve these methods for accuracy, some of them; they have modified these methods for the better accuracy, stability and consistency [6]. From time to time some changes have been made in numerical methods for getting better performance according to our needs. In this research paper, a hybrid numerical method has been introduced for solving the ordinary differential equations (ODEs); which are in the nature of initial value problems (IVPs).That shows the better performance as compared to other second order well-known methods present in the literature.

2. RESEARCH METHODOLOGY

Consider the initial value problems (IVPs) as

$$\frac{dy}{dx} = f(x, y) \;;\; y(x_0) = y_0 \tag{1}$$

For solving eq. (1), the most simplest and old method is explicit forward Euler's method of 1st order; which is given by equation (2).

$$y_{s+1} = y_s + f(\mathrm{x}_s, \mathrm{y}_s)\mathrm{h} \tag{2}$$

Another implicit backward Euler's method is

$$y_{s+1} = y_s + h\; f(\mathrm{x}_{s+1}, \mathrm{y}_{s+1}) \tag{3}$$

Eq. (3) can be written as

$$y_{s+1} = y_s + h\,.\,f(\mathrm{x}_s + \mathrm{h}, \mathrm{y}_s + \mathrm{h}\,.\,\mathrm{f}(\mathrm{x}_s, \mathrm{y}_s)) \tag{4}$$

Modified and Improved Euler's methods (MEM and IEM) are given by eq. (5) and eq. (8) respectively [3].

$$y_{s+1} = y_s + h\,.\,f(\mathrm{x}_s + \frac{1}{2}.\mathrm{h}, \mathrm{y}_s + \frac{1}{2}\mathrm{h}.\mathrm{f}(\mathrm{x}_s, \mathrm{y}_s)) \tag{5}$$

Eq. (5) can be written as

$$y_{s+1} = y_s + hK \tag{6}$$

$$\text{Where} \quad K = f(\mathrm{x}_s + \frac{1}{2}.h, \mathrm{y}_s + \frac{1}{2}h.\mathrm{f}(\mathrm{x}_s, \mathrm{y}_s)) \tag{7}$$

$$y_{s+1} = y_s + \frac{1}{2}h.[\,f(\mathrm{x}_s, \mathrm{y}_s) + f(\mathrm{x}_s + h, \mathrm{y}_s + \mathrm{h}.f(\mathrm{x}_s, \mathrm{y}_s))] \tag{8}$$

In [7], a second stage of second order contra harmonic mean method (C_0M) is

$$y_{s+1} = y_s + \left[\frac{K_1^2 + K_2^2}{K_1 + K_2}\right]h \tag{9}$$

Where $\mathrm{K}_1 = f(\mathrm{x}_s, \mathrm{y}_s)$; $K_2 = f(\mathrm{x}_s + \mathrm{h}, \mathrm{y}_s + h \times f(\mathrm{x}_s, \mathrm{y}_s))$

Eq. (9) can be written as

$$y_{s+1} = y_s + hR \tag{10}$$

Where,

$$R = \left[\frac{K_1^2 + K_2^2}{K_1 + K_2}\right] \tag{11}$$

Now, replace $f(\mathrm{x}_s, \mathrm{y}_s)$ and $f(\mathrm{x}_s + \mathrm{h}, \mathrm{y}_s + h \times f(\mathrm{x}_s, \mathrm{y}_s))$ of eq. (8) by eq. (7) and eq. (11) respectively. We get,

$$y_{s+1} = y_s + \frac{1}{2}.h\left[\,f(\mathrm{x}_s + \frac{1}{2} \times h, \mathrm{y}_s + \frac{1}{2} \times h \times \mathrm{f}(\mathrm{x}_s, \mathrm{y}_s)) + \left[\frac{K_1^2 + K_2^2}{K_1 + K_2}\right]\right] \tag{12}$$

