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Applications of First Order Ordinary Differential Equation as Mathematical Model

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Abstract

The major purpose of this paper is to show the application of first order ordinary differential equation as a mathematical model particularly in describing some biological processes and mixing problems. Application of first order ordinary differential equation in modeling some biological phenomena such as logistic population model and prey-predator interaction for three species in linear food chain system have been analyzed. Furthermore, the application in substance mixing problems in both single and multiple tank systems have been demonstrated. Finally, it is demonstrated that the logistic model is more power full than the exponential model in modeling a population model.

Keywords: mathematical model; logistic model; exponential model

1. Introduction

Many real life problems in science and engineering, when formulated mathematically give rise to differential equation. In order to understand the physical behavior of the mathematical representation, it is necessary to have some knowledge about the mathematical character, properties and the solution of the governing differential equation Lambe and Tranter (2018). Many of the principles, or laws, underlying the behavior of the natural world are statements or relations involving rates at which things happen. When it is expressed in mathematical terms, the relations are equations and the rates are derivatives (Logan, 2017) . If we want to solve a real life problem (usually of a physical nature), we first have to formulate the problem as a mathematical expression in terms of variables, functions, and equations. Such an expression is known as a mathematical model of the given problem. The process of setting up a model, solving it mathematically, and interpreting the result in physical or other term is called mathematical modeling (Bajpai et al., 2018).

Generally a mathematical model is an evolution equation which can potentially describe the evolution of some selected aspects of the real-life problem. The description obtained in solving mathematical problems is generated by the application of the model to the description of real physical behaviors (Bellomo et al., 2007). Since rates of change are represented mathematically by derivatives, mathematical models often involve equations relating an unknown function and one or more derivatives. Such equations are differential equations (Boyce et al., 2017). Many applications, however, require the use of two or more dependent variables, each a function of a single independent variable (typically time) such a problem leads naturally to a system of simultaneous ordinary differential equation (Edwards et al., 2016).

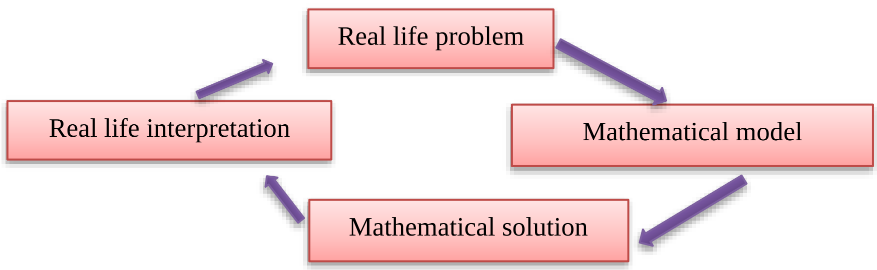
The mathematical model to represent a real-life problem is almost always simpler than the actual situation being studied as simplified assumptions are usually required to obtain a mathematical problem that can be solved. Agarap (2016) used a mathematical method to represent a simple, hypothetical coin-operated vending machine. A mathematical model of thin film flow with the numerical solution method of solving a third order ordinary differential equations is discussed by (Mechee et al., 2013). Different modeling applications of differential equation are also discussed by [9] - [11].

Kolmanovskii, and Myshkis (2013) investigated the basic principles of mathematical modeling in applying differential equations on qualitative theory, stability, periodic solutions and optimal control. Cesari (2012) presented the application of differential equations in economics and engineering by examining concrete optimization problems. Oksendal (2013) and Simeonov (2007) have analysed the applications of a special type of differential equations called stochastic differential equation and impulsive differential equation, respectively. Frigon and Pouso (2017) dealt with the theory and applications of first-order ordinary differential equations by which the usual derivatives are replaced by Stieltjes derivatives. Scholz and Scholz (2015) discussed the application of first-order ordinary differential equations on Bouguer-Lambert-Beer law in spectroscopy, time constant of sensors, chemical reaction kinetics, radioactive decay, relaxation in nuclear magnetic resonance, and the RC constant of an electrode. In this paper, the application of first order differential equation for modeling population growth or decay, prey predator model, single and multiple tank mixing problems are considered.

