

On the Stability Analysis of a Geometric Mean 4^{th} Order Runge-Kutta Formula

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Abstract
In the solution of non-stiff initial-value problems, sometimes stepsize is restricted by stability rather than by accuracy. Just as solutions to differential equations evolve with time, so do numerical approximations progress in small time steps. In each step an error is made and it is important to keep these errors small. But the error caused in one time-step may have an effect on the accuracy of later steps. It may be more important to control the buildup of errors as it is to control the size of the errors themselves. In numerical analysis the smallness of the individual errors is called accuracy and the ability to keep the effect of errors under control is called stability.

Physical systems give rise to system of ordinary differential equations with widely varying eigenvalues resulting precisely to the concept of stiffness. This paper seek to establish the region of absolute stability of a special 4^{th} Runge-Kutta formula first derived and implemented in Agbeboh et al (2007).

Keywords: Absolute Stability, geometric mean, one-third 4^{th} order Runge-Kutta formula and boundary locus.

1. Introduction and Definition of Terms

1.1 Introduction and Definition

The scheme developed here is adaptable to mildly stiff ordinary differential equation. A careful study the work of Hall (1986) Sangui and Evans (1986) gave rise to the derivation as can be seen in the work Agbeboh (2006). This scheme can cope with mildly stiff system because of some basic stability properties that they possessed. These properties are to be discussed here in broad sense, but before we examine them, we will consider some essential definitions needed for the analysis. According to Fatunla (1988) the conventional one-step numerical integrator for IVP (1.4) can be described as follows $y_{n+1} - y_n = h\phi(x_n, y_n; h) where, \phi(x_n, y_n; h) = \sum_{j=1}^{R} c_j k_j$ (1.1.1)

Where $\phi(x_n, y_n; h)$, the increment function, and h the meshsize adopted in the subinterval $[x_n, x_{n+1}]$. The absolute stability property of one step numerical process is normally investigated by applying the scheme to the scalar initial value problem $y' = \lambda y$? (1.1.2)

Where y is a complex constant with negative real party the resultant equation is the first order differential equation $y_{n+1} = r(\mu) y_n$, $\mu = \lambda h$, $y_{n+1} = \mu(X) y_n$, $z = \lambda h$ (1.1.3)

The function $r(\mu)$ is called the stability function which is either a polynomial or a rational function in μ. The parameters in the one-point method are sometimes chosen as to ensure that μ z) is an approximation to e^{μ}.

For many numerical methods, stability for the "zero problem" $y' = 0$, is just as important as high order.

The significance of this scheme can be seen in linear multistep methods. But all Runge–Kutta methods are stable for the zero problems. However, the interested here is in stability for linear problems like $y' = \lambda y$ because for some values of λ the stepsize needs to be restricted to get acceptable answers.

The stability region is the set of points in the complex plane such that for $z = \lambda h$, we get a bounded sequence of approximations. Since we pass on r quantities as we go from step to step, these quantities can have a variety of possible meanings. We express this in terms of a mapping S from the space of possible initial values to a space with r times the dimension. When the required number of steps is completed, we have to map back on to the state space to give a single final result. Butcher (1987) considered the stability regions for explicit method by using a Runge-Kutta method given by

$$Y_1 = y_{n-1}$$

$$Y_2 = y_{n-1} + ha_{12} f(Y_1)$$

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$$Y_s = y_{n-1} + h[a_{s,1} f(Y_1) + a_{s,2} f(Y_2) + \ldots + a_{s,s-1} f(Y_{s-1})]$$

$$y_n = y_{n-1} + h[b_1 f(Y_1) + b_2 f(Y_2) + \ldots + b_s f(Y_s)]$$

Using the standard test problem and writing $z = hq$ as equal, we see that these can be rewritten as

$$Y = y_{n-1} e + zAY \qquad (1.1.4a)$$

$$y_n = y_{n-1} + zb^T Y \qquad (1.1.4b)$$

Where $e = [1,1,\ldots,1]^T$, $Y = [Y_1, Y_2, \ldots, Y_s]^T$ *and* $b^T = [b_1, b_2, \ldots, b_s]$.

The polynomial r which determine the stability of this method is given by

$r(z) = \dfrac{y_n}{y_{n-1}} = 1 + zb^T(y_{n-1}^{-1} Y)$, and , from (1.1.4a), $y_{n-1}^{-1} Y = (I + zA + z^2 A^2 + \ldots + z^{s-1} A^{s-1})e$.

Hence, we have $r(z) = 1 + (I + zA + z^2 A^2 + \ldots + z^{s-1} A^{s-1})e.$

He said if the method in question is of order p then, because using this fact, we find that

$$r(z) = 1 + z + \frac{z^2}{2!} + \ldots + \frac{z^p}{p!} + c_{p+1} z^{p+1} + \ldots + c_s z^s,$$

Where the coefficients $c_{p+1} \ldots c_s$ are equal to $\Phi([p^\tau]_p), \ldots \Phi([s-1^\tau]_{s-1})$.

Following the procedure stated above, we can we can establish the stability region of this new Runge-Kutta formula as will be seen in section 3. Before doing that we first define some very relevant terms that will make for clearer understanding of work.

1.2. **Definition of terms**

Definition 1: The initial value problem in $y' = f(x, y)$ $a \le x \le b$, $y(a) = y_0$ (1.2.1)

is said to be stiff in the interval R = a<x<b, if for $X \in R$, if the eigenvalues λi (x) of the Jacobian $\dfrac{\partial y}{\partial x}$ satisfy

Re λi (x) < 0, i= 1,2 …………………m (1.2.2)

$$\frac{\max|\mathrm{Re}(\lambda i)|}{\min|\mathrm{Re}(\lambda i)|} \gg 1 \tag{1.2.3}$$

The ratio : $S = \frac{\max|\mathrm{Re}(\lambda i)|}{\min|\mathrm{Re}(\lambda i)|}$ (1.2.4)
called the stiffness Ratio.

If the partial derivative appearing in the Jacobian $\frac{\partial f}{\partial y}$ are continuous and bounded in an approximate region, then the Lipschitz constant L may be defined as $L = \left\| \frac{\partial f}{\partial y} \right\|$ (1.2.5)

if $\max|\mathrm{Re}\lambda| >> 0$, it follows that $L >> 0$.