Then Eq. (12) becomes

$$y_{s+1} = y_s + \frac{1}{2} \times h\left[K + \left[\frac{(K_1^2 + K_2^2)}{(K_1 + K_2)}\right]\right] \tag{13}$$

Where $K = f(\mathrm{x}_s + \frac{1}{2} \times h, \mathrm{y}_s + \frac{1}{2} \times h \times \mathrm{f}(\mathrm{x}_s, \mathrm{y}_s))$;

$\mathrm{K}_1 = f(\mathrm{x}_s, \mathrm{y}_s)$; $K_2 = f(\mathrm{x}_s + \mathrm{h}, \mathrm{y}_s + hf(\mathrm{x}_s, \mathrm{y}_s))$; K, K_1 and K_2 should not be initially zero at initial conditions $y(x_0) = y_0$. The autonomous IVPs (initial value problems) which have negative coefficient of the highest power of y and y_0 is positive are best to solve by using eq.

(13). This method is also best to solve non autonomous IVPs (initial value problems) in which the coefficient of y should be equal to initial y_0.
Eq. (13) is known as a hybrid numerical method with greater efficiency for solving initial value problems.

3. STABILITY ANALYSIS

The stability polynomial function of developed hybrid numerical method can be obtained by using following Dahlquist's test problem: [4-5]

$$\frac{dy}{dx} = \lambda y_n \; ; \quad y(0) = \mathrm{y}_0 \; ; \; \lambda \in C \tag{14}$$

Using developed method (13) for this test problem, we get

$$y_{s+1} = y_s + \frac{1}{2}.h\left[f(\mathrm{x}_s + \frac{1}{2}.h \, , \, \mathrm{y}_s + \frac{1}{2}.h \, .\mathrm{f}(\mathrm{x}_s, \mathrm{y}_s)) + \left[\frac{K_1^2 + K_2^2}{K_1 + K_2} \right] \right]$$

After applying test problem in developed method;

$$y_{s+1} = y_s + \frac{1}{2} \times h\left[\lambda(\mathrm{y}_s + \frac{1}{2} \times h \times \lambda \, \mathrm{y}_s) + \left[\frac{[\lambda y_s]^2 + [\lambda(\mathrm{y}_s + \mathrm{h}\lambda \, \mathrm{y}_s)]^2}{\lambda y_s + [\lambda(\mathrm{y}_s + \mathrm{h}\lambda \, \mathrm{y}_s)]} \right] \right]$$

$$y_{s+1} = y_s + \frac{1}{2}h\left[\lambda \, \mathrm{y}_s + \frac{\lambda^2 y_s}{2}h + \left[\frac{\lambda^2 y_s{}^2 + \lambda^2(\mathrm{y}_s + \mathrm{h}\lambda \, \mathrm{y}_s)^2}{\lambda y_s + \lambda \, \mathrm{y}_s + \mathrm{h}\lambda^2 \, \mathrm{y}_s} \right] \right]$$

On further simplification we get,

$$y_{s+1} = \left[1 + \frac{h\lambda}{2} + \frac{h^2\lambda^2}{4} + \frac{h\lambda}{(2 + h\lambda)} + \frac{h^2\lambda^2}{(2 + h\lambda)} + \frac{h^3\lambda^3}{2(2 + \mathrm{h}\lambda)} \right] y_s$$

Letting $Z = h\lambda$, stability function becomes

$$\psi(\mathrm{z}) = 1 + \left(\frac{1}{2}\right)z + \left(\frac{1}{4}\right)z^2 + \frac{z}{(2+z)} + \frac{z^2}{(2+z)} + \left(\frac{1}{2}\right)\frac{z^3}{(2+z)}$$

After, further simplification we get,

$$\psi(\mathrm{z}) = 2 + \frac{1}{2}z + \frac{3}{4}z^2 - \frac{4}{2(2+z)} \tag{15}$$

