An accelerated hybrid Euler's method for numerical integration of ordinary differential equations.

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1. Abstract

The theme of this study is to develop hybrid Euler's method from the chain of Euler's methods, to compute the numerical resolution of initial value problems (*IVPs*) of ordinary differential equations (ODEs). In this work number of *(IVPs)* are taken from literature to check absolute error and stability of method. It is observed that computed results from the proposed method are consistent and stabile under the circumstance of region. From the results it is concluded that the defined algorithm minimizes the error of computation for specified accuracy of solution.

2. Keywords:

Ordinary differential equations, *IVP's*, Improved, Modified Euler's,

3. Introduction

Mathematics is that tool, which uses for computing solution, at the same time it uses as the language of science and engineering. It contains a region that is thought as differential equations which is widely spread in various fields of science. As stated in [15] there is no unique method exist to solve every form of differential equations. The number of ODEs cannot be solved manually and computer languages have importance [8]. The scholars and researchers derived numerical schemes/techniques, when ODEs are divided into classes and furthermore subclasses [14]. The most use of mathematical models in planning and evaluation of experiment, understanding and developing the mechanism of complex systems including balance models in integrals and differential forms [3]. Differential equations play a very important role to interpret the physical phenomena into a mathematical model. It is categorized as ordinary and partial differential equations (PDEs). Whereas involve single independent variable in ordinary differential equations and is differentiable with relevancy of that variable and partial differential equations are differentiable with multi variables.

The number of mathematical models haven't proper solutions to explore better possible approximate resolution as described in [4]. The opinion of Cláudio Faria Lopes Junior, MATLAB is scientific methodology for solving numerical problems of ODE's. As the ordinary differential equations are beneficial tool for modelling and learning the physical phenomena in terms of mathematics, moreover used to resolve real world problems of Science and engineering [6]. In the modern study of science, the scenario describes as the ordinary differential equations frequently arise to moderate all around the circle of physical Science. Unfortunately, various types of differential equations cannot be solved exactly. For the sake of that the numerical methods have importance.

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The greatest significant techniques frequently developed in continues time dynamic for numerical resolution of (ODEs). The procedure of numerical techniques is way to obtain information about solution. Various methods have been developed as well proposed to solve initial value problems like Euler's, Improved Euler's and Runge-Kutta schemes etc. Runge in (1901) developed 2nd order methods which gives best accuracy regarding to Euler's method. Moses A. Akanbi proposed Euler's third order method for better performance of numerical solution of ODE's [6]. The most of present numerical schemes produce linear formula to resolve initial value problems of ordinary differential equation during this scope planed methodology is additionally linear implicit methodology to interpret the solution of IVPs of ordinary differential equations. The basic idea is taken from Euler's method to develop planned algorithm. The planed methodology improves its ability to maintain the consistency of accuracy of IVPs as well overall performance.

4. Methodology

Here our interest is to study the general equation of first order ordinary differential equations of IVPs, it is defined as below,

$$u' = f(t,u) \qquad u(t_o)=u_o \qquad (1)$$

Therefore, the conditions for above equation are categorized as below,

Where f is a function, which used to satisfy all conditions in (1) have only one of kind solution, the subintervals $[t_i, t_{i+1}]$ with $t_0 = a$ & $t_i = t_0 + ih$, from the interval $[a,b]$, where h is step size subintervals, then the problem is solved up to t_i.

As we know that the Euler's developed as the foundation scheme for solving ODEs, to compute the solution of initial value problems using h as a step size. As in [6] Moses A. Akanbi advised a third order method from the sequence of Euler's method.

In order to know about Euler's method, it generate the simplest idea for numerically solving 1st order ODEs. Let the first order differential equation defined as like

$$u' = f(t,u) \qquad u(t_o)=u_o \qquad (1.1)$$

Whenever the resolution of first order differential equation is smooth, then to approximate the derivative $u'(t)$ by finite differences,

$$u'(t) = \frac{u(t+h)-u(t)}{h}$$

Thus, let $\Delta t = h$, here chose Δt as a time step size. Whereas $t_n = t_0 + nh$ and t_n is time step. Then $u'(t_n)$ is denoted by u_n as for $t_n < t < t_{n+1}$, $u(t)$ is approximated by a linear function, thus it approximate $u' = f(t_n, u_n)$ and $f(t_n, u_n) = \frac{u(t+h)-u(t)}{h}$.