Thus, stiff system is occasionally referred to as system with large Lipschitz constants. Associated with stiff systems is the absolute stability requirement of numerical process which however use discretization process with step length h.

Definition 2: A one-step scheme is said to be absolutely stable at a point μ in the complex plane provided the stability function $y_i(\gamma\mu)$ defined in 5 satisfies the following condition

$$|\lambda(\mu)| < 1? \tag{1.2.6}$$

and the corresponding region of absolute stability is $R = \{\mu : |\lambda y(\mu)| < 1\}$ (1.2.7)

Definition 3: The numerical integration scheme is said to be A-stable provided that the region of absolute stability in (1,1.3) include the entire left hand of the complex plane.

A-stability concept was introduced by Dalhquist and Bjorck (1974) as a very desirable property for any numerical integration algorithm, particularly if the IVP are stiff and highly oscillatory. But A-stability requirement is rather too stringent, weaker and a less desirable stability criterion which accommodates higher orders has since been proposed. These include A(α)-stability introduce by Jain (1967) stiff- Stability Lambert (1995), A(0)-stability and A(0)-stability,
Butcher (1987)

Definition 4: A numerical process is said to be A(α)-stable for x $\in (0, \pi/2)$ if it's solution $\{y_n\}$ $\to 0$ as n $\to \infty$ when this process is applied with fixed position into the test problem in (1.2,1) where in this case $\lambda \in (s(\alpha))$ and $s(\alpha) = \{z \in c : = 0, |\arg(-z)| < \alpha\}$ or if its region of absolute stability is continuous in the infinite wedge. S $W_\alpha = \{q = \lambda h, -\alpha < \pi - \arg q < \alpha\}$. See Agbeboh and Omokaro, (2011).

Also a numerical process is said to be A(0)-stable if it is A(α)-stable for all (some) $\alpha \in (0, \pi/2)$, such that $o < \alpha < \pi/2$.

Definition 5: A numerical method is said to be stiffly stable if (i) Its region of absolute stability contains R_1 and R_2 as shown in the figure 1 below: (ii) It is accurate for all q $\in R_2$ when applied to the scalar test equation (1.4) where $R_1 = \{\lambda h \mid R_c(\lambda h) < -a\}$

$R_2 = \{\lambda h | -a \le R_C(\lambda h) \le b_j, .. c \le |\mu(\lambda h) \le c\}$

and a, b and c are positive constants. The reason for this definition is to represent eigenvalue with rapidly decaying terms in the transient solution by corresponding λy in R_1.

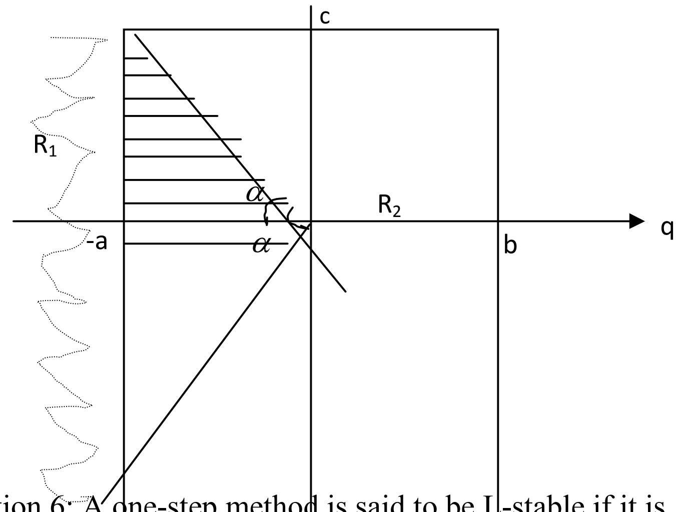


Figure 1

Definition 6: A one-step method is said to be L-stable if it is A-stable and, in addition when applied to the stable test equation $y' = \lambda y$, λ is a complex constant with $\operatorname{Re}\lambda < 0$, it yields $y_{n+1} = R(\lambda h)y_n$, when $|\operatorname{Re}(\lambda h)| \to 0$ as $\operatorname{Re}\lambda h \to \infty$

An L-stable one-step numerical method is necessarily A-stable and A(0)-stable. Based on this recognition, we shall establish the L-stable property in the explicit R – K proposed in sections four.

With the above definition and theorems, we will now analyze the stability properties of the family of three stage scheme.

We may now state the existence and uniqueness theorem without proof because the proof can be found in many textbooks on differential equations, such as, Ince (1956), Henrici (1962), Pontryagin (1962), and Codington and Levinson (1955).

Theorem1. Let f(x,y) be defined and continuous on R given by $R^{m+1} = [a,b]X\{y \mid y_\infty \le \tau < \infty\}$ and in addition satisfy the inequalities $f(x,y) - f(x,z)_\infty \le Ly - z_\infty$, where $\cdot_\infty$ is the maximum norm defined as $y_\infty = \max_{1\le r\le m}|y_r|$. Then the initial value problem (1.2.1) has a unique solution in R.

This theorem will be our guide in the proof of the stability region for the new method.

2.1. The Analysis of the Method

To analyze the stability properties, we shall adopt the definitions and theorems stated above and the derivation in Agbeboh et al (2007a, 2007b). where it was asserted that the geometric mean 4th order Runge-Kutta method can be obtained with rigorous expansion of the k_i's arising from the general RKM given as follows:

$$y_{n+1} - y_n = h\phi(x_n, y_n; h) where, \phi(x_n, y_n; h) = \sum_{j=1}^{R} c_j k_j \tag{2.1.1}$$

$$k_1 = f(x,y), \quad k_i = f(x + a_j h, y_n + h\sum_{J=1}^{R} b_{ji} k_i), \text{ for } j=i=2,3,4. \tag{2.1.2a}$$

$$a_j = \sum_{i=1}^{j=i} b_{ji} \quad \forall j = 2,3,...R \tag{2.1.2b}$$

Such that $y_{n+1} - y_n = \frac{h}{3}\left(\sqrt{k_1 k_2} + \sqrt{k_2 k_3} + \sqrt{k_3 k_4}\right)$ (2.1.3)

which is geometric in nature with:

$$k_1 = f(x_n, y_n) = f_n \tag{2.1.4a}$$

$$k_2 = f(x_n + a_2 h, y + h(b_{21}k_1)) = f(x_n + a_2 h, h a_2 k_1) \tag{2.1.4b}$$