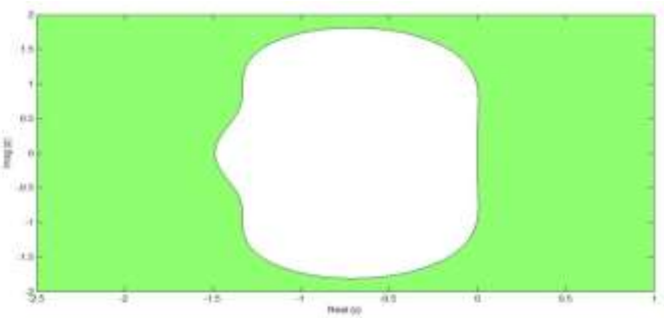


Figure 1

In the Figure 1, the shaded region shows the unstable region where composed method is not stable. But in the unshaded region composed method is stable.

4. CONSISTENCY ANALYSIS

For eq. (1) the numerical formula given by [8] $y_{s+1} = y_s + h\,[\phi(\mathrm{x}_s \, , \, \mathrm{y}_s \, , \, \mathrm{h})]$ will be consistent with the initial value problems (IVPs) if

$$\lim_{h \to 0} \phi(\mathrm{x}_s \, , \, \mathrm{y}_s \, , \, \mathrm{h}) = f(\mathrm{x}_s, \mathrm{y}_s)$$

Hence the defined method (12) checked for consistency criteria as:

$$y_{s+1} = y_s + \frac{1}{2}h\left[f(\mathrm{x}_s + \frac{1}{2}h, \mathrm{y}_s + \frac{1}{2}h\, \mathrm{f}(\mathrm{x}_s, \mathrm{y}_s)) + \left[\frac{K_1^2 + K_2^2}{K_1 + K_2}\right]\right]$$

$$\text{Here } \phi(\mathrm{x}_s, \mathrm{y}_s, \mathrm{h}) = \frac{1}{2}\left[f(\mathrm{x}_s + \frac{1}{2}h, \mathrm{y}_s + \frac{1}{2}h\, \mathrm{f}(\mathrm{x}_s, \mathrm{y}_s)) + \left[\frac{K_1^2 + K_2^2}{K_1 + K_2}\right]\right]$$

$$\lim_{h\to 0} \phi(\mathrm{x}_s, \mathrm{y}_s, \mathrm{h}) = \lim_{h\to 0} \frac{1}{2}\left[f(\mathrm{x}_s + \frac{1}{2}\times h, \mathrm{y}_s + \frac{1}{2}\times h \times \mathrm{f}(\mathrm{x}_s, \mathrm{y}_s)) + \left[\frac{K_1^2 + K_2^2}{K_1 + K_2}\right]\right]$$

$$\lim_{h\to 0} \phi(\mathrm{x}_s, \mathrm{y}_s, \mathrm{h}) = \lim_{h\to 0} \frac{1}{2}\left[2.f(\mathrm{x}_s, \mathrm{y}_s)\right] = f(\mathrm{x}_s, \mathrm{y}_s) \qquad (16)$$

It shows that the hybrid numerical method with greater efficiency for solving initial value problems is consistent.

5. ANALYSIS OF LOCAL TRUNCTION ERROR

LTE (local truncation error) in [4], is defined as

$$L.T.E = Ch^{q+1} y^{(\mathrm{q}+1)}(\mathrm{x}) + O(\mathrm{h}^{q+2}), \qquad (17)$$

Where “C” is an error constant and “q” is order of accuracy.

