

A new iterative technique for solving Van der pol equation

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Abstract

In this article we proposed a new iterative technique namely a semi analytic iterative technique for solve van der pol equation. The Semi analytic iterative technique suggested by Temimi and Ansari in 2011 and it is solved several problems in different areas which accuracy and efficiency in the results. The solution is acquired converge to the exact solution. Also an approximate solution is accuracy as we well show that in figures and tables for the analysis of maximum error reminder various. The software which used in the study for the calculations was MATHEMATICA 12.

Keywords: A sami analytic iterative technique, iterative techniques, Van der pol equation, maximum error reminder, exact solution.

1. Introduction

The Nonlinear differential equations become from the most important matter in different sciences various of the problem that arise in engineering fields and physics can be depicted by non linear ordinary differential equation such as Duffing equations (AL-Jawary & Abd- AL- Razaq, 2016), Emden- fowler equations, Falkner-Skan equation (Helmi Temimi & Ben-Romdhane, 2018), Harry Dym equation (Kumar et al., 2013) and many others.
Van der pol equation (VDP) was find out by Balthazar van der pol in 1920. This equations introduced in first to depicted the vibration for triode in the electrical circuit. In 1945, Littelwood and cartwright reduced the (VDP) equation with considerable value of non linear parameters and presented that the singular solution exist (Hafeez & Ndikilar, 2015). The problem of (VDP) equation have been studied in many aspects, such as the vibration amplitude control and synchronization dynamics (Khan et al., 2012a).
The mathematical model of this equation is:

$$y`` - (1 - y^2)y` + y = f(x) \tag{1}$$

The initial condition:

$y(0) = a_1. \quad y`(0) = a_2$

Van der pol equation clearly it is nonlinear differential equation from second order. This equation solved by various researcher in many approaches for example the Laplace decomposition method (Khan et al., 2012a), Homotopy analysis method (Mishra et al., 2016), Homtopy prtupition transform method (Khan et al., 2012a) and many others. In this article solving van der pol equation by a new iterative technique namely Temimi, Ansari method (TAM) (H Temimi & Ansari, 2011). The TAM solved many problems likes Duffing equations (M. A. Al-Jawary & Al-Razaq, 2016), Linear and Nonlinear Partial Differential Equations (Mohammed, 2017), differential algebraic equation (M. Al-Jawary & Hatif, 2018) and others.

2. The basic thought of TAM

We write the differential equation as Eq. (1):

$$L\,(y(x)) + N\,(y(x)) + g(x) = 0, \qquad \ldots(2)$$

The boundary conditions

B(y,y')

Where x refer to the independent variable, L is a linear operator, N is the nonlinear operator y(x) is an unknown function, g(x) is a known function, and B is a boundary condition [8]. Definitely, here that L be the linear part of the deferential equation but it is similar to take view linear parts and add them to nonlinear part as necessary. The approach applied as follows, start by suppose that $y_0(x)$ is an initial valuation of the solution to the equation y(x) and is the solution of the equation:

$$L\,(y(x)) \;+\; f(x) \;=\; 0. \qquad \mathrm{B}\,(y_0 . y`_0(x)) \tag{3}$$

For mange following iterate to the solution, then solve the next equation:

$$L\left(y(x)\right) + f(x) \;+\; N\,(y_0(x)) \;=\; 0. \qquad B(y_1 . y`_1(x)) \tag{4}$$

Now, getting a simple easy iterative step which is effecting the solution of a linear set of equation

$$L\left(y_{n+1}(x)\right) + \; f(x) + \; N\left(y_n(x)\right) = \; 0. \quad B\left(y_{n+1} . y`_{n+1}(x)\right) \tag{5}$$

It is noted that every of the $y_i(x)$ our unique solutions to Eq. (2). We confirm that this iterative execution, though very easy to apply has advantage in that any solution is a perfection of the last iterate and as more and more iterations are taken the solution converges to the solution of Eq. (2). We propose that the convergence of the iteration be monitored using standard error observation procedures such as the maximal error reminder (MER) (H Temimi & Ansari, 2011).

3. TAM for solving van der pol equation

Here, will show the effectiveness the semi analytic iterative technique through using this method in solve examples of Van der pol equation. We calculate the error remainder by using the maximal error remainder (MER).

