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New Type of Generalized Closed Sets

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Abstract. In this paper, we introduce and study a new type of sets , namely strongly $b\tau$-closed (briefly, $b^{*}\tau$ - closed) set. This class is strictly between the class of closed sets and the class of gsg- closed sets. It is shown that the class of $b^{*}\tau$ - open sets forms a topology finer than τ . Relationships with certain types of closed sets are discussed and basic properties and characterizations are investigated. Further, new characterizations of normal spaces are provided and several preservation theorems of normality are improved.

***Key words and phrases**: $b\tau$ - closed sets, $b\tau$ - open sets, normal spaces.*

1. Introduction and Preliminaries.

The concept of generalized closed (briefly g-closed) sets was first introduced by Levine [16]. Arya et al. [2] defined generalized semi-closed (briefly gs- closed) sets. Bhattacharyya and Lehiri [3] introduced the class of semi- generalized closed sets (sg-closed sets). Maki, et al. [17, 18] introduced generalised α- closed and α-generalized closed closed sets (briefly, gα-closed, αg-closed). Ganster et al [2] introduced $b\tau$ - closed sets. Lellis et al. [13] introduce the class of gsg- closed sets. Jafari [11] and Donchev [8] introduced the concept of sg-compact spaces and studied their properties using sg-open and sg-closed sets.

Throughout this paper X and Y are topological spaces on which no separation axioms

are assumed unless stated explicitly. This paper consists of four sections.

In section 2, we introduce and study the class of $b\tau$ - closed sets and investigate its relations with certain types of closed sets.

In section 3, we derive several properties and characterizations of $b\tau$ -closed sets and $b\tau$ -open sets.

In section 4, we provide some applications of $b\tau$ - closed sets.

Let us recall the following definitions which are useful in the following sections.

Definition 1.1. A subset A of a topological space (X, τ) is called:

(1) a semi-open set [15] if $A \subseteq cl(int(A))$.

(2) a pre-open set [20] if $A \subseteq int(cl(A))$.

(3) an α-open set[22] if $A \subseteq int(cl(int(A)))$.

(4) a b- open set[2] if $A \subseteq \text{int}(cl(A) \cup cl(\text{int}(A)$

(5) a semi- pre-open set (β-open set) [1] if $A \subseteq cl(int(cl(A)))$.

(6) a regular open set [23] if $A = int(cl(A))$.

The complements of the above mentioned sets are called their respective closed sets. The semi closure [10] (resp. α-closure [18], b- closure[2]) of a subset A of X denoted by $scl(A)$ (resp. $\alpha cl(A)$, $bcl(A)$) is defined to be the intersection of all semi-closed (resp. α-closed, b- closed) sets containing A. The semi interior [10](resp. b- interior) of A denoted by $sint(A)$(resp. $bint(A)$[2]) is defined to be the union of all semi-open(resp. b- open) sets contained in A.

Definition 1.2 Let (X,τ) a topological space and A be a subset of X, then A is called

(1) generalized closed set [16](briefly g- closed) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.

(2) semi- generalized closed set [4] (briefly sg- closed) if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in X.

(3) a generalized semi-closed set [3](briefly gs- closed) if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.
(4) α- generalized closed set [17](briefly αg- closed) if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.
(5) a generalized α- closed set [17](briefly gα- closed) if $\alpha cl(A) \subseteq$ U whenever $A \subseteq U$ and U is α-open in X.
(6) $b\tau$ - closed set [9] if $bcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.
(7) w- closed set [23] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in X.
(8) α^{**}g- closed set [18] if $\alpha cl(A) \subseteq Int(cl(U))$ whenever $A \subseteq U$ and U is open in X.
(9) generalized semi- pre- closed set [7] (briefly gsp- closed)if $spcl(A) \subseteq U$ when- ever $A \subseteq U$ and U is open in X.
(10) a generalized pre- closed set [19] (briefly gp- closed) if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.
(11) a generalized sg- closed set [13](briefly gsg- closed) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is sg-open in X.

The complements of the above mentioned sets are called their respective open sets.

2. $b^*\tau$ - Closed Set and its Relationships.