2. Preliminaries

2.1 Basic principles and laws of modeling

The process of mathematical modeling can be generalized as



Population law of mass action: The rate of change of a population $x(t)$ due to interaction with a population $y(t)$ is proportional to the product of the populations $x(t)$ and $y(t)$ at a given time t. That is, for a proportionality constant a,

$$\frac{dx}{dt} = axy. \qquad (1)$$

Balance law for population (Zill [9]): The net rate of change of the population $p(t)$ is equal to the rate of change of a population in to the ecosystem minus the rate of change of population out of the ecosystem at a time t. That is

$$\frac{dp}{dt} = \left(\frac{dp}{dt}\right)_{in} - \left(\frac{dp}{dt}\right)_{out}. \qquad (2)$$

First order rate law: The rate at which a population $p(t)$ grows or decays in a first order process is proportional to its population at that time. That is That is, for proportionality constant γ,

$$\frac{dp}{dt} = \gamma p(t) \qquad (3)$$

Law of conservation of mass: Let $m(t)$ be the mass of a substance at a time t, then we have

$$\frac{dm}{dt} = 0. \qquad (4)$$

2.2 Linearization of nonlinear system

Definition 1: Linearization is the process of finding the linear approximation of a nonlinear function (system) at a given point. In the study of dynamical systems, linearization is a method for assessing the local stability of an equilibrium point of a system of non-linear differential equations or discrete dynamical system.

Consider a nonlinear system of m first order ordinary differential equations with n variables

$$\frac{dX_i(t)}{dt} = f_i(x_1, x_2, \dots, x_n), i = 1,2,3 \dots, m \qquad (5)$$

The Jacobian matrix of the system (5) is the matrix of all first-order partial derivatives of a vector-valued function, $f_i(x_1, x_2, \dots, x_n)$.It is denoted by J and defined as:

$$J = \frac{\partial(f_1, f_2, \dots, f_m)}{\partial(x_1, x_2, \dots, x_n)} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Definition 2: We say that the point $x_0 = ({x_1}^0, {x_2}^0, \dots, {x_n}^0)$ is an equilibrium point or fixed point if

$$f_i({x_1}^0, {x_2}^0, \dots, {x_n}^0) = 0, \ \forall i.$$

The importance of definition 2 lies in the fact that it represents the best linear approximation to a differentiable function near a given point. Based on Jordan and Smith (2007) the linearization form of the non-linear system (5) is given by

$$\frac{dx(t)}{dt} = Ju(t), \qquad (6)$$

where

$$J_f(x_0) = \begin{bmatrix} \frac{\partial f_1(x_0)}{\partial x_1} & \frac{\partial f_1(x_0)}{\partial x_2} & \cdots & \frac{\partial f_1(x_0)}{\partial x_n} \\ \frac{\partial f_2(x_0)}{\partial x_1} & \frac{\partial f_2(x_0)}{\partial x_2} & \cdots & \frac{\partial f_2(x_0)}{\partial x_n} \\ & \vdots & & \\ \frac{\partial f_n(x_0)}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_{n(x_0)}}{\partial x_n} \end{bmatrix},$$

$$u(t) = (u_1, u_2, \dots, u_n)^T$$

$$u_1 = (x_1 - {x_1}^0), u_2 = (x_2 - {x_2}^0), \dots, u_n = (x_n - {x_n}^0)$$

In order to analyze stability of the system, computation of eigenvalues of the corresponding system is the first step.

Definition 3 (Slavik, 2013): The eigenvalues of a tridiagonal matrix $A = [a_{ij}]$ are contained in the union of the intervals$[a_{ij} - r_i , a_{ij} + r_i]$, where

$$r_i = \sum_{j\epsilon\{1,\dots,n\}\setminus i} |a_{ij}|,\ 1 \le i \le n.$$

Given an $n \times n$ tridiagonal matrix $T_n(x)$of the form

$$T_n(x) = \begin{bmatrix} x & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & x & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & x & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & x & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & x & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & x \end{bmatrix} \quad (7)$$

and its associated determinant $D_n(x) = det|T_n(x)|$ (Jeffrey, 2010). Furthermore, the eigenvalue of $T_n(x)$ is given by $\lambda_m = x - 2\cos\left(\frac{m\pi}{n+1}\right), m = 1,2,\dots,n$ and the eigenvector of $T_n(x)$ is given by $u^m = \ (u_1^m), (u_2^m), \dots, (u_n^m)$, for $m = 1,2,\dots,n$.

3. Results and Discussion

3.1 Population Growth or Decay Model

Let $p(t)$ denotes the size of population of a country at any time t, then by Balance law for population, we have

$$\frac{dp}{dt} = B(p,t) - D(p,t) + M(p,t), \quad (8)$$

where

$B(p,t)$ represents inputs (birth rates),

$D(p,t)$ represents outputs (death rates),

$M(p,t)$ represents net migration.