Example (1)

Let as write the Van der pol equation (Khan et al., 2012b):

$$y`` - y + y^2 + (y`)^2 = 1 \qquad \ldots(6)$$

with initial condition:

y(0) =2, y`(0) = 0,

now, apply the semi analytical iterative method as

$$L\,(y) = y``. \qquad N\,(y) = -y + y^2 + (y`)^2 \; and \; f\,(x) = \; 1 \tag{7}$$

Thus the initial problem is

$$\mathrm{L}\,(\,\mathrm{y}_0) = \; 1. \;\mathrm{with}\; \mathrm{y}_0(0) = \; 2\,. \quad \mathrm{y'}_0(0) = \; 0 \tag{8}$$

The next problems can be acquired from the following problem

$$L\,(y_{n+1}(x)) + f\,(x) + N\left(\,y_n(x)\right) = 0. \quad y_{n+1}(0) = 2. \;\; y'_{n+1}(0) = 0 \tag{9}$$

Solving the initial problem as follows

$$\mathrm{y''}_0(\mathrm{x}) = 1. \tag{10}$$

Now, we integrate the two parties as

$$\int_0^x y''_0(x)dx = \int_0^x dx. \tag{11}$$

Then we have after we apply the condition $y'_0(0) = 0$ in Eq. (11)

$$y'_0(x) = x \tag{12}$$

Now we integrate again as

$$\int_0^x y'_0(x)dx = \int_0^x (t)\, dx \tag{13}$$

$$y_0(x) - y_0(x) = \frac{x^2}{2} \tag{14}$$

Then we have after applied the initial condition

$$y_0(x) = 2 + \frac{x^2}{2} \tag{15}$$

In this way we solve the other problems in this article.

The second iteration will be

$$y''_1 = 1 + y_0 - {y_0}^2 - (y_0`)^2 \text{ , with } y_1(0) = 2 \text{ and } y'_1(0) = 0 \tag{16}$$

And has a solution

$$y_1(x) = 2 - \frac{x^2}{2} - \frac{5x^4}{24} - \frac{x^6}{120} \tag{17}$$

The second iteration u_2 is:

$$y''_2 = 1 + y_1 - {y_1}^2 - (y_1`)^2 \text{ with } y_2(0) = 2 \text{ and } y'_2(0) = 0 \tag{18}$$

We solve it than we have

$$y_2(x) = 2 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{31x^6}{720} - \frac{11x^8}{630} - \frac{389x^{10}}{259200} - \frac{43x^{12}}{950400} - \frac{x^{14}}{2620800} \tag{19}$$

And so on

$$y(x) = \lim_{n\to\infty} y_n(x) \tag{20}$$

$$y(x) = 2 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320} - \frac{3511x^{10}}{3628800} - \frac{1549x^{12}}{8553600} - \cdots \tag{21}$$

This solution lead to the following form:

$$y(x) = 1 + \cos x \tag{22}$$

This form is the exact solution of Eq (6) (Khan et al., 2012b).

Example (2)

Let as write the Van der pol equation (Khan et al., 2012b):

$$y`` + y` + y^2y` + y = 2\cos x - cos^3x \tag{23}$$

The initial condition is:

$y(0) = 0.\quad y`(0) = 1$

$f(x)$ is series of cos x

$$f(x) = 2 * \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots\right) - \left(1 - \frac{3x^2}{2} + \frac{7x^4}{8} - \frac{61x^6}{240} + \cdots\right) \tag{24}$$

Then

$$f(x) = 1 + \frac{x^2}{2} - \frac{19x^4}{24} + \frac{181x^6}{720} - \frac{1639x^8}{40320} + \cdots \tag{25}$$

Now, apply the semi analytic iterative method as:

$$L\,(y) = y'',\qquad N\,(y) = y` + y^2y` + y \text{ and } f\,(x) = 1 + \frac{x^2}{2} - \frac{19x^4}{24} + \frac{181x^6}{720} - \frac{1639x^8}{40320} + \cdots \tag{26}$$

The initial problem is

$$L\,(y_0) = 1 + \frac{x^2}{2} - \frac{19x^4}{24} + \frac{181x^6}{720} - \cdots,\text{ with } y_0(0) = 0 \text{ and } y`_0(0) = 1 \tag{27}$$

From acquired the iterative problem generating relation we getting the next problems

$$L\,(y_{n+1}(x)) + f(x) + N\left(y_n(x)\right) = 0.\quad y_{n+1}(0) = 0.\quad y`_{n+1}(0) = 1 \tag{28}$$

After Solving eq. (28) we getting:

$$y``_0(x) = x + \frac{x^2}{2} + \frac{x^4}{24} - \frac{19x^6}{720} + \frac{181x^8}{40320} - \frac{1639x^{10}}{3628800} + \frac{14761x^{12}}{479001600} + \cdots, \tag{29}$$

The second iteration will be

$$y''_1 = 1 + \frac{x^2}{2} - \frac{19x^4}{24} + \frac{181x^6}{720} - y`_0 - {y_0}^2 y_0` - y_0\text{ , with } y_1(0) = 0 \text{ and } y'_1(0) = 1 \tag{30}$$