Taylor Series Expansion for $y(\mathrm{x}_s + \mathrm{h})$ will be expanding as mentioned bellow for obtaining LTE (local truncation error) of composed method.

$$y(\mathrm{x}_{s+1}) = y(\mathrm{x}_s + \mathrm{h}) = \mathrm{y}(\mathrm{x}_s) + \mathrm{y}'(x_s)\times \mathrm{h} + y''(x_s)\times \frac{h^2}{2!} + y'''(x_s)\times \frac{h^3}{3!} + O(h^4) \qquad (18)$$

$$\begin{aligned} y(\mathrm{x}_{s+1}) = y(\mathrm{x}_s + \mathrm{h}) &= \mathrm{y}(\mathrm{x}_s) + \mathrm{hf} + \frac{h^2}{2!}(f_x + ff_y) \\ &+ \frac{h^3}{3!}\times(\mathrm{f}_{xx} + 2\,\mathrm{ff}_{xy} + \mathrm{f}^2\mathrm{f}_{yy} + \mathrm{ff}_y{}^2 + \mathrm{f}_x\mathrm{f}_y) + O(h^4) \end{aligned} \qquad (19)$$

Now expanding the slopes [3] used in (13) by (18);

$$K_1 = f(\mathrm{x}_s, \mathrm{y}_s) = f\,;$$

$$\begin{aligned} K_2 &= f(\mathrm{x}_s + \mathrm{h}, \mathrm{y}_s + \mathrm{h}\times \mathrm{f}(\mathrm{x}_s, \mathrm{y}_s)) \\ &= f + h(\mathrm{f}_x + \mathrm{ff}_y) + \frac{h^2}{2!}(\mathrm{f}^2\mathrm{f}_{yy} + 2\,\mathrm{ff}_{xy} + \mathrm{f}_{xx}) + \mathrm{O}(\mathrm{h}^3) \end{aligned}$$

$$\begin{aligned} K &= f(\mathrm{x}_s + \frac{1}{2}\times \mathrm{h}, \mathrm{y}_s + \frac{1}{2}\mathrm{h}\times \mathrm{f}(\mathrm{x}_s, \mathrm{y}_s)) \\ &= \mathrm{f} + \frac{1}{2}\mathrm{h}(\mathrm{f}_x + \mathrm{ff}_y) + \frac{1}{8}h^2(\mathrm{f}_{xx} + 2\,\mathrm{ff}_{xy} + \mathrm{f}^2\mathrm{f}_{yy}) + \mathrm{O}(\mathrm{h}^3) \end{aligned}$$

$$\begin{aligned} K + \frac{K_1{}^2 + K_2{}^2}{K_1 + K_2} &= 2f + h(\mathrm{f}_x + \mathrm{ff}_y) + \\ & \mathrm{h}^2\left[\frac{1}{8}f_{xx} + \frac{1}{4}ff_{xy} + \frac{1}{8}f^2 f_{yy} + \frac{1}{2.f}\left(\begin{gathered} f\left(\frac{1}{2}f^2 f_{yy} + ff_{xy} + \frac{1}{2}f_{xx}\right) + \\ \left(f_x + ff_y\right)^2 - \frac{1}{2}\left(f_x + ff_y\right)^2 \end{gathered}\right)\right] + O(\mathrm{h}^3) \end{aligned} \qquad (20)$$

Substitute (19) in (13), we get

$$y_{s+1} = y_s + \frac{1}{2}.h.\left[2f + h.(\mathrm{f}_x + \mathrm{ff}_y) + \mathrm{h}^2.\left[\frac{1}{8} f_{xx} + \frac{1}{4} ff_{xy} + \frac{1}{8} f^2 f_{yy} + \frac{1}{2f} \begin{pmatrix} f\left(\frac{1}{2} f^2 f_{yy} + ff_{xy} + \frac{1}{2} f_{xx}\right) + \\ \left(f_x + ff_y\right)^2 - \frac{1}{2}\left(f_x + ff_y\right)^2 \end{pmatrix} \right] + O(\mathrm{h}^3) \right] \qquad (21)$$

Comparing equations (18) and (20), the local truncation error with leading term [9] as found as

$$LTE = y(\mathrm{x}_s + \mathrm{h}) - \mathrm{y}_{s+1} \qquad (22)$$

$$LTE = h^3\left[\frac{-1}{48} \times f_{xx} - \frac{1}{24} \times ff_{xy} - \frac{1}{48} \times f^2 f_{yy} + \frac{1}{24} \times ff_y{}^2 - \frac{1}{12} \times f_x f_y - \frac{1}{8} \times \frac{f^2{}_x}{f} \right] + O(\mathrm{h}^4) \qquad (23)$$

Hence the composed Hybrid numerical method with greater efficiency for solving IVPs (initial value problems) is of order 2 and local truncation error is $O(\mathrm{h}^3)$.