One of the simplest cases is that assuming a model (8) for birth and death rates are proportional to the population and no migrants. Thus

$$B(p,t) = bp(t), \quad D(p,t) = dp(t), \quad M(p,t) = 0$$

Hence equation (8) can be reduced to

$$\frac{dp}{dt} = (b-d)p = \gamma p, \qquad (9)$$

where $b - d = \gamma$ is a proportionality constant which indicates population growth for $\gamma > 0$ and population decay for$\gamma < 0$. Since equation (9) is a linear differential equation, we can get a solution of the form:

$$p(t) = p_0 e^{\gamma t}.$$

where $p(t_0) = p_0$ is the initial population and γ is called the growth or the decay constant. As a result, the population grows and continues to expand to infinity if $\gamma > 0$, while the population will shrink and tend to zero if $\gamma < 0$. However, populations cannot grow without bound there can be competition for food, resources or space. Suppose an environment is capable of sustaining no more than a fixed number $k$of individuals in its population. The quantity k is called the carrying capacity of the environment. Thus, for other models, equation (9) can be expected to decrease as the population p increases in size.

The assumption that the rate at which a population grows (or decreases) is dependent only on the number $p(t)$ present and not on any time-dependent mechanisms such as seasonal phenomena can be stated as

$$\frac{dp}{dt} = pf(p). \qquad (10)$$

Now, assume that $f(p)$ is linear

$$f(p) = \alpha p + \beta$$

with conditions

$$\lim_{p(t)\to 0} fp(t) = \gamma, f(k) = 0 \text{ which leads } f(p) = \gamma - \left(\frac{\gamma}{k}\right)p.$$

Equation (10) becomes

$$\frac{dp}{dt} = p\left(\gamma - \frac{\gamma}{k}p\right) \qquad (11)$$

This is called the logistic population model with growth rate γ and carrying capacity k. Clearly, when assuming $p(t)$ is small compared to k, then the equation reduces to the exponential one which is nonlinear and separable. The constant solutions $p = 0$ and $p = k$ are known as equilibrium solution. From equation (11), we can have

$$p(t) = \frac{kp_0}{p_0 + (k - p_0)e^{-\gamma t}} \qquad (12)$$

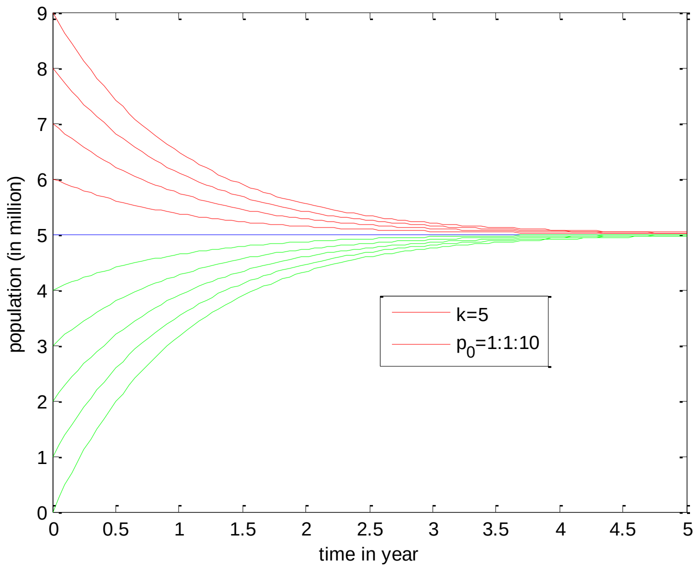


Figure 1: logistic population model with $\gamma = 1$

From Fig. 1, the following behaviors can be observed with the variation in the initial population as $t \to \infty$.

Value	Long term behavior of population	
$p_0 = 0$	$\lim_{t\to\infty} p(t) = 0$	➢ no population
$0 < p_0 < k$	$\lim_{t\to\infty} p(t) = k$	➢ population grows towards the balance population $p = k$
$p_0 = k$		➢ population level or perfect balance with its surroundings
$p_0 > k$		➢ population decreases towards the balance population $p = k$

3.2 Prey predator model

In this model, we completely characterize the qualitative behavior of a linear three species food chain. Suppose that three different species of animals interact within the same environment or ecosystem. The ecosystem that we wish to model is a linear three species food chain, where the lowest-level prey species x is preyed up on by a mid-level species$y$, which, in turn, is preyed up on by a top-level predator species z. Examples of such three species ecosystems include: mouse-snake-owl and worm-robin- falcon (Paullet et al., 2002).