$$\therefore a_2 = \sum_{i=1}^{z-1} b_{2i} = b_{21} \tag{2.1.4c}$$

$$k_3 = f(x_n + a_3 h, y + h(b_{31}k_1 + b_{32}k_2)) \tag{2.1.4d}$$

$$k_4 = f(x_n + a_4 h, y + h(b_{41}k_1 + b4_{42}k_2 + b_{43}k_3)) \tag{2.1.4e}$$

Where 4th order accuracy is obtained by choosing a_1 = ½, a_2 = 0, a_3 = ½, a_4 = 0 a_5 = 0 and a_6 = 1. We developed the geometric 4^{th} order formula as given below.
According to Jain (1983), the principle of Runge-Kutta process can be obtain from the mean-value theorem such that any solution of (1.1) satisfies

$$y(x_{n+1}) = y(x_n) + hy'(s_n) = y(x_n) + hf(s_n, y(s_n)) \tag{2.1.5}$$

Where $x_n = x_n + \theta_n, h \qquad 0 < \theta_n < 1$

If we put $\theta_n = 1/2$, by Euler's method with spacing $h/2$, we get

$$y(x_n, h/2) = y_n + \frac{h}{2} f(x_n, y_n). \tag{2.1.6}$$

Thus we have the approximation

$$y_{n+1} = y_n + hf(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)) \tag{2.1.7}$$

Equation (2.1.7) is regarded as $y_{n+1} = y_n + h$ (Average slope)
This is the underlying principle of the Runge-Kutta method.
In the above analysis the average slope are represented by k_i and is arithmetic in nature as the,(Classical Runge-Kutta Method).
In general we find the slope at x_n and at several other points by averaging these slopes, multiply by h, and add the result to y_n. Thus the Runge-Kutta method with v-shape can be written as

$$k_i = hf\left(x_n + c_i h, y_n + \sum_{j=1}^{i=1} a_{ij} k_j\right), c_i = 0, i = 1,2,3,......r \tag{2.1.8}$$

$$y_{n+1} = y_n \sum_{t+1}^{r} w_i k_j \tag{2.1.9}$$

Where the parameters c_i, a_{ji} and w_i are arbitrary.
From (2.1.8) we may interpret the increment function as the linear combination of the slopes at x_n and other points between x_n and x_{n+1}.
To obtain specific values for the parameters we expand y_{n+1} in powers of h such that it agrees with the Taylor series expansion to a specific number of terms.

Using the above principle we derived our new Rung-Kutta method by summarizing k_i as given in Agbeboh et al (2007).

2.2. SUMMARY OF K_2, K_3, K_4 and K_i^n for i = n = 2, 3, 4

$$k_2 = k_1 + ha_1k_1f_y + \frac{h^2}{2}a_1^2k_1^2f_{yy} + \frac{13}{6}a_1^3k_1^3f_{yyy} \quad (2.2.1)$$

$$k_2^2 = k_1^2 + 2ha_1k_1^2f_y + h^2a_1^2k_1^3f_{yy} + h^2a_1^2k_1^2f_y^2 + h^3a_1^3k_1^3f_yf_{yy} + \frac{h^3}{3}a_1^3k_1^4f_{yyy} \quad (2.2.2)$$

$$k_2^3 = k_1^3 + 3ha_1k_1^3f_y + 3h^2a_1^2f_y^2 + \frac{3}{2}h^2a_1^2k_1^4f_{yy} + \frac{1}{2}h^3a_1^3k_1^5f_{yyy} + 3h^3a_1^3k_1^4f_yf_{yy} + h2a_1^3k_1^3f_y^3 \quad (2.2.3)$$

$$k_3 = k_1 + h(a_2 + a_3)k_1f_y + h^2a_1a_3k_1f_y^2 + \frac{h^3}{2}\{a_1^2a_3 + 2(a_1a_2^3 + a_1a_2a_3)\}k_1^2f_yf_{yy}$$
$$+\frac{h^2}{2}(a_2 + a_3)^2k_1^2f_{yy} + \frac{h^3}{6}(a_2 + a_3)^3k_1^3f_{yyy} \quad (2.2.4)$$

$$k_3^2 = k_1^2 + 2h(a_2 + a_3)k_1^2f_y + h^2\{2a_1a_3 + (a_2 + a_3)^2\}k_1^2f_y^2 + h^2(a_2 + a_3)^2k_1^3f_{yy}$$
$$+2h^3a_1a_3(a_2 + a_3)k_1^2f_y^3 + \frac{h^3}{6}(a_2 + a_3)^3k_1^3f_{yyy} + h^3\left\{\begin{matrix}a_1^2a_3 + 2(a_1a_3^2 + a_1a_2a_3)\\ +(a_2 + a_3)^3\end{matrix}\right\}k_1^3f_yf_{yy} \quad (2.2.5)$$

$$k_4 = k_1 + h(a_4 + a_5 + a_6)k_1f_y + h^2\{a_1a_5 + a_6(a_2 + a_3)\}k_1f_y^2 + \frac{h^2}{2}(a_4 + a_5 + a_6)^2k_1^2f_{yy}$$
$$+h^3a_1a_3a_6k_1f_y^3 + \frac{h^3}{2}\{a_1^2a_5 + a_6(a_2 + a_3)^2 + 2(a_1a_5 + a_6(a_2 + a_3))(a_4 + a_5 + a_6)\}k_1^2f_yf_{yy}$$
$$+\frac{h^3}{6}(a_4 + a_5 + a_6)^3k_1^3f_{yyy} \quad (2.2.6)$$

$$k_4^2 = k_1^2 + 2h(a_4 + a_5 + a_6)k_1^2f_y + h^2\{2a_1a_5 + 2a_6(a_2 + a_3) + (a_4 + a_5 + a_6)^2\}k_1^2f_y^2$$
$$+h^2(a_4 + a_5 + a_6)^2k_1^3f_{yy} + \frac{h^3}{3}(a_4 + a_5 + a_6)^3k_1^4f_{yyy} + 2h^3\left\{\begin{matrix}a_1a_3a_6(a_4 + a_5 + a_6)\\ +(a_1a_5 + a_6a_2 + a_6a_3)\end{matrix}\right\}k_1^2f_y^3$$
$$+h^3\{a_1^2a_5 + a_6(a_2 + a_3)^2 + 2(a1a5 + (a_2a_6 + a_3a_6)(a_4 + a_5 + a_6)) + (a_4 + a_5 + a_6)^3\}k_1^3f_yf_{yy} \quad (2.2.7)$$

