An Exponential-Type Distribution for Modeling Failure Rate.

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ABSTRACT

In this paper we introduced the Truncated Exponential Skew Symmetric Gumbel II (TESSG II) distribution which generalizes the Gumbel II distribution using the method proposed by Nadarajah et al (2013). Unlike the Gumbel II distribution which exhibits a monotone decreasing failure rate, the new distribution is useful for modeling unimodal (Bathtub-shaped) failure rates which has a wider class of applications in solving real life problems. Structural properties of the new distribution namely, density function, hazard function, moments and moment generating function were obtained. The maximum likelihood was employed to estimate parameters of the new distribution. Real data set for failures of Air conditional system of Jet air planes were used to validate its tractability, we discovered that the Truncated Exponential Skew Symmetric Gumbel II (TESSG II) has a better fit than Gumbel II distribution.

Keywords: Quantile function, Bathtub-shaped failure rate, Renyl entropy, moments

1.0 INTRODUCTION

The Gumbel distribution, belong to the class of type-1 extreme value distribution which is often used for extreme value analysis of extreme events. Another type of this distribution is the Gumbel II distribution which is not widely used in statistically modelling because of its lack of fits. For further studies see Okorie et al [7], Pinheiro Ferrari [12].

The cumulative density function cdf (x) of the Truncated Exponentiated Skew symmetric family of distributions according to Nadarajah et al. [10] is defined by

$$F(x) = \frac{1-\exp[-\lambda G(x)]}{1-\exp(-\lambda)} \quad (1)$$

Differentiating the equation (1) above will yield the probability density of Truncated exponentiated Skew Symmetric family of distribution given as

$$f(x) = \frac{\lambda g(x) \exp[-\lambda G(x)]}{1 - \exp(-\lambda)} \qquad (2)$$

A random variable X is said to follow the Gumbel type-2 distribution if its cumulative density function (cdf) (x) is given as defined Gumbel [2–5],

$$F(x) = 1 - e^{-\eta x^{-\alpha}} \qquad (3)$$

And the probability density function given as

$$f(x) = \alpha \eta x^{-\alpha-1} e^{-\eta x^{-\alpha}} \qquad (4)$$

Substituting equation (3) in (1), we have the cumulative density function of Truncated Exponential Skew Symmetric Gumbel Type-II (TESSG II) distribution given as

$$G(x) = \frac{1-e^{-\lambda\left(1-e^{-\eta x^{-\alpha}}\right)}}{1-e^{-\lambda}} \qquad x > 0, \alpha, \eta > 0 \qquad (5)$$

The equation above when differentiated gives the probability density function (pdf) of the TESSG II distribution is given as

$$g(x) = \frac{\alpha \eta \lambda x^{-\alpha-1} e^{-\eta x^{-\alpha}} e^{\left[\lambda\left(1-e^{-\eta x^{-\alpha}}\right)\right]}}{1 - e^{-\lambda}} \qquad x > 0, \alpha, \eta > 0 \qquad (6)$$

The graph of pdf of TESSG II distribution for different values of the parameters is given below