The model of predator and prey association includes only natural growth or decay and the predator-prey interaction itself. We assume all other relationships (factors) to be negligible. The prey population grows according to a first order rate law in the absence of predators, while the predator population declines according to a first order rate law if the prey population is extinct. If there were no predators in the ecosystem, then the prey's species would, with an added assumption of unlimited food supply, grow at a rate that is proportional to the number of prey species present at time$t$ (first order rate law):

$$\frac{dx}{dt} = ax, \qquad a > 0 \tag{13}$$

But when predator species are present, the prey species population is decreased by $bxy,\ b > 0$, that is, decreased by the rate at which the preys population are eaten during their encounters with the predator species: adding this rate to equation (13) gives the model for the prey species population:

$$\frac{dx}{dt} = ax - bxy \qquad (14)$$

If there were no prey species in the ecosystem, then one might expect that the mid- level species, lacking an adequate food supply, would decline in number according to:

$$\frac{dy}{dt} = -cy \qquad c > 0 \qquad (15)$$

When prey species are present in the environment, it seems reasonable that the number of encounters or interactions between these two species per unit time is jointly proportional to their populations (the product xy).Thus, when prey species are present, there is a supply of food, so mid-level species are added to the system rate $exy\ ,\ \ e > 0$. But when top-level predator species are present, the mid-level species population is decreased by $gyz,\ g > 0$, decreased by the rate at which the mid-level species population are eaten during their encounters with the top predator species: Adding this rate to equation (15) gives a model for the mid-level species population:

$$\frac{dy}{dt} = -cy + exy - gyz. \qquad (16)$$

Similarly, a model for the top-level species population can be found as

$$\frac{dz}{dt} = -hz + lyz\ , l > 0. \qquad (17)$$

Equations (14), (16) and (17) constitute a system of nonlinear ordinary differential equations. Then the model we proposed to study becomes

$$\begin{aligned} \frac{dx}{dt} &= ax - bxy \\ \frac{dy}{dt} &= -cy + exy - gyz \\ \frac{dz}{dt} &= -hz + lyz \end{aligned} \qquad (18)$$

where x, y and z take nonnegative values because populations are nonnegative and

- a - natural growth rate of the prey in the absence of mid-level predator y.
- b - effect of predation on the prey x.
- c - natural death rate of the mid-level predator y in the absence of prey x.
- e - efficiency and propagation rate of the mid-level predator y in the absence of prey x.
- g - effect of predation on the species y by species z.
- h - natural death rate of predator z in the absence of prey.
- l - efficiency and propagation of rate of predator z in the presence of prey.

In the absence of top predator ($z = 0$), the model reduces to:

$$\begin{cases} \dfrac{dx}{dt} = ax - bxy \\ \dfrac{dy}{dt} = -cy + exy \end{cases}$$

whose solution in the xy plane is of the form:

$$y^a x^c = Ee^{ex+by}$$

where E is integration constant.

In the absence of mid-level species ($y = 0$), the model reduces to:

$$\begin{cases} \dfrac{dx}{dt} = ax \\ \dfrac{dz}{dt} = -hz \end{cases}$$

which gives $z = Mx^{\frac{-h}{a}}$ for an integration constant M.

Finally, in the absence of bottom-level prey species ($x = 0$), the model reduces to:

$$\begin{cases} \dfrac{dy}{dt} = -cy - gyz \\ \dfrac{dz}{dt} = -hz + lyz \end{cases}$$

And has solution in the yz plane of the form $z^c y^{-h} = Ne^{-gz-ly}$ for an integration constant N. The equilibrium points of (18) are solution of the algebraic system

$$\begin{cases} (a - by)x = 0 \\ (-c + ex - gz)y = 0. \\ (-h + ly)z = 0 \end{cases}$$

As a result, we have three equilibrium points (x_0, y_0, z_0) located at $(0,0,0)$, $(c/e\,, a/b\,, 0)$ and $(0, h/l\,, -c/g)$. Since the right hand sides of system (18) have continuous partial derivatives in x, y and z, it can be linearized at one of the equilibrium (x_0, y_0, z_0) and the associated linearized system takes the form

$$\begin{cases} \dfrac{dx}{dt} \approx (a - by_0)(x - x_0) - bx_0(y - y_0) \\ \dfrac{dy}{dt} \approx ey_0(x - x_0) + (-c + ex_0 - gz_0)(y - y_0) - gy_0(z - z_0) \\ \dfrac{dz}{dt} \approx lz_0(y - y_0) + (-h + ly_0)(z - z_0) \end{cases}$$

whose Jacobian matrix is given by

$$J(x_0, y_0, z_0) = \begin{bmatrix} a - by_0 & -bx_0 & 0 \\ ey_0 & -c + ex_0 - gz_0 & -gy_0 \\ 0 & lz_0 & -h + ly_0 \end{bmatrix}. \qquad (19)$$