Definition 2.1. A subset A of a topological space X is called a strongly $b\tau$ -closed set (briefly, $b^*\tau$ -closed) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $b\tau$ -open in X.

The family of all $b^*\tau$ -closed subsets of X is denoted by $B\tau^*C(X)$.

It is easy to prove.

Proposition 2.2. Let X a topological space and A be a subset of X, then:
(1) Every closed set is $b^*\tau$ - closed.
(2) Every $b^*\tau$ -closed is gsg- closed.

Remark 2.3. The converse of part (1) of Propositions 2.2 is true in general as shown in the following example.

Example 2.4. Let $X = \{a, b, c\}$ with $\tau = \{X, \varphi, \{a, c\}\}$.The set $\{a, b\}$ is $b^*\tau$ - closed but not closed set.

Question: Is there a set which is gsg- closed but not $b^*\tau$ - closed?

From Remark 3.2 of [13], Proposition 2.2 and well- known results we have the following relations.

Proposition 2.5. For any topological space X, we have,
(1) Every $b^*\tau$ -closed set is g-closed.
(2) Every $b^*\tau$ -closed set is w-closed.
(3) Every $b^*\tau$ -closed set is αg-closed.
(4) Every $b^*\tau$ -closed set is α^{**}g–closed.
(5) Every $b^*\tau$ -closed set is gα-closed and pre-closed.

(6) Every $b^*\tau$ -closed set is sg- closed and β-closed.

(7) Every $b^*\tau$ -closed set is gs- closed and gp- closed.

Definition 2.6. A subset *A* of *X* is called $b^*\tau$ -open if and only if A^c is $b^*\tau$ -closed, where A^c is complement of *A*. The family of all $b^*\tau$ - open subsets of *X* is denoted by $B^*\tau O(X)$.

From Proposition 2.2 and Remark 2.5 we have the following result.

Theorem 2.7. (1) Every open set is $b^*\tau$ - open.

(2) Every $b^*\tau$ - open set is gsg- open set.

(3) Every $b^*\tau$ - open set is g- open and ω- open.

(4) Every $b^*\tau$ - open set is gs- open, sg-open, β- open and gsp- open.

(5) Every $b^*\tau$ - open set is gα- open, pre- open and αg- open.

3. Characterizations and Properties of $b^*\tau$ - Closed and $b^*\tau$ - Open Sets.

First we prove that the union of two $b^*\tau$ - closed sets is $b^*\tau$ - closed.

Theorem 3.1. If *A* and *B* are $b^*\tau$ -closed subsets of *X* then $A \cup B$ is $b^*\tau$ -closed in X. **Proof.** Let $A \cup B \subseteq U$ and U be any $b\tau$ - open set. Then $A \subseteq U$, $B \subseteq U$. Hence $cl(A) \subseteq U$ and $cl(B) \subseteq U$. But $cl(A \cup B) = cl(A) \cup cl(B) \subseteq U$. Hence $A \cup B$ is $b^*\tau$ –closed. □

Theorem 3.2. (1) If a set *A* is $b^*\tau$ - closed, then $cl(A) - A$ contains no non empty closed set (2) If a set *A* is $b^*\tau$ - closed if and only if $cl(A) - A$ contains no non empty $b\tau$ - closed set

Proof. (1)Let *F* be a closed subset of $cl(A) - A$. Then $A \subseteq F^c$. Since *A* is $b^*\tau$ - closed then $cl(A) \subseteq F^c$. Hence $F \subseteq (cl(A))^c$. We have $F \subseteq cl(A) \cap (cl(A))^c = \varphi$ and hence *F* is empty. Similarly, we prove (2). □

Remark 3.3. The converse of Theorem 3.2 need not be true as the following example shows.

Example 3.4. Let $X = \{a, b, c\}$ with $\tau = \{X, \varphi, \{a\}, \{b, c\}\}$. If $A = \{a, b\}$ then $cl(A) - A = X - \{a, b\} = \{c\}$ does not contain non empty closed.

Theorem 3.5. If *A* is $b^*\tau$ - closed in *X* and $A \subseteq B \subseteq cl(A)$ then *B* is $b^*\tau$ - closed in *X*.

