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ORIENTATION OF MANIFOLDS AND SMOOTH FIBRE BUNDLES

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Abstract

For a smooth fibre bundle $F = (T, \rho, P, L)$ and a vertical subalgebra of $B_X(T)$, it is shown that an isomorphism $g: B_K(T) \otimes_T B_X(T) \xrightarrow{\cong} B(T)$ of graded algebras is given by the multiplication map $\omega \otimes \Upsilon \mapsto \omega \wedge \Upsilon$. If η and ω are two n-forms in $B^s(T)$ and their induced forms are $i_X\eta$, $i_X\omega$ in $B^r(L_x)$, then the orientations of $F = (T, \rho, P, L)$ by η and ω are identical if and only if the orientations of X_T by $i_X\eta$ and $i_X\omega$ are identical. Finally, if the bundle $F = (T, \rho, P, L)$, the manifolds P and T are oriented by an n-form ω, $d_P \in B^r(P)$ and the $(r+s)$-form $d_T = \rho^* d_P \wedge \omega$, respectively, then d_T depends on ω and d_P.

Keywords: Smooth fibre bundle, manifold, vector bundle, bundle isomorphism, bundle orientation, graded subalgebra.

1. Introduction

Consider the manifolds T and P such that $\rho: T \to P$ is a smooth map between them. If the smooth map ρ has the local product property for a manifold L, then there exists an open covering $\{X_a\}$ of the manifold P and a family of diffeomorphisms $\{\gamma_a\}$, where γ_a is given by

$$\gamma_a: X_a \times L \to \rho^{-1}(X_a)$$

such that $\rho\gamma_a(\alpha, \beta) = \alpha$ for $\alpha \in X_a, \beta \in L$. For the manifolds T, P, L and the smooth map $\rho: T \to P$, a four-tuple (T, ρ, P, L) is said to be a smooth fibre bundle if ρ has the local product property.

Let $F = (T, \rho, P, L)$ and $F' = (T', \rho', P', L')$ be two vector bundles. Also, suppose that $f: T \to T'$ is a smooth fibre-preserving map, then the map $f: F \to F'$ is said to be a bundle map if $f_x: L_x \to L'_{g(x)}$ is linear for $x \in P$ and the smooth map $g: P \to P'$ induced by the map f. The composition of two bundle maps is also a bundle map ([2], [7]).

If a bundle map $f: F \to F'$ is a diffeomorphism, then it is said to be a bundle isomorphism and its inverse is also a bundle isomorphism. If there exists a bundle isomorphism $f: F \xrightarrow{\cong} F'$ between the vector bundles F and F', then they are called isomorphic ([8], [9]).

Let K_T be a subbundle of i_T. Assume that $F = (T, \rho, P, L)$ is a smooth fibre bundle. If $i_T = K_T \oplus X_T$, then the subbundle K_T is called horizontal. For a smooth fibre bundle $F = (T, \rho, P, L)$, let $Z \in \Upsilon_X(T)$. Suppose that $\omega \in K(T)$ is a differential form. Then, ω is called horizontal if $f(Z)\omega = 0$. All these horizontal forms are a graded subalgebra of $B(T)$ as it is obvious that each $f(Z)$ is a homogeneous antiderivation. This kind of algebra will be called the horizontal subalgebra and will be denoted by $B_K(T)$ ([1], [3], [11]).

Now we define the vertical subalgebra of $B(T)$. For this, we choose a horizontal subbundle K_T of i_T. The $\mho(T)$-module of horizontal vector fields on T is $Y_H(T)$. Let us define a graded subalgebra $B_X(T) \subset B(T)$ by

$$B_X(T) = \{\omega \in B(T) : f(X)\omega = 0\},$$

where $X \in \mathcal{X}_H(T)$ and $B_X(T)$ is dependent on the choice of K_T. Then, the graded subalgebra $B_X(T)$ is called the vertical subalgebra of $B(T)$.

Assume that $\dim P = r$, $\dim L = s$ and $F = (T, \rho, P, L)$ is a smooth fibre bundle. Let σ_x denote the inclusion given by

$$\sigma_x : L_x \to T,$$

where $x \in P$, and L_x is the fibre at x. Also, L_x is a submanifold of T.