The Jacobian matrix (19) at the equilibrium point $(0, 0, 0)$ takes the form

$$J(0,0,0) = \begin{bmatrix} a & 0 & 0 \\ 0 & -c & 0 \\ 0 & 0 & -h \end{bmatrix}$$

with eigenvalues $\lambda_1 = a$, $\lambda_2 = -c$ and$\lambda_3 = -h$ and corresponding eigenvectors $\langle 1,0,0\rangle, \langle 0,1,0\rangle$ and $\langle 0,0,1\rangle$ respectively.Since there is an eigenvalue with positive real part, namely a, the equilibrium point $(0,0,0)$ is unstable. The general solution is written in the form of the linear combination of eigenvalue and its corresponding eigenvector. That is,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{at} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{-ct} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-ht}$$

where c_1, c_2, c_3 are arbitrary constants.

The long term behavior of this general solution is

$$\text{As} \quad t \to \infty, \quad \begin{cases} x(t) \to \infty \\ y(t) \to 0 \\ z(t) \to 0 \end{cases}$$

The physical interpretation of this solution is that of the mid-level and top-level predators were eradicated, the prey population would grow in this simple model.

The Jacobian matrix (19) at the equilibrium point $c/e, a/b$ is of the form

$$J(c/e, a/b, 0) = \begin{bmatrix} 0 & \frac{-cb}{e} & 0 \\ \frac{ae}{b} & 0 & \frac{-ag}{b} \\ 0 & 0 & \frac{-hb+al}{b} \end{bmatrix}$$

with eigenvalues

$$\lambda_1 = \frac{la - hb}{b}, \qquad \lambda_2 = -i\sqrt{ac}, \qquad \lambda_3 = i\sqrt{ac}$$

and corresponding eigenvectors respectively:

$$\langle 1, \frac{(hb-al)e}{b^2c}, \frac{ab^2ce + b^2eh^2 - 2abehl + a^2el^2}{ab^2cg} \rangle, \langle 1, \frac{ie\sqrt{ac}}{bc}, 0 \rangle, \langle 1, \frac{-ie\sqrt{ac}}{bc}, 0 \rangle$$

Thus, the equilibrium point is stable if $la - hb < 0$ and unstable if $la - hb > 0$.

For $la - hb = 0$, the jacobian matrix evaluated at this equilibrium point have three eigenvalues with zero real part. Thus, each such equilibrium point is stable. The stability of this equilibrium point is of significance. Since it is stable, non-zero populations might be attracted towards the equilibrium point and as such the dynamics of the system might lead towards the extinction of all the three species for many cases of initial population levels.

In this case, the general solution becomes

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ \frac{(hb-al)e}{b^2c} \\ \frac{ab^2ce + b^2eh^2 - 2abehl + a^2el}{ab^2cg} \end{pmatrix} e^{\left(\frac{la-hb}{b}\right)t} + c_2 \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cos\sqrt{ac}t - \begin{pmatrix} 0 \\ \frac{e\sqrt{ac}}{bc} \\ 0 \end{pmatrix} \sin\sqrt{ac}t \right]$$

$$+ c_3 \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} sin\sqrt{ac}t + \begin{pmatrix} 0 \\ \frac{e\sqrt{ac}}{bc} \\ 0 \end{pmatrix} \cos\sqrt{ac}t \right]$$

where $c_{1,}c_{2,}c_3$ are arbitrary positive constants.