$$k_4^3 = k_1^3 + 3h(a_4 + a_5 + a_6)k_1^3f_y + 3h^2\{a_1a_5 + a_6(a_2 + a_3) + (a_4 + a_5 + a_6)^2\}k_1^3f_y^2$$
$$+\frac{3}{2}h^2(a_4 + a_5 + a_6)^2k_1^4f_{yy} + \frac{h^3}{2}(a_4 + a_5 + a_6)^3k_1^5f_{yyy}$$
$$+h^3\{3a_1a_3a_6 + 6(a_4 + a_5 + a_6)(a_1a_5 + a_6a_2 + a_6a_3) + (a_4 + a_5 + a_6)^3\}k_1^3f_y^3$$
$$+\frac{3}{2}h^3\{a_1^2a_5 + a_6(a_2 + a_3)^2 + 2(a_1a_5 + a_6a_2 + a_6a_3(a_4 + a_5 + a_6)) + 2(a_4 + a_5 + a_6)^3\}k_1^4f_yf_{yy} \quad (2.2.8)$$

accordingly, we evaluate

$$y_{n+1} = y_n + \frac{h}{3}\left(\sqrt{k_1k_2} + \sqrt{k_2k_3} + \sqrt{k_3k_4}\right), \quad (2.2.9)$$

to get values the parameters a_i using a binomial expansion strategy with the help of the reduce formula manipulation package. To do this set the binomial expansion of

$$(1+x)^{1/2} = 1+\frac{1}{2}x^2 - \frac{1}{8}x^2 + \frac{1}{16}x^3 ..... \quad (2.2.10)$$

To do this we write $\sqrt{k_{i-1}k_i} = f(1+x)^{\frac{1}{2}} \forall i = 1,2,3,4$

Such that $(k_1k_2)^{1/2} = f(1+x)^{1/2} \Rightarrow \frac{k_1k_2}{f^2}1 = x$

Evaluate Equation (ii) with x as given in (12) and by using equation (10) we obtain

$$\sqrt{k_1k_2} = 1+\frac{1}{2}\frac{(k_1k_2-1)}{f^2} = \frac{1}{8}\frac{(k_1k_2-1)^2}{f^2} + \frac{1}{16}\frac{(k_1k_2-1)^3}{f^2} + ...$$

$$\therefore \sqrt{k_1k_2} = 1+\frac{h}{2}a_1f_y + \frac{1}{4}h^2a_1^2k_1f_{yy} + \frac{h^3}{4}a_1^3k_1^2f_{yyy} - \frac{h^2}{8}a_1^2f_y^2 - \frac{h^3}{8}a_1^3k_1f_yf_{yy} + \frac{h^3}{16}a_1^3f_y^3 \quad (2.2.11)$$

Also

$$\sqrt{k_2k_3} = 1+\frac{1}{2}\left(\frac{k_2k_3}{f^2}-1\right) - \frac{1}{8}\left(\frac{k_1^2k_3^2}{f^4}\right) + \frac{1}{4}\left(\frac{k_2k_3}{f^2}\right) - \frac{1}{8} + \frac{1}{10}\left(\frac{k_2^3k_3^3}{f^6}\right) - \frac{3}{16}\left(\frac{k_2^2k_3^2}{f^4}\right) + \frac{1}{16}\left(\frac{k_2k_3}{f^2}\right) - \frac{1}{6}$$

$$\sqrt{k_2k_3} = 1+\frac{b}{2}(a_1+a_2+a_3)f_y + \frac{b^2}{16}\{8a_1a_3 + 4a_1(a_1+a_3) - 2a_1^2(a_2+a_3)\}f_y^2$$

$$+\frac{b^2}{4}\{a_1^2(a_2+a_3)^2\}k_1f_{yy} + \frac{b^3}{32}\begin{Bmatrix} 4a_1^2a_3 + 4(a_1a_3^2 + a_1a_2a_3) + 4a_1(a_2+a_3)^2 \\ +4a_1^2(a_2+a_3) - 4(a_2+a_3)3 - 4a_1^3k_1f_yf_{yy} \end{Bmatrix}$$

$$+\frac{b3}{12}\{(a_2+a_3)^3 + a_1^2\}k_1^2f_{yyy} + \frac{b^3}{16}\begin{Bmatrix} 4a_1^2a_3 - a_1(a_2+a_3)^2 - a_1^2(a_2+a_3) + 4a_1a_3 \\ +(a_2+a_3)^3 - a_1^3 \end{Bmatrix}f_y^3$$

Similarly,

$$\sqrt{k_3k_4} = 1+\frac{h}{2}\{(a_2+a_3)+(a_4+a_5+a_6)\}f_y$$

$$+\frac{h^2}{8}\{4(a_1a_3+a_1a_5) + 4a_6(a_2+a_3) + 4(a_4+a_5+a_6)((a_2+a_3)-(a_4+a_5+a_6)) - (a_2+a_3)^2\}f_y^2$$

$$+\frac{h^2}{4}\left\{(a_2+a_3)^2+(a_4+a_5+a_6)^2\right\}k_1 f_{yy}+\frac{h3}{16}\left\{\begin{array}{l}8a_1a_3a_6+4a_6(a_2+a_3)^2+4a_1a_5(a_2+a_3)\\+4a_1a_3(a_4+a_5+a_6)+(a_2+a_3)^3+(a_4+a_5+a_6)^3\\+4a_1a_5(a_4+a_5+a_6)-4a_1a(a_2+a_3)\\-4a_6(a_2+a_3)(a_4+a_5+a_6)-(a_2+a_3)(a_4+a_5a_6)^2\\-(a_2+a_3)^2(a_4+a_5+a_6)\end{array}\right\}f_y^3$$

$$+\frac{h^3}{12}\left[(a_4+a_5+a_6)^3+(a_2+a_3)^3\right]k_1^2 f_{yyy}+\frac{h3}{8}\left\{\begin{array}{l}2a_1^2a_3+2a_1^2a_5+4a_1a_3(a_2+a_3)\\+4a_1a_5(a_4+a_5+a_6)+2a_6(a_2+a_3)^2\\+4a_6(a_2+a_3)(a_4+a_5+a_6)\\+(a_2+a_3)^2(a_4+a_5+a_6)-(a_2+a_3)^3\\-(a_4+a_5+a_6)^3\end{array}\right\}$$