Proof. Let $B \subseteq U$, where *U* is $b\tau$ - open set. Since $A \subseteq B$, so $cl(A) \subseteq U$. But $B \subseteq cl(A)$, so $cl(B) \subseteq cl(A)$). Hence $cl(B) \subseteq U$. Thus *B* is $b^*\tau$ - closed in *X*. □

Theorem 3.6. Let $A \subseteq Y \subseteq X$ and *A* is $b^*\tau$ - closed in *X*, then *A* is $b^*\tau$ - closed relative to Y.

Proof. Let $A \subseteq Y \cap G$ and *G* is $b\tau$ -open in *X*. Then $A \subseteq G$ and hence $cl(A) \subseteq G$ Then $Y \cap cl(A) \subseteq Y \cap G$. Thus *A* is $b\tau$ -closed relative to *Y*. □

Theorem 3.7. In a topological space *X*, $B\tau O(X) = \mathcal{F}$ if and only if every subset of *X* is a $b^*\tau$ - closed, where $B\tau O(X)$ is the collection of all $b\tau$ - open sets in X and $\mathcal{F}$ is the set of closed sets in *X*.

Proof: If $B\tau O(X) = \mathcal{F}$. Let *A* is a subset of *X* such that $A \subseteq U$ where $U \in B\tau O(X)$, then $cl(A) \subseteq cl(U) = U$. Hence A is $b^*\tau$ - closed in *X*.

Conversely, if every subset of *X* is a $b^*\tau$ - closed. Let $U \in B\tau(X)$. Then $U \subseteq U$ and *U* is $b^*\tau$ -closed in *X*, hence

$cl(U) \subseteq U$. Thus $cl(U) = U$. Therefore $B\tau O(X) \subseteq \mathcal{F}$. Now, if $S \in \mathcal{F}$. Then S^c is open and hence it is $b\tau$-open. Therefore $S^c \in B\tau O(X) \subseteq \mathcal{F}$ and hence $S \in \mathcal{F}^c$. Therefore $B\tau O(X) = \mathcal{F}$. □

Theorem 3.8. If A is $b\tau$ - open and $b^*\tau$ - closed in X, then A is closed in X.

Proof. Since A is $b\tau$ - open and $b^*\tau$ - closed in X then, $cl(A) \subseteq A$ and hence A is closed in X.
□

Theorem 3.9. For each $x \in X$ either $\{x\}$ is $b\tau$ - closed or $\{x\}^c$ is $b^*\tau$ - closed in X.

Proof. If $\{x\}$ is not $b\tau$ - closed in X, then $\{x\}^c$ is not $b\tau$ - open and the only $b\tau$-open set containing $\{x\}^c$ and its closure is the space X. Hence $\{x\}^c$ is $b^*\tau$ - closed in X. □

Definition 3.10. The intersection of all $b\tau$ - open subsets of X containing A is called the $b\tau$ - *kernel* of A and is denoted by $b\tau$ - *ker(A)*.

Lellis et al. [13] defined *sg- ker(A)* to be the intersection of all sg-open subsets of containing A.

Remark 3.11. It is clear that $b\tau$ -*ker(A)* $\subseteq$ *sg- ker(A)*.

Theorem. 3.12. A subset A of X is $b^*\tau$ - closed if and only if $cl(A) \subseteq b\tau$ - *ker(A)*.

Proof. Let A be a $b^*\tau$ - closed in X. Let $x \in cl(A)$. If $x \notin b\tau$ - *ker (A)* then there is a $b\tau$ -open set U containing A, such that $x \notin U$. Since U is a $b\tau$ - open set containing A, we have $cl(A) \subseteq U$, hence $x \notin cl(A)$, which is a contradiction.

Conversely, let $cl(A) \subseteq b\tau$ - *ker(A)*. If U is any $b\tau$ - open set containing A, then $cl(A) \subseteq b\tau$ - *ker(A)* $\subseteq U$. Therefore A is $b^*\tau$ - closed. □

Jankovic and Reilly[12] pointed out that every singleton $\{x\}$ of a space X is either nowhere dense or preopen. This provides another decomposition appeared in [5], namely $X = X_1 \bigcup X_2$ where $X_1 = \{x \in X : \{x\}$ is no where dense$\}$ and $X_2 = \{x \in X : \{x\}$ is preopen$\}$. □

Analogous to Proposition 4.1 in [13], we have the following.