The long term behavior of the general solution is illustrated in the table bellow

For $la - hb > 0$

	$x(t) \to \infty$
As $t \to \infty$	$y(t) \to -\infty$
	$\begin{cases} z(t) \to \infty, \text{if } ab^2ce + b^2eh^2 - 2abehl + a^2el > 0 \\ z(t) - \infty, \text{if } ab^2ce + b^2eh^2 - 2abehl + a^2el < 0 \end{cases}$

3.3 Single Tank Mixture Problem Model

Suppose that we have two chemical substances where one is solvable in the other, such as salt and water. Suppose that we have a tank containing a mixture of these substances, and the mixture of them is poured in and the resulting "well-mixed" solution pours out through a valve at the bottom. Now, let's consider Fig. 2 with the following denotations

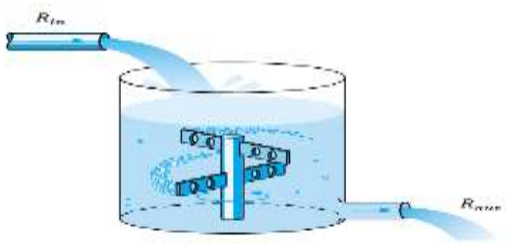


Figure2: mixing Solutions in the tank

- c_{in} ≡ concentration of salt in the solution being poured into the tank.
- c_{out} ≡ concentration of salt in the solution being poured out of the tank.
- R_{in} ≡ rate at which the salt is being poured into the tank.
- R_{out} ≡ rate at which the salt is being poured out of the tank.

There are two types of flow rates for which we can set up differential equations in connection with the mixing problem. Each of these is used to describe what is happening within the tank of liquid. These are volume flow rate and mass flow rate. The volume flow rate equation tells us how the amount of liquid in the tank is changing. The net rate of change of the volume in the tank is given by

$$\frac{dv}{dt} = \left(\frac{dv}{dt}\right)_{in} - \left(\frac{dv}{dt}\right)_{out}, \qquad (20)$$

where $v(t)$ is the volume of liquid present in the tank at time t measured from some chosen moment. By the law of conservation of salt, the two rates in the difference represent the constant rate at which liquid is being added to (input flow rate) and at which it is being drained from (output flow rate) the tank. The mass flow rate equation describes the net rate of change of the mass of dissolved substance in the tank.

$$\frac{dm}{dt} = \left(\frac{dm}{dt}\right)_{in} - \left(\frac{dm}{dt}\right)_{out} \qquad (21)$$

By definition of concentration, the solution found in the tank at a given time t has a mass of the substance given by the formula

$$m(t) = v(t).c(t).$$

Thus, for any of the mixing problems, the mass flow rate equation is given by

$$\frac{d}{dt}m(t) = \frac{d}{dt}[v(t).c(t)] = \left[\left(\frac{dv}{dt}\right)_{in}.c_{in}\right] - \left[\left(\frac{dv}{dt}\right)_{out}.c(t)\right] \qquad (22)$$

When the rate at which liquid is added is the same as the rate at which liquid is drained out of the tank, That is

$$\left(\frac{dv}{dt}\right)_{in} = \left(\frac{dv}{dt}\right)_{out} = \left(\frac{dv}{dt}\right)_{flow}$$

we have,

$$\frac{d}{dt}v(t) = \left(\frac{dv}{dt}\right)_{flow} - \left(\frac{dv}{dt}\right)_{flow} = 0$$

which is the same as to say $v(t)$is constant and $v(t) = v(0) = v_0$. On the other hand, the condition of having equal rates of filling and draining permits us to rewrite the mass flow rate equation (22) as

$$\frac{d}{dt}m(t) = \left[\left(\frac{dv}{dt}\right)_{flow}.c_{in}\right] - \left[\left(\frac{dv}{dt}\right)_{flow}.\frac{m(t)}{v(t)}\right] = \left(\frac{dv}{dt}\right)_{flow}.\left[c_{in} - \frac{m(t)}{v_0}\right].$$

Then, it follows that

$$\frac{dm}{c_{in} - \frac{m}{v_0}} = \left(\frac{dv}{dt}\right)_{flow} dt, \quad \text{for} c_{in} \neq \frac{m}{v_0}$$

with solution

$$-v_0 \ln\left|c_{in} - \frac{m}{v_0}\right| = \left(\frac{dv}{dt}\right)_{flow} t + C$$

Assuming the initial condition $m_0 := m(0) = v(0).c(0) = v_0.c_0$ gives

$$\ln\left|\frac{v_0 c_{in} - m_0}{v_0 c_{in} - m}\right| = kt, \ \text{where} \ k = \frac{\left(\frac{dv}{dt}\right)_{flow}}{v_0}.$$

cases	Substance mass	$m(t=0)$	$\lim_{t\to\infty} m(t)$
$c_{in} = \frac{m}{v_0}$	$m(t) = m_0 = v_0 c_{in}$	m_0	$v_0 c_{in}$
$c_{in} > \frac{m}{v_0}$	$m(t) = v_0 c_{in} - (v_0 c_{in} - m_0)e^{-kt}$	m_0	$v_0 c_{in}$
$c_{in} < \frac{m}{v_0}$	$m(t) = v_0 c_{in} + (m_0 - v_0 c_{in}).e^{-kt}$	m_0	$v_0 c_{in}$