The solution of Eq. (30) is:

$$y_1(x) = x - \frac{x^3}{3} - \frac{x^4}{12} - \frac{13x^5}{120} - \frac{5x^6}{72} - \frac{41x^7}{5040} - \cdots. \tag{31}$$

The second iteration y_2 is

$$y''_2 = 1 + \frac{x^2}{2} - \frac{19x^4}{24} + \frac{181x^6}{720} - y`_1 - {y_1}^2 y_1` - y_1 \text{ with } y_2(0) = 0 \text{ and } y'_2(0) = 1 \tag{32}$$

Now solve it than we get

$$y_2(x) = x - \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{30} + \frac{x^6}{20} + \frac{41x^7}{1680} + \frac{43x^8}{6720} + \cdots. \tag{33}$$

And so on

$$y(x) = \lim_{n\to\infty} y_n(x) \tag{34}$$

$$y(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{2520} - \frac{x^8}{6720} - \frac{5x^9}{10368} \tag{35}$$

The closed form for y(x) is:

$$y(x) = \sin x \tag{36}$$

This form is the exact solution of Eq. (23) (Khan et al., 2012b).

Example (3)

We next consider the Van der pol equation (Mishra et al., 2016):

$$y`` - y` + y^2 y` + y = 0 \tag{37}$$

The initial condition is:

$y(0) = 1, \quad y'(0) = 0$

We apply TAM as follows

$$L(y) = y''; \quad N(y) = -y` + y^2 y` + y \text{ and } \; f(x) = 0 \tag{38}$$

The initial problem is:

$$L(y_0) = 0 \quad with \;\; y_0(0) = 1. y'_0(0) = 0 \tag{39}$$

From the iterative problem generating relation getting the following problem

$$L\left(y_{n+1}(x)\right) + f(x) + N\left(y_n(x)\right) = 0 \; with \; y_{n+1}(0) = 1. \quad y'_{n+1}(0) = 0 \tag{40}$$

We solve eq. then we have

$$y_0 = 1 \tag{41}$$

Then the next iteration can be writing as the following problem:

$$y''_1 = y_0` - {y_0}^2 y_0` - y_0; \quad \text{with } \; y_1(0) = 1 \text{ and } y'_1(0) = 0. \tag{42}$$

And has a solution

$$y_1 = 1 - x^2 \tag{43}$$

Applying the process of TAM for y_2 as following:

$$y''_2 = y_1` - {y_1}^2 y_1` - y_1; \;\; \text{with } \; y_2(0) = 1. \quad y'_2(0) = 0. \tag{44}$$

Now, solving Eq. (44), then have

$$y_2 = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^5}{20} + \frac{x^7}{168}. \tag{45}$$

And so on

The series form for this solution is:

$$y(x) = \lim_{n\to\infty} y_n(x) \quad (46)$$

$$y(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^5}{20} - \frac{x^6}{720} + \frac{11x^7}{840} - \frac{251x^8}{40320} - \frac{17x^9}{10080} + \cdots. \quad (47)$$

Eq. (37) have an approximate solution this solution can be moreover studied by evaluating the maximal error remainder this shown in table (1). From figure (1) we can see clearly the points are lying on a linear line that mean the convergence is achieved from exponential rate.

Table 1: The MER for Eq. (37) using the semi analytic method.

N	MER
1	0.0040025
2	1.52497×10^{-6}
3	2.08438×10^{-10}
4	1.16573×10^{-14}
5	2.22045×10^{-16}

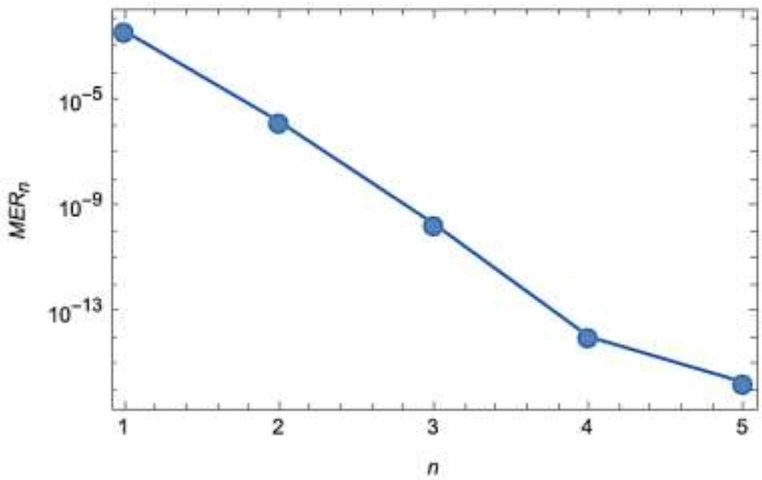

Figure 1- The MER in logarithmic plots when n is 1 through 5.