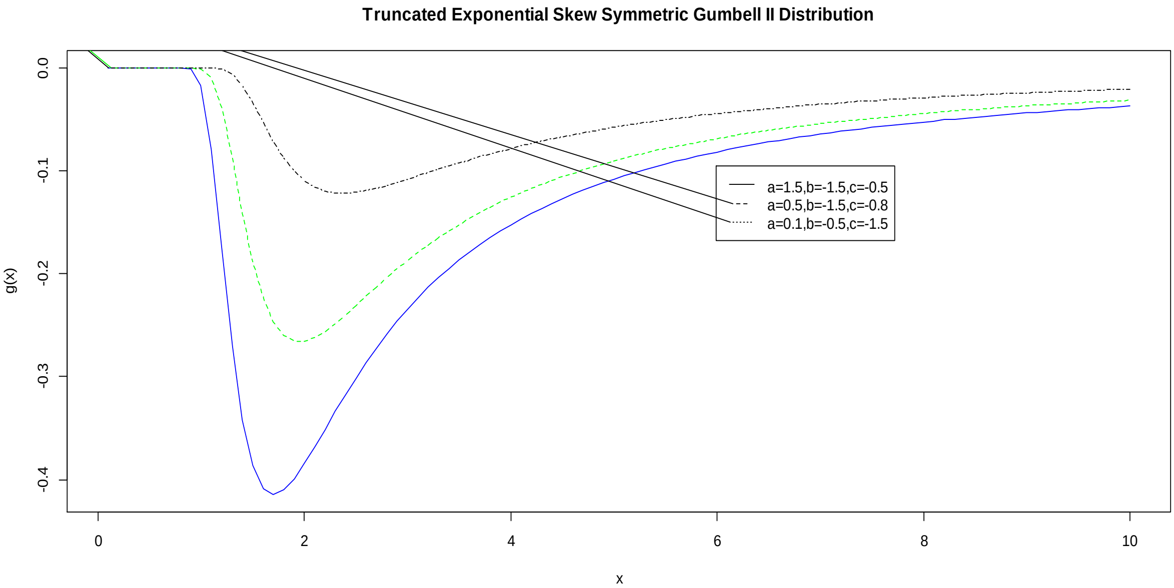

Figure 1.0 The graph of the TESSG II distribution

2.0 Reliability Function

The reliability function or the survival function of a random variable X is given by $R(x) = P(X > x) = 1 - G(x)$. This could be interpreted as the probability of a system not failing before some

specified time t, Lee and Wang [5]. The reliability function of the TESSG II distribution is given
by

$$R(x) = 1 - \frac{1 - e^{-\lambda\left(1-e^{-\eta x^{-\alpha}}\right)}}{1 - e^{-\lambda}} \qquad x > 0\,, \alpha, \eta, \lambda > 0 \qquad (7)$$

The graph of the Reliability function of TESSG II distribution is given below for various values of the parameters.