In other words, the initial mass of substance in the tank is either $m_0 < v_0 c_{in}$ or $m_0 > v_0 c_{in}$both asymptotically approach the constant solution function $m(t) = v_0 c_{in}$

Analogously, equation (22) for the rate of change of concentration in the tank can be rewritten as

$$\frac{dc}{dt} = \frac{\left(\frac{dv}{dt}\right)_{flow}}{v_0}.(c_{in} - c)$$

whose solution is

$$\frac{c_{in} - c}{|c_{in} - c_0|} = e^{-kt}, \quad \text{for} c_{in} \neq c_0.$$

Upon examining the solution of the equation for concentration rate of change, we have the following properties

case	Concentration	$c(t=0)$	$\lim_{t\to\infty} c(t)$
$c_{in} = c$	$c(t) = c_0$	c_0	c_{in}
$c_{in} > c$	$c(t) = c_{in} - (c_{in} - c_o)e^{-kt}$	c_0	c_{in}
$c_{in} < c$	$c(t) = c_{in} + (c_0 - c_{in})e^{-kt}$	c_0	c_{in}

Similarly, the concentration functions satisfy the initial condition in the tank is either $c_0 < c_{in}$ or $c_0 > c_{in}$ both approach the constant solution function asymptotically. When the rate at which liquid added is faster than the rate at which liquid is drained out of the tank, we have

$$\frac{d}{dt}v(t) = \left(\frac{dv}{dt}\right)_{in} - \left(\frac{dv}{dt}\right)_{out} > 0$$

By assuming the input and output rates are constant, and letting $\Delta v = \left(\frac{dv}{dt}\right)_{net} = \left(\frac{dv}{dt}\right)_{in} - \left(\frac{dv}{dt}\right)_{out}$, The mass flow rate equation for this case becomes

$$\frac{dm}{dt} = \left[\left(\frac{dv}{dt}\right)_{in}.c_{in}\right] - \left[\left(\frac{dv}{dt}\right)_{out}.\frac{m(t)}{v_0 + \Delta v.t}\right] \qquad (22)$$

with solution

$$m(t) = \frac{C}{(v_0 + \Delta v.t)^{\Omega}} + c_{in}.(v_0 + \Delta v.t)$$

where C is integration constant and $\Omega = \frac{\left(\frac{dv}{dt}\right)_{out}}{\Delta v}$. Now, applying the initial condition $m(0) = m_0$ gives

$$m(t) = (m_0 - v_0.c_{in}).\left(\frac{v_0}{v_0 + \Delta v.t}\right)^{\Omega} + c_{in}.(v_0 + \Delta v.t) \qquad (23)$$

With the input flow rate being larger than the output flow rate, that it is generally impossible to have $\frac{dm}{dt} \to 0$. That is, the volume of liquid being added to the tank keeps increasing and, if it carries a nonzero concentration of dissolved substance, the mass of dissolved substance in the tank can only increase.

When the rate at which liquid is drained out of the tank is faster than the rate at which liquid is added, we have

$$0 < \left(\frac{dv}{dt}\right)_{out} - \left(\frac{dv}{dt}\right)_{in} = -\Delta v, \text{where } v(t) = C - \Delta v.t.$$

Applying the initial conditions, the solution of mass function is given by

$$m(t) = (m_0 - v_0.c_{in}).\left(\frac{v_0 - \Delta v.t}{v_0}\right)^{\Omega} + c_{in}.(v_0 - \Delta v.t) \qquad (24)$$

As $t \to \infty,\ m(t) \to -\infty$. Since the output flow rate being larger than the input flow rate, the volume of liquid being flow out of the tank keeps increasing and, if it carries a zero concentration of dissolved substance, the mass of dissolved substance in the tank can only decrease.

3.4 Multiple Tank Mixture Problem Model

Consider n tanks filled with brine, which are connected by a pair of pipes. One pipe brings brine from the i^{th} tank to the $(i+1)^{th}$ tank at a given rate, while the second pipe carries brine in the opposite direction (to the i^{th} tank) at the same rate, for $i = 1, 2, \dots, n-1$. Assuming that the initial concentrations in all tanks are known and that we have a perfect mixing, finding the concentrations in all tanks after a given period of time leads to a system of linear ordinary differential equations.