Example (4)

Consider (VDP) equation (Jafari et al., 2012):

$$y`` + 0.5(y^2 - 1)y` + y = 0 \quad (48)$$

The initial condition is:

y(0)= 0 ,   y`(0) = 1,

Applying the TAM as earlier as

$$L(y) = y``, \quad N(y) = 0.5(y^2 - 1)y` + y \quad and \quad f(x) = 0 \tag{49}$$

The initial problem is

$$\mathrm{L}(\mathrm{y}_0) = 0 \quad \text{with} \quad \mathrm{y}_0(0) = 0. \quad \mathrm{y}`_0(0) = 1. \tag{50}$$

From the iterative problem generating relation getting the following problem

$$L\big(y_{n+1}(x)\big) + N\big(y_n(x)\big) = 0. with \;\; y_{n+1}(0) = 0. \; y`_{n+1}(0) = 1 \tag{51}$$

We solve Eq. (51) then we have

$$\mathrm{y}_0(\mathrm{x}) = \mathrm{x} \tag{52}$$

Then the next iteration can be write as the following problem:

$$\mathrm{y}``_1 = 0.5({y_0}^2 - 1){y_0}` + y_0 \quad \text{with} \quad \mathrm{y}_1(0) = 0 \quad \text{and} \quad \mathrm{y}`_1(0) = 1. \tag{53}$$

And has a solution

$$\mathrm{y}_1(\mathrm{x}) = 1.x + 0.25x^2 - 0.166667x^3 - 0.041667x^4. \tag{54}$$

Applying the process of TAM for y_2 as following

$$\mathrm{y}``_2 = 0.5({y_1}^2 - 1){y_1}` + y_1 \qquad \text{with } \mathrm{y}_2(0) = 0 \quad \text{and} \quad \mathrm{y}`_2(0) = 1 \tag{55}$$

This has a solution

$$\mathrm{y}_2(\mathrm{x}) = 1.x + 0.25x^2 - 0.125x^3 - 0.08333x^4 - 0.020833x^5 + \cdots. \tag{56}$$

And so on

The series form for this solution is:

$$\mathrm{y}(\mathrm{x}) = \lim_{\mathrm{n}\to\infty} \mathrm{y}_\mathrm{n}(\mathrm{x}) \tag{57}$$

$$\mathrm{y}(\mathrm{x}) = 1.x + 0.25x^2 - 0.125x^3 - 0.08333x^4 - 0.020833x^5 + \cdots. \tag{58}$$

The solution of Eq. (48) is a numerical solution it is obtained in a series form. This solution explain the highest accuracy level for result by using the proposed method. Table 2 and figure 2 shows the highest accuracy of Eq. (48).

Table 2- The MER for Eq. (48) using the semi analytic method.

N	MER
1	0.0196171
2	0.00048987
3	8.15524×10^{-6}
4	1.01773×10^{-7}
5	1.01563×10^{-9}

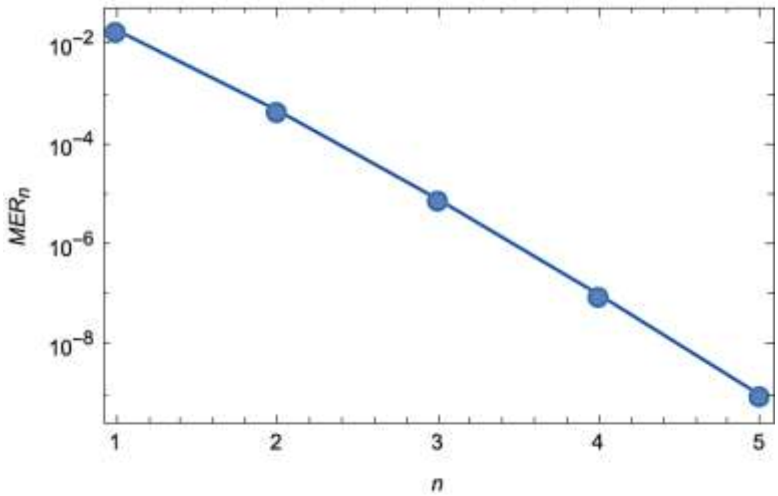

Figure 2- The MER in logarithmic plots when n is 1 through 5.

4. Conclusions

Here in this article we introduce a new iterative analytic method. This method namely (TAM) proposed to solve (VDP) equation. The TAM is characterized by simplicity in apply and find the exact solution also the high accuracy of the result in numerical examples for that considered the best of the many other methods such as Variational iteration method that needs to be complicated and many calculations. The software which used in the study for the calculations was MATHEMATICA 12.

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