Let ω be a differential s-form in $B^s(T)$. For each $\omega \in B^s(T)$, and $x \in P$, $\sigma_x^*\omega$ is a differential r-form in $B^r(L_x)$. Since $\sigma_x^*\omega \in B^r(L_x)$, so $\sigma_x^*\omega$ orients the fibre at $x \in P$, i.e., L_x. If $\omega_1, \omega_2 \in B^s(T)$, then $\sigma_x^*\omega_1, \sigma_x^*\omega_2 \in B^r(L_x)$. If, for every $x \in P$, the orientations on L_x induced by $\sigma_x^*\omega_1$ and $\sigma_x^*\omega_2$, respectively, are identical, then the differential forms ω_1 and ω_2 are called equivalent ([5], [12]).

Consider the smooth fibre bundle $F = (T, \rho, P, L)$ and an r-form ω on T. For every $x \in F$, the r-form ω indeces an r-form $\sigma_x^*\omega \in B^r(L_x)$. Then, the smooth fibre bundle $F = (T, \rho, P, L)$ is orientable if $\sigma_x^*\omega$ orients L_x for every $x \in F$. In this case, an orientation for $F = (T, \rho, P, L)$ is an equivalence class of the r-form ω.

Let P be an oriented base, then the vector bundle $F = (T, \rho, P, L)$ is an oriented bundle over P. Let $\omega \in B^s(T)$ and $d_P \in B^r(P)$. If the orientation of the bundle $F = (T, \rho, P, L)$ is represented by ω and the orientation of the oriented base P is represented by d_P, then the orientation of the manifold T is represented by $\rho^* d_P \wedge \omega$. The orientation represented by $\rho^* d_P \wedge \omega$ is said to be the local product orientation ([4], [6], [10]).

For the oriented bases P and $\hat{P}$, let us consider the oriented vector bundles $F = (T, \rho, P, L)$ and $\hat{F} = (\hat{T}, \hat{\rho}, \hat{P}, \hat{L})$ over P and $\hat{P}$, respectively. Assume that $g: T \to \hat{T}$ is a fibre-preserving map and $h: P \to \hat{P}$ is a local diffeomorphism induced by g. Then the map g is called a local diffeomorphism if g is restricted for $z \in P$ to the local diffeomorphism

$$g_z: L_z \to \hat{L}_{h(z)}.$$

2. Main Results

Theorem 1. Assume that $\omega \in K(T)$ is a differential n-form and $B_X(T)$ is the vertical subalgebra of $B(T)$. Then, an isomorphism

$$g: B_K(T) \otimes_T B_X(T) \xrightarrow{\cong} B(T)$$

of graded algebras is given by the multiplication map $\omega \otimes \Upsilon \mapsto \omega \wedge \Upsilon$.

Proof. Consider the decomposition $i_T = K_T \oplus X_T$. The following maps are the projections induced by this decomposition:

$$K_x: T_x(T) \to K_x(T)$$

and

$$X_x: M_x(T) \to X_x(T).$$

For $X \in \mathcal{X}_H(T)$, the graded subalgebra $B_X(T) \subset B(T)$ given by

$$B_X(T) = \{\omega \in B(T) : f(X)\omega = 0\}$$

is the vertical subalgebra of $B(T)$, which depends on K_T, so it is obvious that the map from $B_K(T) \otimes_T B_X(T)$ to $B(T)$ is a homomorphism of graded algebras, that is, the map g given by

$$g: B_K(T) \otimes_T B_X(T) \xrightarrow{\cong} B(T)$$

is a homomorphism of graded algebras.