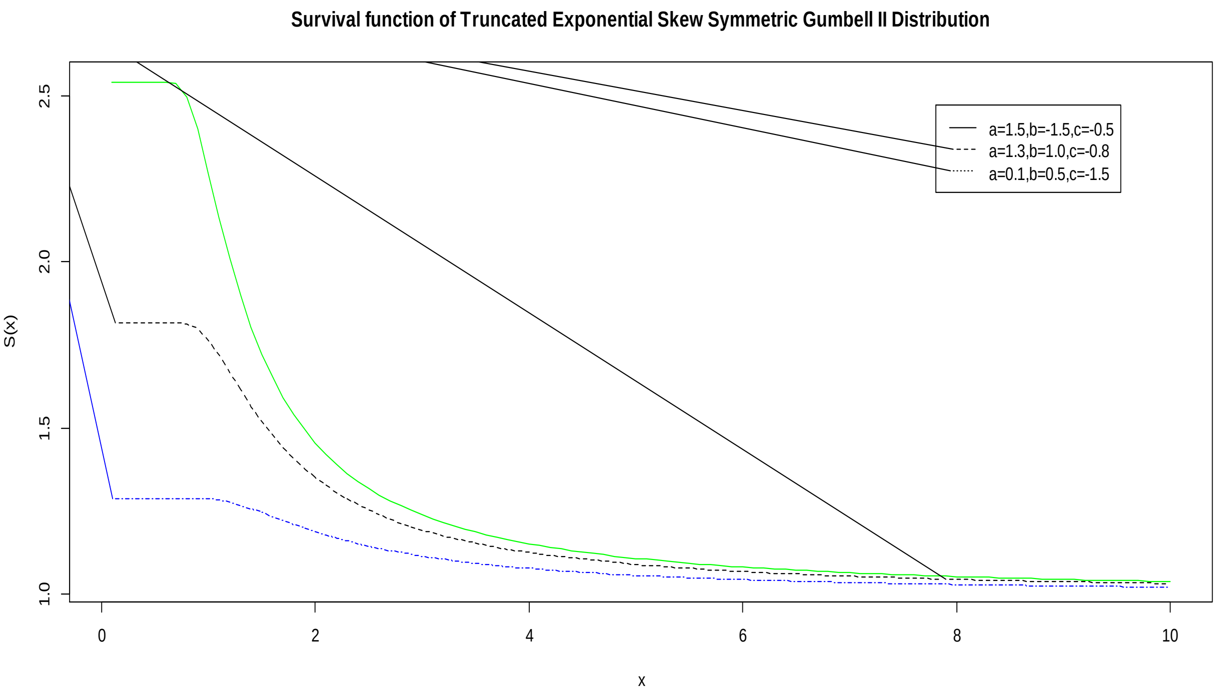

Figure 2.0 The graph of survival function of TESSG II distribution

3.0 Hazard Rate Function

The hazard rate function $h(x)$ or the instantaneous failure rate of a random variable X is the probability that a system fails given that it has survived up to time t and is given by

$$h(x) = \frac{f(x)}{1 - F(x)} = \frac{f(x)}{R(x)} \qquad (8)$$

Then the hazard rate function of TESSG II distribution is given as

$$h(x) = \frac{\alpha\eta\lambda x^{-\alpha-1} e^{-\eta x^{-\alpha}} e^{\left[\lambda\left(1-e^{-\eta x^{-\alpha}}\right)\right]}}{e^{-\lambda} + e^{-\lambda\left(1-e^{-\eta x^{-\alpha}}\right)}} \qquad (9)$$

The graph of the hazard rate function of TESSG II Distribution for various parameters values is drawn below

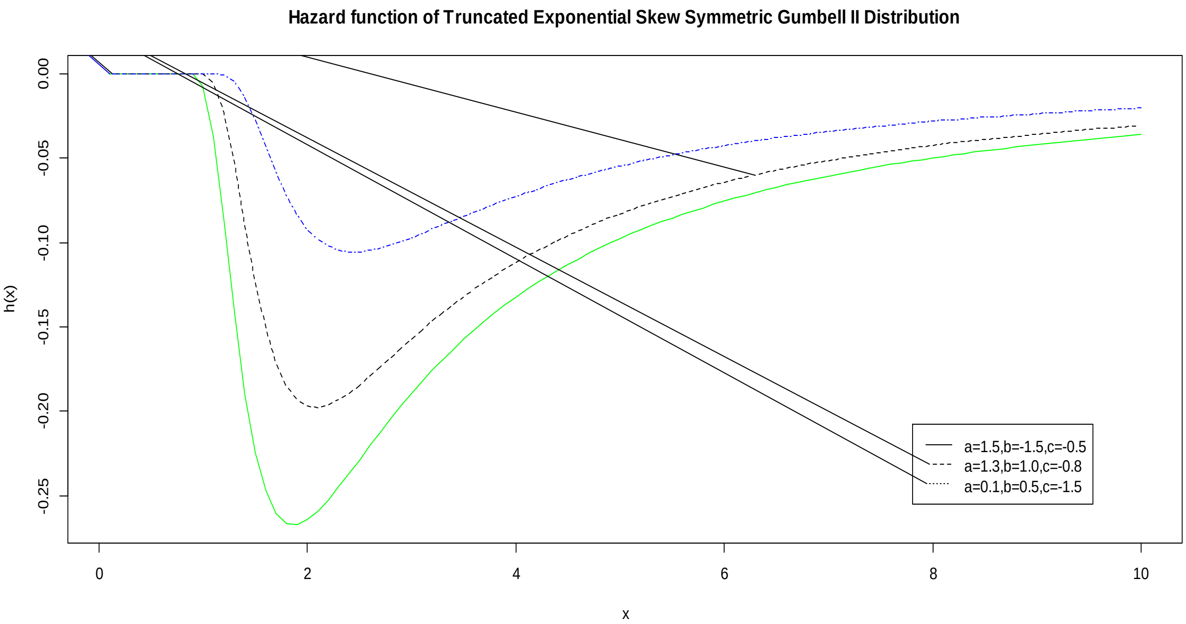

Figure 3.0 The graph of function of TESSG II distribution

- The graph of the hazard function of TESSG II drawn above indicates that the distribution can effectively be used to model real life data that possesses non-monotone failure rate.