Substituting in

$$y_{n+1}-y_n=\frac{h}{3}\left(\sqrt{k_1k_2}+\sqrt{k_2k_3}+\sqrt{k_3k_4}\right)$$

we have

$$y_{n+1}-y_n=h+\frac{h^2}{6}\{2a_1+2(a_2+a_3)+(a_4+a_5+a_6)\}f_y+\frac{h3}{12}\left\{2a_1^2+2(a_2+a_3)^2+(a_4+a_5+a_6)^2\right\}k_1 f_{yy}$$

$$+\frac{h^3}{24}\left\{\begin{array}{l}12a_1^2+4a_1a_3+2a_1(a_2+a_3)+4a_1(a_5+a_6)+4a_6(a_2+a_3)\\+2(a_2+a_3)(a_4+a_5+a_6)-(a_4+a_5+a_6)-2(a_2+a_3)^2\end{array}\right\}f_y^2$$

$$+\frac{h2}{48}\left\{\begin{array}{l}2a_1^3+4a_1^2a_3+8a_1a_3a_6+4a_1a_3(a_4+a_5+a_6)+4a_1a_5(a_2+a_3)+4a_6(a_2+a_3)^2\\+2(a_2+a_3)^3+(a_4+a_5+a_6)^3-a_1(a_2+a_3)^2-a_1^2(a_2+a_3)-8a_1a_3(a_2+a_3)\\-4a_1a_5(a_4+a_5+a_6)-(a_2+a_3)(a_4+a_5+a_6)\{(a_2+a_3)+(a_4+a_5+a_6)+4a_6\}\end{array}\right\}f_y^3$$

$$+\frac{h4}{36}\left\{2a_1^3+2(a_2+a_3)^3+(a_4+a_5+a_6)^3\right\}k^2 f_{yyy}$$

$$+\frac{h4}{24}\left\{\begin{array}{l}-2a_1^3+2a_1^2a_3+a_1(a_2+a_3)^2+2a_1^2(a_3+a_5)+a_1^2(a_2+a_3)+4a_1a_5(a_4+a_5+a_6)\\+2a_6(a_2+a_3)^2+(a_2+a_3)(a_4+a_5+a_6)\{4a_6+(a_2+a_3)+(a_4+a_5+a_6)\}\\-(a_4+a_5+a_6)^3\end{array}\right\}k_1 f_y f_{yy}$$

$$+0\left(h^5\right) \qquad (2.2.12)$$

By Taylor Series expansion of two variables, we have

$$y_{(x)}=\sum_{j=0}^{\infty}\frac{h^i}{j}\left(y_{(x_0)}\right)$$

So that $y(x)=y_{(x_0)}+hy'_{(x_0)}+\frac{h^2}{2!}y''_{(x_0)}+\frac{h^3}{3!}y'''_{(x_0)}+\frac{h^4}{4!}y^{(iv)}{}_{(x_0)}+0\left(h^5\right)$

in terms of y-functional derivatives, yield

$$y_{(x)} - y_{(x_0)} = hk_1 + \frac{h^2}{2}k_1 f_y + \frac{h^3}{6}kf_{yy} + \frac{h^3}{6}k_1^2 f_y^2 + \frac{h^4}{24}k_1^3 f_{yyy} + \frac{h^4}{6}k_1^2 f_y f_{yy} + \frac{h^4}{24}k_1 f_y^3 \qquad (2.2.13)$$

Comparing equation (2.2.12) with (2.2.13)
We have the following six equations in six unknowns

$$k_1 = 1 \qquad (2.2.14)$$

$$a_1 + 2(a_2 + a_3) + (a_4 + a_5 + a_6) = 3 \qquad (2.2.15a)$$

$$2a_1^2 + 2(a_2 + a_3)^2 + (a_4 + a_5 + a_6)^2 = 2 \qquad (2.2.15b)$$

$$\left\{\begin{array}{l} -2a_1^2 + a_1 a_3 + 2a_1(a_2 + a_3) + 4a_1(a_5 + a_6) + 3a_6(a_2 + a_3) \\ +4(a_2 + a_3)(a_4 + a_5 + a_6) - (a_4 + a_5 + a_6) - 3(a_2 + a_3)^2 \end{array}\right\} = 4 \qquad (2.2.15c)$$

$$\left(2a_1^3 + 2(a_2 + a_3)^3 + (a_4 + a_5 + a_6)^3\right) = 3/2 \qquad (2.2.15d)$$

$$\left\{\begin{array}{l} -2a_1^3 + 2a_1^2 a_3 + a_1(a_2 + a_3)^2 + 2a_1^2(a_3 + a_5) + a_1^2(a_2 + a_3) \\ +8a_1 a_3(a_2 + a_3) + 4a_1 a_5(a_4 + a_5 + a_6) + 2a_6(a_2 + a_3)^2 \\ +(a_2 + a_3)(a_4 + a_5 + a_6)\{4a_6 + (a_2 + a_3) + (a_4 + a_5 + a_6)\} \\ -2(a_2 + a_3)^3 - (a_4 + a_5 + a_6)^3 \end{array}\right\} = 4 \qquad (2.2.15e)$$

$$\left\{\begin{array}{l} 2a_1^3 + 4a_1^2 a_3 + 8a_1 a_3 a_6 + 4a_1 a_3(a_4 + a_5 + a_6) + 4a_1 a_5(a_2 + a_3) + 4a_6(a_2 + a_3)^2 \\ +2(a_2 + a_3)^2 + (a_4 + a_5 + a_6)^3 - a_1(a_2 + a_3)^2 - a_1^2(a_2 + a_3) - 8a_1 a_3(a_2 + a_3) \\ -4a_1 a_5(a_4 + a_5 + a_6) \\ -(a_2 + a_3)(a_4 + a_5 + a_6)\{4a_6 + (a_2 + a_3) + (a_4 + a_5 + a_6) + 4a_6\} \end{array}\right\} = 2 \qquad (2.2.15f)$$

Solving the above equations we obtain the values of the parameters a_1, a_2, a_3, a_4, a_5, and a_6 as given below. Since $k_1 = 1$ and k_1^2, k_1^3 etc remain 1.