Now, we have to show that the homomorphism given by

$$g: B_K(T) \otimes_T B_X(T) \xrightarrow{\cong} B(T)$$

is bijective. Since the decomposition $i_T = K_T \oplus X_T$ induces $K_x: T_x(T) \to K_x(T)$ and $X_x: M_x(T) \to X_x(T)$, so there exist isomorphisms h_K and h_X given by

$$h_K : Sec \wedge K_T^* \xrightarrow{\cong} B_K(T)$$

and

$$h_X : Sec \wedge X_T^* \xrightarrow{\cong} B_X(T).$$

The isomorphisms h_K and h_X are of $\mho(T)$-modules. Thus, if $\omega \in K(T)$ is a differential form and $t_i \in T_x(T)$, then we have the following relations:

$$h_K\omega(x;\ t_1,\ \cdots\ ,t_n) = \ \omega(x;\ K_x t_1,\ \cdots\ ,K_x t_n),$$

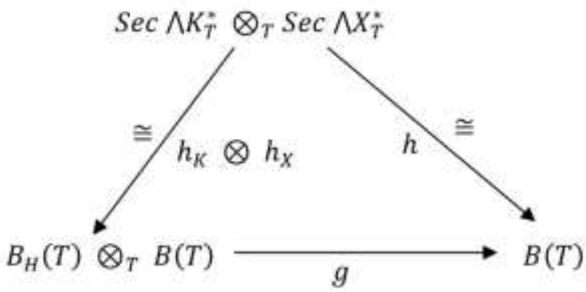
and

$$h_X\Upsilon(x;\ t_1,\ \cdots\ ,t_n) = \ \Upsilon(z;\ K_x t_1,\ \cdots\ ,K_x t_n).$$

The map from $\wedge K_T^* \otimes \ \wedge X_T^*$ to $\wedge i_T^*$ is a bundle isomorphism and this bundle isomorphism induces another isomorphism h. That means h is induced by

$$\wedge K_T^* \otimes \wedge X_T^* \xrightarrow{\cong} \wedge i_T^*.$$

Since the map from Sec $\wedge \mathrm{K}_\mathrm{T}^*$ $\otimes_\mathrm{T}$ Sec $\wedge \mathrm{X}_\mathrm{T}^*$ to $B_H(T) \otimes_T B(T)$ is also an isomorphism, we have the following commutative diagram:



Consequently, the map $g: B_K(T) \otimes_T B_X(T) \xrightarrow{\cong} B(T)$ of graded algebras given by the multiplication map $\omega \otimes \Upsilon \mapsto \omega \wedge \Upsilon$ is an isomorphism. □

Theorem 2. Assume that $F = (T, \rho, P, L)$ is a smooth fibre bundle and ω is a differential form in $B^s(T)$. Let $i_X\omega \in Sec \wedge^s X_T^*$. Then the vector bundle X_T is oriented by $i_X\omega$ if F is oriented by ω. Consider two differential forms η and ω in $B^s(T)$ and their induced forms $i_X\eta$, $i_X\omega$ in $B^r(L_x)$. Then, the orientations of $F = (T, \rho, P, L)$ by η and ω are identical if and only if the orientations of X_T by $i_X\eta$ and $i_X\omega$ are identical.

Proof. Suppose that $F = (T, \rho, P, L)$ is a smooth fibre bundle. Let $\dim P = r$, $\dim L = s$. For $x \in P$, let us consider the inclusion σ_x given by

$$\sigma_x: L_x \to T,$$

where L_x is the fibre at x and is a submanifold of T.

For each $\omega \in B^s(T)$, $\sigma_x^*\omega$ is a differential form in $B^r(L_x)$, where $x \in P$. Since $\sigma_x^*\omega \in B^r(L_x)$, so $\sigma_x^*\omega$ orients the fibre L_x at $x \in P$. If $\omega_1, \omega_2 \in B^s(T)$, then $\sigma_x^*\omega_1, \sigma_x^*\omega_2 \in B^r(L_x)$. If the orientations on L_x induced by $\sigma_x^*\omega_1$

and $\sigma_x^*\omega_2$ are identical for every $x \in P$, then the differential forms ω_1 and ω_2 are equivalent.

For every $x \in F$, the r-form ω indeces an r-form $\sigma_x^*\omega \in B^r(L_x)$. If $\sigma_x^*\omega$ orients L_x for every $x \in F$, then the smooth fibre bundle $F = (T, \rho, P, L)$ is orientable. In this case, an orientation for $F = (T, \rho, P, L)$ is an equivalence class of the r-form ω.