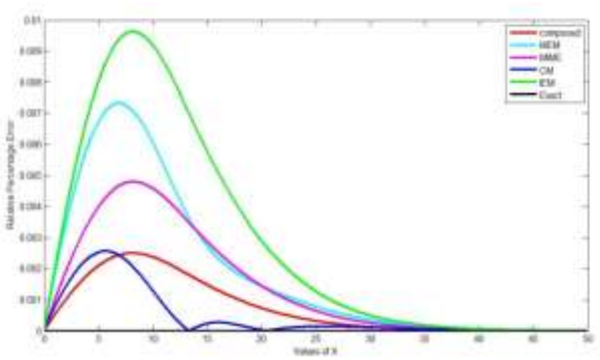
6. RESULTS AND DISCUSSION

In this section, numerical values of y(x) for initial value problems (IVPs) and relative percentage error of composed method and few existing methods has been discussed in the examples 1-3. Tables 1-3 show the results of composed hybrid numerical method, MEM (modified Euler's method), MIME (modified Improved Modified Euler's method), a second stage second order Contra harmonic mean method, Improved Euler's method and their relative percentage errors and exact solution at different x-values. These results and graphs have been tested by using MATLAB software of version 7.5.0.342 (R2007b). The graphs 1.1, 1.2, 2.1, and 3.1 show the errors of the methods given in the tables 1-3 of the examples 1-3.

Example.1 Solve $y' = y/4 - y^2/80; y(0) = 1$ in the range of $0 \le x \le 50$. The theoretical solution is $y(x) = 20/(1 + 19e^{\frac{-x}{4}})$ and h = 0.1

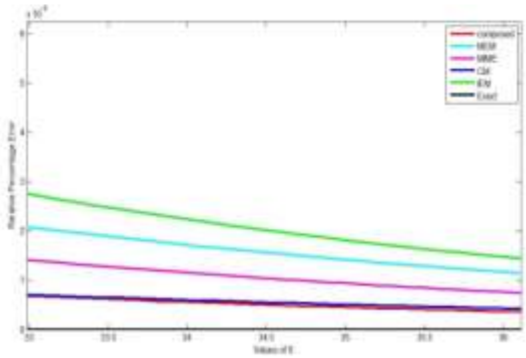
Example: 2 Solve $y'(\mathrm{x}) = 1 - y^2(\mathrm{x})$; $x = 0, y = 0$ in the interval $0 \le x \le 10$. The theoretical solution is $y = \tanh x$ and h = 0.1.

Example: 3 Solve $y'(\mathrm{x}) = x + y(\mathrm{x})$; $x = 0, \mathrm{y} = 1$ in the interval $0 \le x \le 10$. The theoretical solution is $y(\mathrm{x}) = 2e^x - (x) - 1$ and h = 0.1.