Consider the case where n tanks (with $n > 1$) of the same volume v are arranged in linear shape. We have a row of n tanks $T_1, T_2, \dots, T_n$ with neighboring tanks connected by a pair of pipes.

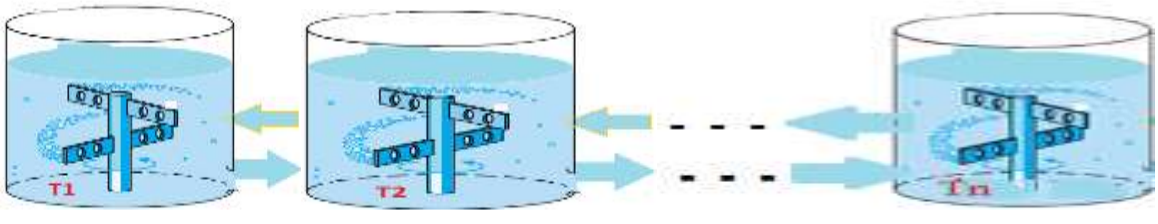


Figure 3: A linear arrangements of n tanks

Assume that the flow of Fig. 3 through each pipe is f gallons per unit of time. Consequently, the volume v in each tank remains constant. Let $x_i(t)$ be the amount of salt in tank T_i at time t. This yields the differential equations

$$x'_1(t) = -f\frac{x_1(t)}{v} + f\frac{x_2(t)}{v}$$

$$x'_i(t) = f\frac{x_{i-1}(t)}{v} - 2f\frac{x_i(t)}{v}) + f\frac{x_{i+1}(t)}{v}, \quad \text{for } 2 \le i \le n-1$$

$$x'_n(t) = f\frac{x_{n-1}(t)}{v} - f\frac{x_n(t)}{v}$$

Without loss of generality, we assume that$f = v$ and switch to the vector form and then the system in matrix form becomes

$$x' = Ax(t),$$

where

$$A = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -1 \end{bmatrix}, \quad x(t) = (x_1, x_1, \ldots, x_n)^T.$$

Clearly, A is tridiagonal and symmetric. Since A is a symmetric matrix, all eigenvalues must be real. According to definition 3 (In this case $a_{11} = a_{nn} = -1, r_1 = r_n = 1$, while $a_{ii} = -2, r_i = 2$), the eigenvalues of A are contained in the intervals of [-4, 0]. Now, applying definition 3 and linear recurrence relation Elayedi (2005) give

$$\det(A - \lambda I) = 2\cot\left(\frac{\gamma}{2}\right)\sin(n\gamma), \gamma\epsilon(0, \pi]$$

when $\gamma = \frac{k\pi}{n}$, for $k\epsilon\{1,2,\ldots,n\}$.

The eigenvalues of the coefficient matrix A is then given by

$$\lambda_k = -2cos\frac{k\pi}{n} - 2, \ k\epsilon\{1,2,\ldots,n\}.$$

It follows that the components of an eigenvector $V = (v_1, v_2, \ldots, v_n)$ corresponding to the eigenvalues λ_k are given by the relations

$$v_2 = -\left(1 + 2cos\frac{k\pi}{n}\right)v_1$$

where $v_1$is an arbitrary nonzero number. In general, while $v_i = -2\left(cos\frac{k\pi}{n}\right)v_{i-1} - v_{i-2}$, for $i\epsilon\{3,4,\ldots,n\}$, the general solution is given by

$$x(t) = \sum_{i=1}^{n} c_i v_i\, e^{\lambda_i t}.$$

Since, $\lambda_n = 0$, the corresponding eigenvectors have all components identical, and the remaining eigenvalues are negative, we conclude that the long term behavior of the general solution always approaches the state with all tanks containing the same amount of salt.

4. Conclusion

This paper attempted to discuss the application of first order ordinary differential equation in modeling phenomena of real world problems. The included models are Population growth and decay, Prey-predator interaction, mixing problems in a single tank and multiple tank systems. It is seen that it is possible to represent the population variations of prey and predator relationship to a certain extent of accuracy by mathematical model which is described by systems of non-linear order ordinary differential equations. The logistic model remedies the weakness of exponential model. That is, the exponential model predicts either the population grows without bound or it decays to extinction. But population cannot grow without bound as there can be competition for food, resources or space and this effect can be modeled by a logistic model by supposing that the growth rate depends on the population. It is further seen that finding the concentration of the mixed solution after a given period of time leads to the resulting well mixed solution. Finally, this paper believed that many problems of future technologies will be solved using ordinary differential equations.

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