Proposition 3.13. For any subset A of X, $X_2 \cap cl(A) \subseteq b\tau$ - *ker(A)*.

Proof. Let $x \in X_2 \cap cl(A)$. If $x \notin b\tau$ - *ker(A)*. Then there is a $b\tau$ - open set U containing A such that $x \notin U$. Then U^c is $b\tau$ - closed containing x. Since $x \in cl(A)$, so $cl\{x\} \subseteq cl(A)$, we have $int(cl(\{x\})) \subseteq int(cl(A))$. Since $x \in X_2$, so $\{x\} \subseteq int(cl(\{x\})$, hence $int(cl(\{x\}) \neq \varphi$. Also $x \in cl(A)$, so $A \cap int(cl(\{x\}) \neq \varphi$. Thus there is $y \in A \cap int(cl(\{x\})$ and hence $y \in A \cap U^c$. This is a contradiction. □

Since $b\tau$ -*ker(A)* $\subseteq$ *sg- ker(A)*, by Remark 3.11. we have the following results which is Proposition 4.1 of [13].

Corollary 3. 15. For any subset A of X, $X_2 \cap cl(A) \subseteq sg$ - *ker(A)*.

Theorem 3.16. A subset A of X is $b^*\tau$ - closed, if and only if $X_1 \cap cl(A) \subseteq A$.

Proof. Suppose that A is $b^*\tau$ -closed, and $x \in X_1 \cap cl(A)$. Then $x \in X_1$ and $x \in cl(A)$. Since $x \in X_1$, $int(cl(\{x\}) = \varphi$. Therefore $\{x\}$ is semi-closed. Hence $int(cl(\{x\}) \subseteq \{x\}$. Since every semi-closed set is gs-closed, hence $b\tau$ - closed [9]. So $\{x\}$ is $b\tau$-closed. If $x \notin A$ and $U = X - \{x\}$, then U is a $b\tau$ - open set containing A and so $cl(A) \subseteq U$, since $x \in cl(A)$. So $x \in U$, which is a contradiction.

Conversely, let $X_1 \cap cl(A) \subseteq A$. Then $X_1 \cap cl(A) \subseteq b\tau$ - *ker(A)*, since $A \subseteq b\tau$ - *ker(A)*. Now $cl(A) = X \cap cl(A) = (X_1 \cup X_2) \cap cl(A) = (X_1 \cap cl(A)) \cup (X_2 \cap cl(A))$. By hypothesis, $X_1 \cap cl(A) \subseteq b\tau - ker(A)$, and by Proposition 3.13, $X_2 \cap cl(A) \subseteq b\tau - ker(A)$. Then $cl(A) \subseteq b\tau - ker(A)$. Hence by Theorem 3.12, A is $b^*\tau$ -closed. □

Since every $b^*\tau$ - closed set is gsg- closed, we have the following result which is a part of Theorem 4.9 in [15].

Corollary 3.17. If $X_I \cap cl(A) \subseteq A$, then A is gsg- closed.

Theorem 3.18. Arbitrary intersection of $b^*\tau$ -closed sets is $b^*\tau$ - closed.

Proof. Let $F = \{Ai: i \in \Lambda\}$ be a family of $b^*\tau$ -closed sets and let $A = \cap_{i\in\Lambda} A_i$. Since $A \subseteq A_i$ for each i, $X_I \cap cl(A) \subseteq X_I \cap cl(A_i)$ for each i. By Theorem 3.16 for each $b^*\tau$ -closed set A_i, we have $X_I \cap cl(A_i) \subseteq A_i$ for each i. So $X_I \cap cl(A_i) \subseteq A$ for each i. Hence $X_I \cap cl(A) \subseteq X_I \cap cl(A_i) \subseteq A$ for each $i \in \Lambda$. That is $X_I \cap cl(A) \subseteq A$. Hence by Theorem 3.16, A is $b^*\tau$ -closed. □

From Theorem 3.1, we have,

Corollary 3.19. If A and B are $b^*\tau$ -open sets then $A \cap B$ is $b^*\tau$ -open.