x-value	Composed Method	Error	MEM	Error	MIME	Error	C/M	Error	IEM	Error	Exact Value
[illegible]	1.0000	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
[illegible]	1.0486	0.0001	1.0486	0.0004	1.0488	0.0002	1.0488	0.0002	1.0486	0.0004	1.0486
[illegible]	1.2660	0.0005	1.2660	0.0018	1.2661	0.0010	1.2661	0.0008	1.2660	0.0020	1.2660
[illegible]	1.5250	0.0009	1.5249	0.0031	1.5250	0.0017	1.5250	0.0013	1.5249	0.0035	1.5250
[illegible]	1.9603	0.0014	1.9603	0.0047	1.9604	0.0026	1.9604	0.0019	1.9603	0.0054	1.9604
[illegible]	2.3435	0.0017	2.3434	0.0056	2.3436	0.0032	2.3436	0.0022	2.3434	0.0065	2.3435
[illegible]	2.7900	0.0020	2.7898	0.0064	2.7901	0.0037	2.7901	0.0025	2.7898	0.0075	2.7900
[illegible]	5.3026	0.0025	5.3024	0.0072	5.3030	0.0048	5.3029	0.0022	5.3022	0.0096	5.3027
[illegible]	8.5382	0.0023	8.5379	0.0056	8.5387	0.0044	8.5384	0.0010	8.5376	0.0089	8.5383
[illegible]	16.4416	0.0010	16.4415	0.0018	16.4421	0.0020	16.4417	0.0002	16.4411	0.0039	16.4418
[illegible]	19.5955	0.0002	19.5955	0.0006	19.5956	0.0004	19.5956	0.0001	19.5954	0.0008	19.5956
[illegible]	19.8416	0.0001	19.8416	0.0003	19.8417	0.0002	19.8416	0.0001	19.8415	0.0004	19.8416
[illegible]	19.9337	0.0000	19.9336	0.0002	19.9337	0.0001	19.9337	0.0001	19.9336	0.0002	19.9337
[illegible]	19.9749	0.0000	19.9748	0.0001	19.9749	0.0000	19.9749	0.0000	19.9749	0.0001	19.9749
[illegible]	19.9947	0.0000	19.9947	0.0000	19.9947	0.0000	19.9947	0.0000	19.9947	0.0000	19.9947
[illegible]	19.9978	0.0000	19.9978	0.0000	19.9978	0.0000	19.9978	0.0000	19.9778	0.0000	19.9978

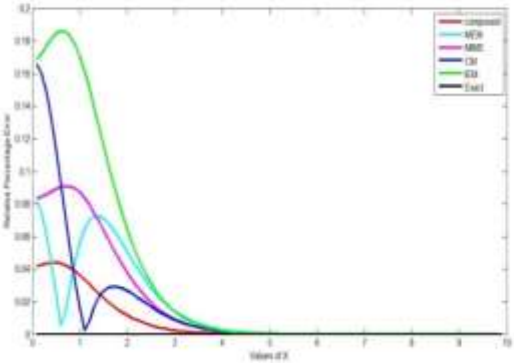
Graph 1.1 shows the errors of methods given in table 1. **Table.1** shows the results of example 1 [10].



Graph 1.2 shows the Zoom In result of graph 1.1.

X-value	Composed Method	Error	MEM	Error	MIME	Error	C_qM	Error	IEM	Error	Exact Value
0.1000	0.0996	0.0419	0.0998	0.0823	0.0998	0.0835	0.0995	0.1660	0.0995	0.1686	0.0997
0.4000	0.3798	0.0441	0.3801	0.0446	0.3803	0.0875	0.3794	0.1329	0.3793	0.1812	0.3799
1.0000	0.7613	0.0363	0.7612	0.0566	0.7623	0.0871	0.7615	0.0162	0.7608	0.1705	0.7616
1.8000	0.9467	0.0155	0.9462	0.0605	0.9473	0.0478	0.9471	0.0293	0.9460	0.0832	0.9468
2.6000	0.9890	0.0048	0.9888	0.0252	0.9892	0.0170	0.9892	0.0152	0.9888	0.0269	0.9890
3.7000	0.9988	0.0008	0.9987	0.0049	0.9988	0.0031	0.9988	0.0032	0.9987	0.0043	0.9988
4.6000	0.9998	0.0002	0.9998	0.0011	0.9998	0.0007	0.9998	0.0007	0.9998	0.0009	0.9998
5.3000	0.9999	0.0000	0.9999	0.0003	1.0000	0.0002	1.0000	0.0002	0.9999	0.0002	1.0000
6.8000	1.0000	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
8.7000	1.0000	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
9.8000	1.0000	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000