For convenience and ease of solution we set $(a_2 + a_3) = 1/2$ and $(a_4 + a_5 + a_6) = 1$ and obtain a_1, after which we substitute in the equations to obtain values for other parameters. From equation (2.2.15a)
(2.2.15b) and (2.2.15d)
it can be seen that the value of a_1 remains consistently $1/2$ using the fact that
$a_1 = 1/2, (a_2 + a_3) = 1/2$ and $(a_4 + a_5 + a_6) = 1$ we continue as follows.

Solving equation (2.2.15c), (2.2.15e) and (2.2.15f) simultaneously with $a_1 = 1/2$,
we have $a_2 = -1/16$, $a_3 = 9/16$, $a_4 = -1/8$ $a_5 = 5/24$ and $a_6 = 11/12$

Substituting these values of a_i in equation (2,1,3) – (2.1.4) we have the required formula given by

$$y_{n+1} = y_n + \frac{h}{3}\left(\sqrt{k_1 k_2} + \sqrt{k_2 k_3} + \sqrt{k_3 k_4}\right). \qquad (2.2.16)$$

with

$$k_1 = f(y_n) \tag{2.2.17a}$$

$$k_2 = f\left(y_n + \frac{h}{2}k_1\right) \tag{2.2.17b}$$

$$k_3 = f(y_n + \frac{h}{16}h[-k_1 + 9k_2]) \tag{2.2.17c}$$

$$k_4 = f(y_n + \frac{h}{24}[-3k_1 + 5k_2 + 22k_3]) \tag{2.2.17d}$$

3. Stability of the Method

We establish the stability region for method by proving the following theorem.

Theorem 1 : If the initial value problem

$$y' = f(x, y), \quad y(a) = y_0. \quad where\ y = (y_1, y_2, \ldots y_m)^T, \quad y_0 = (\eta_1, \eta_2 \ldots, \eta_m)^T$$

(3.1)

fulfills the hypothesis of the existence theorem (cf Fatunla 1988), the method (2.2.16) is stable in the sense of definition 1.4.

Proof

Using test equation $y' = \lambda y$

We restate equations (2.2.16) to (2.2.17), so that

$$k_1 = \lambda y \tag{3.2}$$

$$k_2 = \lambda y\left(1 + \frac{1}{2}\lambda h\right) \tag{3.3}$$

$$k_3 = \lambda y\left(1 + \frac{1}{2}\lambda h + \frac{9}{32}\lambda^2 h^2\right) \tag{3.4}$$

$$k_4 = \lambda y\left(1 + \lambda h + \frac{9}{16}\lambda^2 h^2 + \frac{33}{128}\lambda^3 h^3\right) \tag{3.5}$$

And substituting these into the formula (2.2.16), we have

$$y_{n+1} - y_n = \frac{h}{3}\{\left(\lambda^2 y^2\left(1 + \frac{1}{2}\lambda h\right)\right)^{\frac{1}{2}} + \left(\lambda^2 y^2\left(1 + \frac{1}{2}\lambda h\right)\left(1 + 1 + \frac{1}{2}\lambda h + \frac{9}{32}\lambda^2 h^2\right)\right)^{\frac{1}{2}} + \left(\lambda^2 y^2\left(1 + \frac{1}{2}\lambda h + \frac{9}{32}\lambda^2 h^2\right)\left(1 + \lambda h + \frac{9}{16}\lambda^2 h^2 + \frac{33}{128}\lambda^3 h^3\right)\right)^{\frac{1}{2}}\} \tag{3.6}$$

$$y_{n+1}-y_n=\frac{\lambda hy}{3}\left[\begin{array}{l}\left(\left(1+\frac{1}{2}\lambda h\right)\right)^{\frac{1}{2}}+\left(\left(1+\frac{1}{2}\lambda h\right)\left(1+1+\frac{1}{2}\lambda h+\frac{9}{32}\lambda^2h^2\right)\right)^{\frac{1}{2}}\\+\left(\left(1+\frac{1}{2}\lambda h+\frac{9}{32}\lambda^2h^2\right)\left(1+\lambda h+\frac{9}{16}\lambda^2h^2+\frac{33}{128}\lambda^3h^3\right)\right)^{\frac{1}{2}}\end{array}\right] \quad (3.7)$$

Let $\lambda h=\mu$ and dividing throughout by y_n

$$\frac{y_{n+1}-y_n}{y_n}=\frac{\mu}{3}\left[\begin{array}{l}\left(\left(1+\frac{1}{2}\mu\right)\right)^{\frac{1}{2}}+\left(\left(1+\frac{1}{2}\mu\right)\left(1+\frac{1}{2}+\frac{9}{32}\lambda^2h^2\right)\right)^{\frac{1}{2}}\\+\left(\left(1+\frac{1}{2}\lambda h+\frac{9}{32}\lambda^2h^2\right)\left(1+\lambda h+\frac{9}{16}\lambda^2h^2+\frac{33}{128}\lambda^3h^3\right)\right)^{\frac{1}{2}}\end{array}\right] \quad (3.8)$$

Expanding the exponential terms, we have

$$\left(1+\frac{1}{2}\mu\right)^{\frac{1}{2}}=1+\frac{1}{4}\mu-\frac{1}{32}\mu^2+\frac{1}{128}\mu^3 \quad (3.9)$$

$$\left(1+\mu+\frac{17}{32}\mu^2+\frac{9}{64}\mu^3\right)^{\frac{1}{2}}=1+\frac{1}{2}\mu+\frac{9}{64}\mu^2 \quad (3.10)$$