From Theorem 3. 18, we have the following,

Corollary 3.20. Arbitrary union of $b^*\tau$ - open sets is $b^*\tau$ - open.

From Corollary 3.19, Corollary 3.20 and Proposition 2.2(1), we have,

Corollary 3.21. The class of $B^*\tau O(X)$ forms a topology on X finer than τ.

Theorem 3.22. A set A is $b^*\tau$ -open if and only if $F \subseteq int(A)$, where F is $b\tau$ -closed and $F \subseteq A$.

Proof. Let $F \subseteq int(A)$ where F is $b\tau$ -closed and $F \subseteq A$. Then $A^c \subseteq F^c$ where F^c is $b\tau$ -open. Since $F \subseteq int(A)$. So $cl(A^c) \subseteq (F^c)$. Thus A^c is $b^*\tau$ - closed. Hence A is $b^*\tau$ -open. Conversely, if A is $b^*\tau$ - open, $F \subseteq A$ and F is $b\tau$ - closed. Then F^c is $b\tau$ - open and $A^c \subseteq F^c$. Therefore $cl(A^c) \subseteq F^c$. Hence $F \subseteq int(A)$. □

Theorem 3.23. If $A \subseteq B \subseteq X$ where A is $b^*\tau$ - open relative to B and B is $b^*\tau$ - open in X, then A is $b^*\tau$ - open in X.

Proof. Let F be a $b\tau$ - closed set in X and let F be a subset of A. Then $F = F \cap B$ is $b\tau$ - closed in B. But A is $b^*\tau$ - open relative to B. Therefore $F \subseteq int_B(A)$. Since $int_B(A)$ is an open set relative to B. We have $F \subseteq G \cap B \subseteq A$, for some open set G in X. Since B is $b^*\tau$ - open in X, We have $F \subseteq int(B) \subseteq B$. Therefore $F \subseteq int(B) \cap G \subseteq B \cap G \subseteq A$. Hence $F \subseteq int(A)$. Therefore A is $b^*\tau$ -open in X.
□

Theorem 3.24. If $int(B) \subseteq B \subseteq A$ and A is $b^*\tau$ -open in X, then B is $b^*\tau$ -open in X.

Proof. Suppose that $int(A) \subseteq B \subseteq A$ and A is $b^*\tau$ -open in X then $A^c \subseteq B^c \subseteq cl(A^c)$ and since A^c is $b^*\tau$ -closed in X, by Theorem 3.6, B is $b^*\tau$ -open in X. □

Lemma 3. 25. The product of two $b\tau$ - open sets is $b\tau$ - open.

Proof. Let $A \in B\tau O(X)$, $B \in B\tau O(Y)$ and $W = A \times B \subseteq X \times Y$. Let $F \subseteq W$ be a closed set in $X \times Y$, then there exist two closed sets $F_1 \subseteq A$, $F_2 \subseteq B$ and so, $F_1 \subseteq b\,\mathrm{int}(A)$, $F_2 \subseteq b\,\mathrm{int}(B)$. Since $F_1 \times F_2 \subseteq A \times B$ and $F_1 \times F_2 \subseteq b\,\mathrm{int}(A) \times b\,\mathrm{int}(B) = b\,\mathrm{int}(A \times B)$. Therefore $A \times B \in B\tau O(X \times Y, \tau \times \sigma)$.
□

Theorem 3.26. Let (X,τ) and (Y,σ) be topological spaces, B be a subset of (Y,τ), If $X \times B$ is $b^*\tau$ - closed in the product space $(X \times Y, \tau \times \sigma)$, then *B* is $b^*\tau$ - closed in (Y, σ).