The bundle $F = (T, \rho, P, L)$ is oriented by ω. If we choose an element y in the fibre L_x and $x \in P$, then we have

$$(\sigma_x^*\omega)(y) \neq 0.$$

Since the differential form ω in $B^s(T)$ induces the differential form $i_X\omega$ in $B^r(L_x)$, consequently, for $y \in T$, we have

$$(i_X\omega)(y) \neq 0.$$

Therefore, X_T is oriented by $i_X\omega$.

It is obvious that for $y \in T$, the nonzero scalars j_y are unique. Since the bundle $F = (T, \rho, P, L)$ is oriented by η and ω, then for $x \in P$, $y \in L_x$ and the nonzero unique scalars j_y, we have

$$(\sigma_x^*\omega)(y) = j_y \cdot (\sigma_x^*\eta)(y).$$

The both conditions are equivalent to $j_y > 0$, $y \in T$, since in this case we have

$$(i_T\omega)(y) = j_y \cdot (i_T\omega)(y).$$

Therefore, the orientations of $F = (T, \rho, P, L)$ by η and ω are identical if and only if the orientations of X_T by $i_X\eta$ and $i_X\omega$ are identical. □

Lemma 1. If $F = (T, \rho, P, L)$ is a smooth fibre bundle, then the map $i_X\omega: F \to X_T$ is bijective.

Proof. Consider the smooth fibre bundle $F = (T, \rho, P, L)$ and an r-form ω on T. For every $x \in F$, the r-form $\sigma_x^*\omega \in B^r(L_x)$ is induced by the r-form ω. Then, the smooth fibre bundle $F = (T, \rho, P, L)$ is orientable if $\sigma_x^*\omega$ orients L_x for every $x \in F$. It is obvious from Theorem 2 that the map from orientations of the smooth fibre bundle $F = (T, \rho, P, L)$ to orientations of X_T is one-to-one, that is, the correspondence $\omega \mapsto i_T\omega$ is injective. Let us consider an element $\Gamma \in Sec \wedge^s X_T^*$. Suppose that X_T is oriented by Γ. Let $B_X(T) \subset B(T)$ be the vertical subalgebra corresponding to a particular horizontal subbundle. Then, there is an isomorphism i_T which maps $B_X(T)$ onto $Sec \wedge X_T^*$. Consequently, there exists a unique element in $B_X^s(T)$, say ω, such that

$$i_X\omega = \Gamma.$$

Therefore, $F = (T, \rho, P, L)$ is oriented by ω. Thus, the map $i_X\omega: F \to X_T$ is bijective.

□

Theorem 3. Consider a connected base P and a smooth fibre bundle $F = (T, \rho, P, L)$ over P. Assume that $F = (T, \rho, P, L)$ is oriented by two elements $\eta, \omega \in B^s(T)$ and their induced maps are $\sigma_c^*\eta$ and $\sigma_c^*\omega$ for a fixed $c \in P$. Then, the orientations in $F = (T, \rho, P, L)$ represented by η and ω are identical if the orientations in L_c represented by $\sigma_c^*\eta$ and $\sigma_c^*\omega$ are identical.

Proof. Let us consider any component U of T. For U, L_U is the union of components of L. Then, there exists a smooth bundle (U, ρ_U, P, L_U) if ρ is restricted to U. Let us choose two elements η and ω in $B^s(T)$ such that they orients $F = (T, \rho, P, L)$. Let L_c be the fibre at $c \in P$ and be a submanifold of T, then σ_c denote the inclusion given by

$$\sigma_c: L_c \to T.$$

Also, the maps induced by η and ω are $\sigma_c^*\eta$ and $\sigma_c^*\omega$, respectively. Since the orientations in L_c by $\sigma_c^*\eta$ and $\sigma_c^*\omega$ are identical, so the orientations in $(L_U)_c$ by $\sigma_c^*\eta$ and $\sigma_c^*\omega$ are also identical. As a result, we can conclude that T is connected.