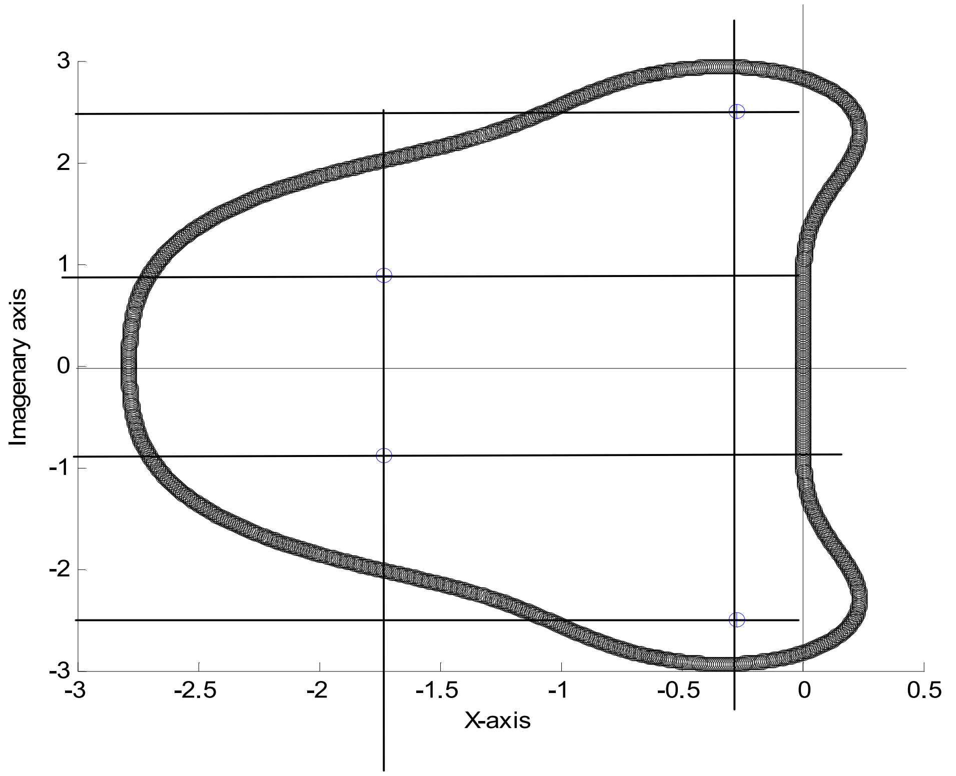
$$\left(1+\frac{3}{2}\mu+\frac{43}{32}\mu^2+\frac{105}{128}\mu^3\right)=1+\frac{3}{4}\mu+\frac{25}{54}\mu^2+\frac{15}{128}\mu^3 \quad (3.11)$$

$$\frac{y_{n+1}}{y_n}-1=\frac{\mu}{3}\left(3+\frac{3}{2}\mu+\frac{1}{2}\mu^2+\frac{1}{8}\mu^3\right) \quad (3.12)$$

$$\frac{y_{n+1}}{y_n}=1+\frac{1}{2}\mu+\frac{1}{6}\mu^2+\frac{1}{24}\mu^3 \quad (3.13)$$

Using MATLAB Package we obtain the following results for the stability region.

μ_1=-0.2708+2.5048i, μ_2= -0.2708-2.5048, μ_3= -1.7294+0.8890i and
μ_4= -1.7294 -0.8890i



MATLAB program for drawing the stability curve.

```
Q=0:0.001:2*pi;
a=zeros(4,length(Q));
for k=1:length(Q)
   c=[1/24 1/6 1/2 1 1-exp(i*Q(k))];
   a(:,k)=roots(c);
 end
hold on
plot(a(1,:),'ko')
plot(a(2,:),'ko')
plot(a(3,:),'ko')
plot(a(4,:),'ko')
P=[1/24 1/6 1/2 1 1];
R=roots(P);
plot(R,'+')
plot(R,'o')
hold off
```

4. Convergence of the Method

According to Lambert (2000), there are various one-step schemes in existence, but a method becomes useful only when it has properties like consistency, convergence and stability inherent in it. In this section, we will establish the consistency and convergence properties of the new algorithm, in line with Ababneh, Ahmed and Ismail (2009), who acknowledged that a one-scheme method is said to be consistent, if the difference equation of the computation formula exactly approximates the differential equation it intends to solve. In line with the general one step method stated in (1.1.1), we prove the following theorem.

Theorem 1: (Consistency and Convergence)

We assert that our method (2.2.16) to (2.2.17) is consistent and converges to a known function

$$y'(x) = f(x, y) \tag{4.1}$$

Proof: Recall Lambert and Johnson (1973) which has it that if a one-step method is proved to be consistent, then it is convergent.

In order to establish the convergence of the method, we show that (2.2.16) to (2.2.17) is consistent with the initial value problem (1.1.2); that is, $f(x, y, 0) = f(x, y)$ (4.2)

In line with Ababneh et al (2009), demonstrated the process of establishing the consistency of the method by first defining the local truncation error of the one-step method as follow:

Given

$$y(x_{n+1}) = y(x_n) + hf(x_n, y_n, h) + T_n(h) \tag{4.3}$$

where $T_n(h)$ =local truncation error, we assert (4.3), is the result of substituting the exact solution into the approximation of the ordinary differential equation by the numerical method. Taking the limits as $h \to 0$ of both sides, we have;

$$\lim_{h\to 0} T_n(h) = \lim_{h\to 0} \frac{y(x_{n+1}) - y(x_n)}{h} - f(x_n, y(x_n), h) \tag{4.4}$$

$$x_0 + nh = x \in (x_0, \beta) \tag{4.5}$$

$$T_n(h) \to y'(x) - f(x, y(x), 0) = 0 \tag{4.6}$$

$$\therefore f(x, y, 0) = f(x, y) \tag{4.7}$$

Hence the one-step method is consistent

Similarly, adopting the above principle we show that the method,

$$y_{n+1} = y_n + \frac{h}{3}\left(\sqrt{k_1k_2} + \sqrt{k_2k_3} + \sqrt{k_3k_4}\right). \tag{4.8a}$$

with

$$\mathrm{K}_1 = \mathrm{f}(\mathrm{y}_\mathrm{n}),\ \mathrm{K}_2 = \mathrm{f}(\mathrm{y}_\mathrm{n} + a_1hk_1),\ \mathrm{k}_3 = \mathrm{f}(\mathrm{y}_\mathrm{n} + \mathrm{h}(a_2\mathrm{k}_1 + a_3\mathrm{k}_2)),$$

$$\mathrm{k}_4 = \mathrm{f}(\mathrm{y}_\mathrm{n} + \mathrm{h}(a_4\mathrm{k}_1 + a_5\mathrm{k}_2 + a_6k_3)) \tag{4.8b}$$

is consistent and hence converges to the exact solution $y(x_n)$ of the initial value problem (3.1).