Proof. Let *M* be $b\tau$ - open subset of *Y* such that $B \subseteq M$. By Lemma 3. 25, $X \times M$ is $b\tau$ - open. Since $X \times B$ is $b^*\tau$ - closed and $X \times B \subseteq X \times M$, so $cl(X \times B) = X \times cl(B) \subseteq X \times M$. Therefore, $cl(B) \subseteq M$. Hence *B* is $b^*\tau$ - closed in (Y, σ). □

4. Applications.

In this section, we introduce a new space namely, $T_{b^*\tau}$. Its relations with some known spaces are discussed and some characterizations are provided. Further we make use of $b^*\tau$ - closed sets to obtain new characterizations of normal spaces.

Let us recall the following concepts.

Definition 4.1. A topological space *X* is called a

(1) $T_{1/2}$- space [16] if every g- closed set is closed.

(2) T_w - space [22] if every *w*- closed set is closed.

(3) T_b- space [6] if every gs- closed set is closed.

(4) T_{gsg}- space [13] if every gsg- closed set is closed.

Definition 4.2. A space *X* is called a $T_{b^*\tau}$ -space if every $b^*\tau$ - closed set is closed.

From Definitions 4.1, 4.2, Proposition 2.2 and Proposition 2.5, we can easily prove that:

Proposition 4.3.

(1) Every $T_{1/2}$ - space is a $T_{b^*\tau}$ - space.

(2) Every T_w - space is a $T_{b^*\tau}$ - space.

(3) Every T_b - space is a $T_{b^*\tau}$ - space.

(4) Every T_{gsg} - space is a $T_{b^*\tau}$ - space.

Remark 4.4.The converse of parts, 1, 2, 3 of Proposition 4.3 is not true in general as shown in the following examples.

Example 4.5. Let $X = \{a, b, c\}$ with $\tau = \{X, \varphi, \{a\}, \{b, c\}\}$. The family of generalized closed sets = P(X) = The family of all *w*- closed sets = The family of all generalized semi- closed sets and $B\tau^*C(X) = \{X, \varphi, \{a\}, \{b, c\}\}$.Then X is a $T_{b^*\tau}$ - space but it is not a $T_{1/2}$- space, not a T_w – space and not a T_b- space.

Question. Is there a $T_{b^*\tau}$ - space which is not T_{gsg}?

Theorem 4.6. For a space *X* the following are equivalent.

(1) (X, τ) is a $T_{b^*\tau}$ - space.

(2) Every singleton of *X* is either $b\tau$ - closed or open.

Proof. (1) $\Rightarrow$ (2) Let $x \in X$. Suppose that the set $\{x\}$ is not a $b\tau$ - closed set in *X*. Then the only $b\tau$ - open set containing $\{x\}^c$ is the space *X* itself and so $\{x\}^c$ is $b^*\tau$ -closed in X. By assumption $\{x\}^c$ is closed in *X* or equivalently $\{x\}$ is open.

(2) $\Rightarrow$ (1) Let *A* be a $b^*\tau$ -closed subset of *X* and let $x \in cl(A)$. By assumption $\{x\}$ is either $b\tau$ - closed or open.

case(1): Suppose $\{x\}$ is $b\tau$ -closed. If $x \notin A$ then $cl(A) - A$ contains a non-empty $b\tau$ - closed set $\{x\}$ which is a contradiction to Theorem 3.2. Therefore $x \in A$.

case(2): Suppose $\{x\}$ is open. Since $x \in cl(A)$, $\{x\} \cap A \neq \varphi$ and therefore $cl(A) \subseteq A$ or equivalently *A* is a closed subset of *X*. □

Finally, we make use of $b^*\tau$ - closed sets to obtain further characterizations and preservation theorems of normal spaces.

Theorem 4.7. The following are equivalent for a space *X:*

(1) *X* is normal.

(2) For any disjoint closed sets *A* and *B*, there exist disjoint $b^*\tau$ - open sets *U, V* such that $A \subset U$ and $B \subset V$.

(3) For any closed set *F* and any open set *G* containing *F*, there exists a $b^*\tau$ - open set *U* of X such that $F \subset U \subset cl(U) \subset G$.

Proof. (1)$\Rightarrow$(2): This is obvious since every open is $b^*\tau$ - open.