4.0 The Moments and Moment generating function

The moments of a random variable X are one of the most important properties of a distribution that could be used to obtain other essential properties such as mean, variance, skewness, and kurtosis statistics which can also be used to describe a probability distribution. The moment of a distribution function can be obtained by using the relation given as

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx \quad (10)$$

For TESSG II distribution, the r^{th} moment is given as

$$E(X^r) = \alpha\eta\lambda \int_{-\infty}^{\infty} x^r \frac{x^{-\alpha-1} e^{-\eta x^{-\alpha}} e^{\left[\lambda\left(1-e^{-\eta x^{-\alpha}}\right)\right]}}{1-e^{-\lambda}} dx \quad (11)$$

Then we have

$$E(X^r) = \frac{\alpha\eta\lambda}{1-e^{-\lambda}} \int_{-\infty}^{\infty} x^{r-\alpha-1} e^{-\eta x^{-\alpha}} e^{\left[\lambda\left(1-e^{-\eta x^{-\alpha}}\right)\right]} dx \quad (12)$$

If we let, $y = \eta x^{-\alpha}, x = \left(\frac{\eta}{y}\right)^{\frac{1}{\alpha}}, dx = \frac{\eta^{\frac{1}{\alpha}}}{\alpha} y^{-\frac{1}{\alpha}-1} dy$ and substitute it in equation (12), we have

$$E(X^r) = -\frac{\eta\lambda}{1-e^{-\lambda}} \eta^{\frac{r}{\alpha}} e^{-\lambda} \int_{-\infty}^{\infty} y^{-\frac{r}{\alpha}} e^{-\lambda(1-e^{-y})} dy \qquad y > 0 \qquad (13)$$

Since

$$e^x = \sum_{i=1}^{k} \frac{x^k}{k!} \qquad (14)$$

Finally, we have

$$E(X^r) = -\frac{\eta\lambda}{1-e^{-\lambda}} \eta^{\frac{r}{\alpha}} e^{-\lambda} \sum_{i=1}^{k} \frac{\lambda^k}{k!} \left(\frac{1}{k}\right)^{1-\frac{r}{\alpha}} \int_{k+1}^{\infty} z^{-\frac{r}{\alpha}} e^{-z} dz \qquad (15)$$

Since X can only take values on the positive real line we can introduce the exponential integral defined by

$$Ei(-X) = -\int_{x}^{\infty} t^{-1} e^{-t} dt \qquad (16)$$

For further study see Chapter 5 of Abramowitz and Stegun [8] and Equation (6.2.6) of Oliver et al. [7]): and applying it to equation (14) will transform to

$$E(X^r) = \frac{\eta}{1-e^{-\lambda}} \eta^{\frac{r}{\alpha}} e^{-\lambda} \sum_{i=1}^{k} \frac{\lambda^{k+1}}{k!} \left(\frac{1}{k}\right)^{1-\frac{r}{\alpha}} \int_{x}^{\infty} t^{\frac{r}{\alpha}} e^{-t} dt \qquad (17)$$

Using a generalised gamma function to summarize equation (17), we obtain the moment TESSG II distribution given as

$$E(X^r) = \frac{\eta}{1-e^{-\lambda}} \eta^{\frac{r}{\alpha}} e^{-\lambda} \sum_{i=1}^{k} \frac{\lambda^{k+1}}{k!} \left(\frac{1}{k}\right)^{1-\frac{r}{\alpha}} \Gamma\left(\frac{r}{\alpha} - 1; x\right) \qquad (18)$$