Again, let us consider T to be connected. For $\eta, \omega \in B^s(T)$, the vector bundle X_T is oriented by the induced maps $i_X\eta$ and $i_X\omega$. If we choose a map $h \in \mho(T)$ such that h has no zeros, then it follows immediately that

$$i_X\eta = h \cdot i_X\omega.$$

Since T is connected, so we have either $h > 0$ or $h < 0$. In this case, we will show that $h > 0$. For $y \in L_x$ and $x \in P$, we have

$$(\sigma_c^*\omega)(y) \neq 0.$$

Since $i_X\omega$ is the map induced by ω, so, for $y \in T$, we have

$$(i_X\omega)(y) \neq 0.$$

Since η and ω orient the bundle $F = (T, \rho, P, L)$, then for $x \in P$, $y \in L_x$ and the nonzero unique scalars j_y, we have

$$(\sigma_c^*\omega)(y) = j_y \cdot (\sigma_c^*\eta)(y).$$

Thus, $(i_T\omega)(y) = j_y \cdot (i_T\omega)(y)$. As a result, we have $j_y > 0$, $y \in T$. Equivalently, there exists $j_y > 0$ and $y \in L_c$ such that

$$(\sigma_c^*\eta)(y) = j_y \cdot (\sigma_c^*\omega)(y).$$

Therefore, $h(y) = j_y > 0$, that is, $h > 0$. Hence, we can conclude that the orientations of F represented by η and ω are identical if the orientations of L_c represented by $\sigma_c^*\eta$ and $\sigma_c^*\omega$ are identical. □

Lemma 2. Let $g: T \to \hat{T}$ be a fibre preserving map for the smooth fibre bundles $F = (T, \rho, P, L)$ and $\hat{F} = (\hat{T}, \hat{\rho}, \hat{P}, \hat{L})$. Assume that if P is connected, then F and $\hat{F}$ are oriented bundles. The bundle orientations are preserved by g if g_z is orientation preserving, where $z \in P$ and g_z is restricted to the following local diffeomorphisms

$$g_z : L_z \to \hat{L}_{h(z)}.$$

Proof. Consider the smooth fibre bundles $F = (T, \rho, P, L)$ and $\hat{F} = (\hat{T}, \hat{\rho}, \hat{P}, \hat{L})$. Let the map

$$h: P \to \hat{P}$$

be induced by a smooth fibre-preserving map $g: T \to \hat{T}$. For $z \in P$, the map $g: T \to \hat{T}$ is restricted to the following diffeomorphism

$$g_z : L_z \to \hat{L}_{h(z)}.$$

Here, g_z is local diffeomorphism. The map g preserves the bundle orientations if $F = (T, \rho, P, L)$ and $\hat{F} = (\hat{T}, \hat{\rho}, \hat{P}, \hat{L})$ are oriented and g_z is orientation preserving.

Consider a differential form ω in $B^s(\hat{T})$. Let the orientation of $\hat{P}$ be represented by ω. Then, $g^*\omega$ orients F if for each $z \in P$, we have

$$\sigma_z^* \, g^*\omega = g_z^* \sigma_{h(z)}^* \omega.$$

Therefore, the bundle orientations are preserved by g if F is oriented by $g^*\omega$.

Equivalently, let g_z be orientation preserving. If the orientation of $\hat{P}$ is presented by ω and the orientation of F is presented by η, then the orientations of L_z presented by $\sigma_c^*\eta$ and $\sigma_c^* g^*\omega$ are identical. Therefore, the orientation of F is presented by $g^*\omega$, that is, the bundle orientations are preserved by g if g_z is orientation preserving. □

Theorem 4. Let T be a manifold. Assume that an r-form ω orients the smooth fibre bundle $F = (T, \rho, P, L)$, $d_P \in B^r(P)$ orients P and the manifold T is oriented by the $(r+s)$-form $d_T = \rho^* \, d_P \wedge \omega$. Then, d_T depends on ω and d_P.

Proof. Assume that P is connected. Consider the smooth fibre bundle $F = (T, \rho, P, L)$ such that the r-form ω orients F. Also assume that $d_P \in B^r(P)$ orients P and the manifold T is oriented by the $(r+s)$-form

$$d_T = \rho^* \, d_P \wedge \omega.$$

Let us choose a fixed element $c \in P$ such that $d_L = \sigma_c^* \omega$. Then, L is oriented by d_L. Since P is connected, we have to consider the case $T = P \times L$. Then, the orientations of F presented by ω and $1 \times d_L$ are identical.