By substituting in equations (2.2.12) we have the following:

$$T_n(h^5) = y_{n+1} - y_n - \frac{h}{4}\left[\begin{array}{l}\left(f(y_n)*f(y_n+ha_1f(y_n))\right)^{\frac{1}{2}} + f\{(y_n+ha_1f(y_n))*f\begin{pmatrix}y_n+ha_2f(y_n)+\\ ha_3f(y_n)+a_1h(y_n)\end{pmatrix}\}^{\frac{1}{2}}\\ \left(f\begin{pmatrix}y_n+ha_2f(y_n)+\\ ha_3f(y_n)+a_1h(y_n)\end{pmatrix}*f\begin{pmatrix}y_n+ha_4f(y_n)+ha_5f(y_n)+\\ a_1h(y_n)+ha_6f\left(y_n+h\begin{pmatrix}a_2f(y_n)\\ +a_3f(y_n)+a_1h(y_n)\end{pmatrix}\right)\end{pmatrix}\right)^{\frac{1}{2}}\end{array}\right]$$

(4.9)

Dividing all through by h :

$$T_n(h) = \frac{y_{n+1} - y_n}{h} - \frac{1}{4}\left[\begin{array}{l}\left(f(y_n)*f(y_n+ha_1f(y_n))\right)^{\frac{1}{2}} + f(y_n+ha_1f(y_n)) + f\begin{pmatrix}y_n+ha_2f(y_n)+\\ ha_3f(y_n)+a_1h(y_n)\end{pmatrix}\\ \left(f\begin{pmatrix}y_n+ha_2f(y_n)+\\ ha_3f(y_n)+a_1h(y_n)\end{pmatrix}*f\begin{pmatrix}y_n+ha_4f(y_n)+ha_5f(y_n)+\\ a_1h(y_n)+ha_6f\left(y_n+h\begin{pmatrix}a_2f(y_n)\\ +a_3f(y_n)+a_1h(y_n)\end{pmatrix}\right)\end{pmatrix}\right)^{\frac{1}{2}}\end{array}\right]$$

(4.10)

Take the limit of both sides as h⟶0, we have

$$\lim_{h\to 0} T_n(h) = \lim_{h\to 0}\left(\frac{y_{n+1}-y_n}{h}\right) = \frac{1}{4}\left[f(y_n)+f(y_n)+f(y_n)+f(y_n)\right]$$

(4.11)

$$y'(y_n) = f(y_n)$$

(4.12)

hence the method is consistent and therefore converges;

4. Conclusion

Having successfully established the Region of Absolute Stability of the new method by using MATLAB to draw the curve of the stability polynomial it can be seen from the above proof and sketch that the method is A-stable. Consequently the method is L-Stable going by definition 6 of section1.2 above. This is because the region of absolute stability lies entirely on the right hand side of the complex plane as shown in the diagram where all the eigenvalues are completely embedded in the Jordan curve. The validity of our new method for numerical solution of first order initial value problems has been both theoretically and practically investigated. The results show that the method is consistent, convergent, stable, and of high accuracy. It is therefore suitable for the solution of real life problems that can possibly be reduced to initial value problems in ordinary differential equations.

REFERENCES

Ababneh O.Y., Amad R. and Ismail E.S. (2009), New Multi-Step Runge-Kutta Method. Applied Mathematical Science. J. Vol.3, No. 45, 2255-2262

Agbeboh G.U. et al. (2007a). On the Performance of a New One Third 4^{th} Order Runge- Kutta method (International Journal of Numerical Mathematics ISSN 1117-1812 IJNM 2(1). Pp 374-398)

Agbeboh G.U., Aashikpelokhai U.S.U. and Aigbedion. I. (2007) Implementation of a New 4^{th} order Runge- Kutta Formula for Solving Initial Value Problems (i.v.ps) (International Journal of Physical Sciences. 2(4) pp.089-098). Available Online at http:/www.academicjournals.org/IJPS. ISSN1992- 19950 Academic Journals).

Agbeboh G.U. & Omonkaro B. (2010). On the Solution of Initial Value Problems in Ordinary Differential Equations Using A New Third Order Inverse Runge-Kutta Method (International Journal of Physical Sciences 5(4) pp 299-307). Available Online at http:/www.academicjournals.org/IJPS. ISSN1992- 19950 Academic Journals).

Butcher J.C. (1987). The Numerical Analysis of Ordinary Differential Equation. Runge- Kutta and
general Linear methods. A Wiley Inter. Science Publications Printed and bound in Great Britain.
Dahlquist, G and Bjorck, A. (1974). "Numerical l Methods." (Translated by Anderson, N), Prentice Hill
Inc. Engle wood Cliffs. N.J.)

Fatunla. S. O. (1988). Numerical Methods for Initial Value Problems in Ordinary Differential Equations (Computer Science and Scientific Computing) Academic press INC 1250 Sixth Avenue, San Diago CA92101.

Hall George (1986). Equilibrium State of Runge-Kutta Scheme. AIM transactions on mathematical
software vol. 12 No. 3 pages 183-192.

Henrici, P. (1962). Discrete Variable Methods in ODEs." New York; John Wiley and Sons. Ince. E.L. (1956), "Ordinary Differential Equations, New York: Dover.

Lambert J.D. (1995). Numerical methods for Ordinary Differential Systems; The initial valu problem John Wiley & Sons New York.

Lambert J. D. (2000). Numerical Methods for Ordinary Differential Systems, the initial value Problem,
John Wiley &Sons Ltd.

Lawson, J.D. (1966). An Order Five R-K Process with Extended Region of Absolute Stability, SIAM Journal on Numerical Analysis 3, 593-597.

Lawson, J.D. (1967a). Generalized R-K Processes for Stable System with Large Lipschitz Constants.
SIAM Journal on Numerical Analysis 4, 372-380.

Pontryagin L.S. (1962). Ordinary Differential Equations, Reading Massachusetts. Addison-Wesley
Publishing Co. Inc.

Sangui, B.B. and Evans D.J. (1986). A New 4th Order Runge-Kutta Method for Initial Value Problems. Journal of Computational Mathematics II Proceedings of the Second International Conference on Numerical Analysis and Application 27-31, January 1986, Benin City, Nigeria.

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