(2) $\Rightarrow$(3): Let *F* be closed and *G* be open set containing *F*. Then *F* and G^c are disjoint closed sets. There exist disjoint $b^*\tau$ - open sets *U* and *V* such that $F \subset U$ and $G^c \subset V$.

Since G^c closed, hence $b\tau$ - closed. So $(\text{int}(V))^c \subset G$. Since $U \bigcap V = \phi$, so $U \bigcap \text{int}(V) = \phi$. Thus $U \subset (\text{int}(V))^c$. Therefore $F \subset U \subset cl(U) \subset G$.

(3) $\Rightarrow$ (1): Let *A* and *B* be disjoint closed sets of *X*. Hence $A \subset B^c$ and B^c is open. So, by (3), there is a $b^*\tau$ - open set *U* of *X* such that $A \subset U \subset cl(U) \subset B^c$. We have $B \subset (cl(U))^c$. Since A closed, hence $b\tau$ - closed and $A \subset \text{int}(U)$. Put G= $\text{int}(U)$ and *W*=$(cl(U))^c$. Hence $G \bigcap W = \phi$. Thus we find two disjoint open sets *G* and *W* containing *A* and *B* respectively. Therefore *X* is normal. □

Definition 4.8. A function $f:(X,\tau) \to (Y,\sigma)$ is said to be almost closed (resp. almost $b^*\tau$ - closed) if for each regular closed set F of X, f(F) is closed (resp. $b^*\tau$ - closed).

It is clear that: closed $\Rightarrow$ almost closed $\Rightarrow$ almost $b^*\tau$ - closed.

Theorem 4.9. A surjection $f:(X,\tau) \to (Y,\sigma)$ is almost $b^*\tau$ - closed if and only if for each subset H of Y and each regular open set U of X containing $f^{-1}(H)$ there exists a $b^*\tau$ - open set V of Y such that $H \subset V$ and $f^{-1}(V) \subset U$.

Proof. Suppose that *f* is almost $b^*\tau$ - closed. Let *H* be a subset of *Y* and $U \in RO(X)$ containing $f^{-1}(H)$. Put $V = (f(U^c))^c$, then *V* is a $b^*\tau$ - open set of *Y* such that $H \subset V$ and $f^{-1}(V) \subset U$.

Conversely, let *F* be any regular closed set of X. Then $f^{-1}(f(F)^c) \subset F^c$ and $F^c \in RO(X)$. Hence there exists a $b^*\tau$ - open set *V* of *Y* such that $f(F)^c \subset V$ and $f^{-1}(V) \subset F^c$. Thus $V^c \subset f(F)$ and $F \subset f^{-1}(V^c)$. Hence $f(F) = V^c$ and $f(F)$ is $b^*\tau$ - closed in Y. Therefore *f* is almost $b^*\tau$ - closed. □

Theorem 4.10. If $f:(X,\tau) \to (Y,\sigma)$ is a continuous almost $b^*\tau$ - closed surjection and X is a normal space, then Y is normal.

Proof. Let A and B be any disjoint closed sets of Y. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint closed sets of X. Since X is normal, there exist disjoint open sets U and V such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. Let *G*= *int*(*cl*(*U*)) and *H*= *int*(*cl*(*V*)), then *G* and *H* are disjoint regular open sets of X such that $f^{-1}(A) \subset G$ and $f^{-1}(B) \subset H$. Hence, by Theorem 4.9, there exist $b^*\tau$ - open sets L and W such that $A \subset L$, $B \subset W$, $f^{-1}(L) \subset G$ and $f^{-1}(W) \subset H$. And L and W are disjoint, since G and H are disjoint. Therefore Y is normal, by Theorem 4.7.
□

The following result is immediate consequence of Theorem 4.2.

Corollary 4.11. [14] If $f:(X,\tau)\to(Y,\sigma)$ is a continuous almost closed surjection and X is a normal space, then Y is normal.

5. Conclusion.

The class of generalized closed sets is one of the significant notions which used in general topology and fuzzy topology spaces. The class of $b^{*}\tau$ - open sets forms a topology finer than τ. This type of closed sets can be used to derive new separation axioms, new forms of continuity and new decompositions of continuity.

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