Using equation (18), we obtain the $1^{st}, 2^{nd}, 3^{rd}, 4^{th}$ moment for $r = 1,2,3,4$, we have

$$\mu_1' = \frac{\eta}{1-e^{-\lambda}} \eta^{\frac{r}{\alpha}} e^{-\lambda} \sum_{i=1}^{k} \frac{\lambda^{k+1}}{k!} \left(\frac{1}{k}\right)^{1-\frac{r}{\alpha}} \left(\frac{1}{\alpha} - 1; x\right) \qquad (19)$$

$$\mu_2' = \frac{\eta}{1-e^{-\lambda}} \eta^{\frac{r}{\alpha}} e^{-\lambda} \sum_{i=1}^{k} \frac{\lambda^{k+1}}{k!} \left(\frac{1}{k}\right)^{1-\frac{r}{\alpha}} \Gamma\left(\frac{2}{\alpha} - 1; x\right) \qquad (20)$$

$$\mu_3' = \frac{\eta}{1-e^{-\lambda}} \eta^{\frac{r}{\alpha}} e^{-\lambda} \sum_{i=1}^{k} \frac{\lambda^{k+1}}{k!} \left(\frac{1}{k}\right)^{1-\frac{r}{\alpha}} \Gamma\left(\frac{3}{\alpha} - 1; x\right) \qquad (21)$$

$$\mu_4' = \frac{\eta}{1-e^{-\lambda}} \eta^{\frac{r}{\alpha}} e^{-\lambda} \sum_{i=1}^{k} \frac{\lambda^{k+1}}{k!} \left(\frac{1}{k}\right)^{1-\frac{r}{\alpha}} \Gamma\left(\frac{r}{\alpha} - 1; x\right) \tag{22}$$

The mean of TESSG II distribution is the first moment about the origin (μ_1') which corresponds to equation (19). It then follows that the variance (μ_2), the coefficient of variation (ρ), the coefficient of skewness (γ_1), and the coefficient of kurtosis (γ_2) of the TESSG II distribution are respectively, obtained as

$$(\mu_2) = \mu_2' - (\mu_1')^2 \tag{23}$$

$$\rho = \frac{\sqrt{\mu_2}}{\mu_1'} = \frac{\sqrt{\mu_2' - (\mu_1')^2}}{\mu_1'}, \tag{24}$$

$$\gamma_1 = \frac{\mu_3}{(\mu_2)^{\frac{3}{2}}} = \frac{\mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^2}{\{\mu_2' - (\mu_1')^2\}^{\frac{3}{2}}} \tag{25}$$

$$\gamma_2 = \frac{\mu_4}{(\mu_2)^2} = \frac{\mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1' - 3(\mu_1')^2}{\{\mu_2' - (\mu_1')^2\}^2} \tag{26}$$

5.0 Moment generating function of TESSG II distribution

The moment generating function of a random variable x is defined by

$$M_x(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx \tag{27}$$

The above expression can further be simplify as

$$M_x(t) = \sum_{k=0}^{\infty} \frac{t^r}{r!} \int_{-\infty}^{\infty} x^r f(x) dx \tag{28}$$

Since,

$$e^{tx} = \sum_{r=0}^{\infty} \frac{t^r x^r}{r!} \tag{29}$$

Inserting equation (18) in equation (29), then we have

$$M_x(t) = \frac{\eta}{1-e^{-\lambda}} e^{-\lambda} \sum_{i=1}^{k} \sum_{r=1}^{\infty} \frac{\lambda^{k+1}}{k!} \frac{t^r}{r!} \eta^{\frac{r}{\alpha}} \left(\frac{1}{k+1}\right)^{1-\frac{r}{\alpha}} \Gamma\left(\frac{r}{\alpha} - 1; x\right) \tag{30}$$

6.0 Estimation of the parameters

In this section method of maximum likelihood is used to estimate the parameters and also we construct a confidence interval for the unknown parameters. Let $X_1, X_2, \ldots, X_n$ be a random sample from

$X \sim TESSG\ I(\alpha, \eta, \lambda)$ with observed values $x, x_2, \ldots, x_n$ then the likelihood function $L \equiv L(\alpha, \eta, \lambda)$ can be written as

$$L = \prod_{i=1}^{n} \frac{\alpha\eta\lambda x^{-\alpha-1} e^{-\eta x^{-\alpha}} e^{\left[\lambda\left(1-e^{-\eta x^{-\alpha}}\right)\right]}}{1-e^{-\lambda}} \qquad (39)$$