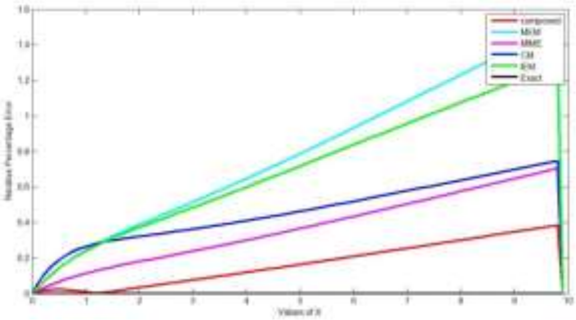
Table 2 shows the results of example 2 [11]



Graph 2.1 shows the results of table 2

X-value	Composed Method	Error	MEM	Error	MIME	Error	CiM	Error	Exact Value
0	1	0	1	0	1	0	1	0	1
0.100000	1.1105	0.010151	1.11	0.030787	1.1105	0.014245	1.1109	0.051088	1.1103
0.600000	2.0447	0.021326	2.0409	0.16535	2.0458	0.076551	2.0485	0.20809	2.0442
1.300000	5.0384	0.0030699	5.0239	0.29255	5.0454	0.13554	5.053	0.28678	5.0386
2.100000	11.774	0.033525	11.732	0.38752	11.799	0.17969	11.816	0.32123	11.778
2.400000	29.07	0.067868	28.947	0.48858	29.155	0.22675	29.192	0.35415	29.089
3.300000	84.509	0.11075	84.078	0.61925	84.846	0.28772	84.941	0.39993	84.602
4.300000	174.29	0.14133	173.29	0.71543	175.11	0.33267	75.29	0.43571	174.53
5.500000	721.7	0.20408	716.54	0.91704	726.26	0.4271	726.89	0.51365	723.17
6.100000	1783.5	0.24528	1769.1	1.0508	1796.7	0.48988	1798	0.56643	1787.9
7.700000	4395.4	0.28687	3455.7	1.1862	4432.4	0.55357	4435.3	0.62029	4408
9.200000	19713	0.35665	19504	1.4136	19915	0.66081	19925	0.71125	19784
9.800000	35918	0.38464	35514	1.5047	36310	0.7039	36326	0.74785	36057

Table 3 shows the results of example 3 [3]



Graph 3.1 shows the Relative Percentage Error of the methods given in table 3

In the Graph 1.1, the composed hybrid numerical method shows the better performance from $0 \le x < 5.5$ and then a second stage second order contra harmonic mean method shows better performance up to x = 33. After that, composed method shows better convergence as compared to others one. The patterns of relative percentage errors from x > 33, shown by the graph 1.2, represent that the composed method shows the better convergence as compared to others.

In the Graph 2.1, MEM (modified Euler's method) and a second stage second order contra harmonic mean method shows the fluctuation at x = 0 to x = 2. But composed method converges smoothly and rapidly as compared to others mentioned methods. From Graph 3.1 it is clear that the composed method is fast convergent as compared to other methods.

CONCLUSION

In this research work, a hybrid numerical method has been developed for solving the IVPs (initial value problems) in ODEs (ordinary differential equations). Developed method has generated by the combination of slopes used in MEM (Modified Euler's method), IEM

(Improved Euler's method) and a 2nd stage 2nd order contra harmonic mean method. The result of developed method has been compared with the MEM, IEM, MIME, and a 2nd stage 2nd order contra harmonic mean method and found that, the developed method has been performing better accuracy as compared to all mentioned methods and others well known second order methods. Developed method has also been analyzed for the stability, consistency and for the local truncation error; and found that composed method is more stable, consistent, and second order accurate.

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