Assume that $F = (T, \rho, P, L)$ is a smooth fibre bundle and ω is a differential form in $B^s(T)$. Let $i_X\omega \in Sec \wedge^s X_T^*$. Then, the vector bundle X_T is oriented by $i_X\omega$ if F is oriented by ω. Consider two differential forms η and ω in $B^s(T)$ and their induced forms $i_X\eta$ and $i_X\omega$ in $B^r(L_x)$. Then, the orientations of $F = (T, \rho, P, L)$ by η and ω are identical if and only if the orientations of X_T by $i_X\eta$ and $i_X\omega$ are identical.

Let $h \in \mho(P \times L)$ such that $h > 0$. If the orientations of $F = (T, \rho, P, L)$ by η and ω are identical, then the orientations of X_T by $i_X\eta$ and $i_X\omega$ are also identical. Therefore, the orientations in X_T represented by $i_X\omega$ and $i_X(1 \times d_L)$ are identical. In this case, we have

$$i_X\omega = h \cdot i_X(1 \times d_L).$$

If $a \in P, b \in L$, then we have

$$(\sigma_c^*\omega)(a) = h(a, b) \cdot d_L(b).$$

Consequently, it follows immediately that

$$\rho^* d_P \wedge \omega = h \cdot \rho^* d_P \wedge \rho_L^* d_L.$$

The orientation presented by the form $\rho^* d_P \wedge \rho_L^* \Delta_L$ orients $P \times L$. Since d_P and d_L represent the orientations of P and L, respectively, hence the orientation represented by $\rho^* d_P \wedge \rho_L^* \Delta_L$ depends on the orientations represented by d_P and d_L. Since the manifold T is oriented by the $(r + s)$-form $d_T = \rho^* d_P \wedge \omega$ and $h > 0$, so, d_T depends on ω and d_P. □

References

[1] Becker, James C., and Daniel H. Gottlieb. "The transfer map and fiber bundles." *Topology* 14.1 (1975): 1-12.

[2] Brown, Lawrence. "Stable isomorphism of hereditary subalgebras of C∗-algebras." *Pacific Journal of Mathematics* 71.2 (1977): 335-348.

[3] Hattori, Akio. "Spectral swquence in the de Rham cohomology of fibre bundles." *Journal of the Faculty of Science, University of Tokyo. Sect. 1, Mathematics, astronomy, physics, chemistry* (1960): 289-331.

[4] Hermann, Robert. "A sufficient condition that a mapping of Riemannian manifolds be a fibre bundle." *Proceedings of the American Mathematical Society* 11.2 (1960): 236-242.

[5] Hicks, Noel J. *Notes on differential geometry*. Vol. 3. Princeton: van Nostrand, 1965.

[6] Kostant, Bertram. "Graded manifolds, graded Lie theory, and prequantization." *Differential geometrical methods in mathematical physics*. Springer, Berlin, Heidelberg, 1977. 177-306.

[7] Matsushima, Yozô, and Shingo Murakami. "On vector bundle valued harmonic forms and automorphic forms on symmetric riemannian manifolds." *Annals of Mathematics* (1963): 365-416.

[8] Meyer, Mark, et al. "Discrete differential-geometry operators for triangulated 2-manifolds." *Visualization and mathematics III*. Springer, Berlin, Heidelberg, 2003. 35-57.

[9] Narasimhan, Mudumbai S., and Conjeeveram S. Seshadri. "Stable and unitary vector bundles on a compact Riemann surface." *Annals of Mathematics* (1965): 540-567.

[10] Okonek, Christian, et al. *Vector bundles on complex projective spaces*. Vol. 3. Boston: Birkhäuser, 1980.

[11] Struik, Dirk Jan. *Lectures on classical differential geometry*. Courier Corporation, 1961.

[12] Wang, Hsien-Chung. "Closed manifolds with homogeneous complex structure." *American Journal of Mathematics* 76.1 (1954): 1-32.