And the log likelihood ($logL = l$) is given as

$$l(\theta) = nlog\left(\frac{\alpha\eta\lambda}{1-e^{-\lambda}}\right) - (\alpha-1)\sum_{i=1}^{n} lnx_i - \eta\sum_{i=1}^{n} x_i{}^{-\alpha} + \lambda\sum_{i=1}^{n}\left(1-e^{-\eta x^{-\alpha}}\right)$$

And the element of the score vector is given as

$$\frac{dl}{d\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} lnx_i - \eta\sum_{i=1}^{n}(lnx_i{}^{-\alpha})(lnx_i)$$

$$\frac{dl}{d\eta} = \frac{n}{\eta} - \sum_{i=1}^{n} x_i{}^{-a} - \sum_{i=1}^{n} x_{i\left(1-e^{-\eta x^{-\alpha}}\right)}$$

$$\frac{dl}{d\lambda} = \frac{n}{\lambda} - \frac{ne^{-\lambda}}{1-e^{-\lambda}} + \lambda\sum_{i=1}^{n}\left(1-e^{-\eta x^{-\alpha}}\right)$$

7.0 Application

We consider the number of failures for the air conditioning system of jet airplanes. These data were reported by Cordeiro and Lemonte [1]: 194,413, 90, 74, 55, 23, 97, 50, 359, 50, 130, 487, 57, 102, 15, 14, 10, 57, 320, 261, 51, 44, 9, 254, 493, 33, 18, 209, 41, 58, 60, 48, 56, 87, 11, 102, 12, 5, 14, 14, 29, 37, 186, 29, 104, 7, 4, 72, 270, 283, 7, 61, 100, 61, 502, 220, 120, 141, 22, 603, 35, 98, 54, 100, 11, 181, 65, 49, 12, 239, 14, 18, 39, 3, 12, 5, 32, 9, 438, 43, 134, 184, 20, 386, 182, 71, 80, 188, 230, 152, 5, 36, 79, 59, 33, 246, 1, 79, 3, 27, 201, 84, 27, 156, 21, 16, 88, 130, 14, 118, 44, 15, 42, 106, 46, 230, 26, 59, 153, 104, 20, 206, 5, 66, 34, 29, 26, 35, 5, 82, 31, 118, 326, 12, 54, 36, 34, 18, 25, 120, 31, 22, 18, 216, 139, 67, 310, 3, 46, 210, 57, 76, 14, 111, 97, 62, 39, 30, 7, 44, 11, 63, 23, 22, 23, 14, 18, 13, 34, 16, 18, 130, 90, 163, 208, 1, 24, 70, 16, 101, 52, 208, 95, 62, 11, 191, 14, 71.

Table 1 gives the exploratory data analysis of the data, Table 2 provides the maximum likelihood estimate of the unknown parameters (and the corresponding standard errors in parentheses) and the measure of goodness-of-fit tests that was used to verify which distribution fits better to the data set between the Truncated Exponentiated Skew Symmetric Gumbel II distribution and the Gumbel II (G II) distribution. We consider the Akaike Information Criterion

(AIC), Bayesian Information Criterion (BIC) and the Hanna Quinn Information Criteria (HQIC) as the selection criteria.

RESULTS

Descriptive Statistics of the failure data

Min	Q_1	Median	$mean$	Q_3	Max	$kurtosis$	Range	Skewness
1.00	2.75	54.00	92.07	118.00	603.00	5.20	600	2.17

Table 2.0

Distributions	$Parameter\ Estimates$			l	AIC	BIC	$HQIC$
$\boldsymbol{TESSG\ II}$ $(\boldsymbol{a},\boldsymbol{\eta},\boldsymbol{\lambda})$	1.4004 (0.0254)	0.0188 (0.0020)	11.2907 (1.0062)	−260.78	527.56	542.28	533.15
$\boldsymbol{G\ II}$ $(\boldsymbol{a},\boldsymbol{\eta})$	1.0287 (0.0179)	0.2854 (0.0131)	−	−448.68	901.37	911.18	905.10

CONCLUSION

Since the Truncated exponential skew symmetric Gumbel II distribution possess the smaller AIB,BIC and HQIC it can be considered as better model than the Gumbel II model most especially in modeling data that exhibits non-monotone failure rate.

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