Therefore, Euler's method is defined as $u_{n+1} = u_n + h\,f(t_n, u_n)$ (2)

Here lookout the method is known as Improved Euler's method, the focused area of this method is to generate the numerical solution. The basic idea is taken from Euler's methods to develop Improved Euler's method. As above, we have discussed Euler's developed method, it generates a numerical solution for an initial value problems of the form, $u'(t) = f(t,u)$

Here is time for a confession, in shortly discussion the inaccuracy and illness of Euler's method encourage to improve the Euler's method. Although interesting point of discussion, it is simple for solving and understanding. The Heun's method based on approximating the integral curve of (1.1) at (t_0, u_0) by the line through (t_0, u_0) with slope

$$m_i = \frac{\left(f\left(t_i, u\left(t_i\right)\right) + f\left(t_{i+1}, u_{i+1}\right)\right)}{2} \qquad (3)$$

At the end point tangents to the integral curve, having m_i average slopes of (t_i, t_{i+1}). The equation of approximating line as below

$$u(t_{i+1}) = u(t_i) + \frac{\left(f\left(t_i, u(t_i)\right) + f\left(t_{i+1}, u_{i+1}\right)\right)}{2} \cdot (t - t_i) \qquad (3.1)$$

Sort out $t = t_{i+1} = t_i + h$ in (3.1)

$$u_{i+1} = u_i + \frac{h}{2}\left(f(t_i, u_i) + f\left(t_{i+1}, u_{i+1}\right)\right) \qquad (4)$$

By replacing $t_i = t_i + \frac{h}{2}$ & $u_i = u_i + \frac{h}{2}$ in (2) we get

$$u_{i+1} = u_i + hf\left(t_i + \frac{1}{2}h, u_i + \frac{1}{2}hf\left(t_i, u_i\right)\right) \qquad (5)$$

Whereas (4) is known as Improved Euler′s method and (5) is known as Modified Euler′s method or Improved Polygon method. The number of slope evaluations requires two per step for Huen′s and Improved Polygon method, While Euler's method require only one. As in [6] defined by inserting missing values of y in equation (5). Get different methods like,

$$u_{i+1} = u_i + hf\left(t_i + \frac{1}{2}h, u_i + \frac{1}{2}hf\left(t_i + h, u_i + hf\left(t_i, u_i\right)\right)\right) \qquad (6)$$

$$u_{i+1} = u_i + hf\left(t_i + \frac{1}{2}h, u_i + \frac{1}{2}hf\left(t_i + \frac{1}{2}h, u_i + \frac{1}{2}hf\left(t_i, u_i\right)\right)\right) \qquad (7)$$

$$u_{i+1} = u_i + hf\left(t_i + \frac{1}{2}h, u_i + \frac{1}{2}hf\left(t_i + \frac{1}{2}h, u_i + \frac{1}{3}hf\left(t_i, u_i\right)\right)\right) \qquad (8)$$

Here the equation (6) & (7) known as Improved Modified Euler′s Method whereas equation (8) is called Third order Euler′s Method because of equation (8) has three slope evaluation. As concern the current study to hybrid above method to propose the method is defined as below, Taking u_n from 4 & 5 then equating, getting results

$$u_{i+1} - \frac{h}{2}\left(f\left(t_i, u_i\right) + f\left(t_i + h, u_i + hf\left(t_i, u_i\right)\right)\right) = u_{i+1} - hf\left(t_i + \frac{1}{2}h, u_i + \frac{1}{2}hf\left(t_i, u_i\right)\right) \qquad (9)$$

As well using 4 & 7 to take u_n and equating both getting results as,

$$u_{i+1} - \frac{h}{2}\left(f\left(t_i, u_i\right) + f\left(t_i + h, u_i + hf\left(t_i, u_i\right)\right)\right) = u_{i+1} - hf\left(t_i + \frac{1}{2}h, u_i + \frac{1}{2}hf\left(t_i + \frac{1}{2}h, u_i + \frac{1}{2}hf\left(t_i, u_i\right)\right)\right)$$

Place u_{n+1} to an alternative sides, $- \neq 0$ gets

$$u_{i+1}+\frac{h}{2}\left(f\left(t_i,u_i\right)+f\left(t_i+h,u_i+hf\left(t_i,u_i\right)\right)\right)=u_{i+1}+hf\left(t_i+\frac{1}{2}h,u_i+\frac{1}{2}hf\left(t_i+\frac{1}{2}h,u_i+\frac{1}{2}hf\left(t_i,u_i\right)\right)\right) \quad (10)$$

from 9 & 10 the calculation elaborate Accelerated hybrid Euler's method,

$$u_{i+1}=u_n+\frac{1}{2}h\left(f\left(t_i+\frac{1}{2}h,u_i+\frac{1}{2}hf\left(t_i,u_i\right)\right)\right)+f\left(t_i+\frac{1}{2}h,u_i+\frac{1}{2}hf\left(t_i+\frac{1}{2}h,u_i+\frac{1}{2}hf\left(t_i,u_i\right)\right)\right)$$

Therefore, the proposed method further elaborates like that

$$u_{i+1}=u_i+\frac{h}{2}\left(k_2+k_3\right)$$

$$here\ k_1=f\left(t_i,u_i\right);\ k_2=f\left(t_i+\frac{h}{2},u_i+\frac{h}{2}\cdot k_1\right);\ k_3=f\left(t_i+\frac{h}{2},u_i+\frac{h}{2}\cdot k_2\right)$$

Whereas $\phi(t)=\frac{h}{2}(k_2+k_3)$ is known as increment function of proposed method.

5. Absolute Stability

Using accelerated Hybrid Euler's method to Dahl-Quist's model problem, as in (7) it gives,

$$k_1=\lambda u_n, \qquad k_2=\left(u_i+\frac{h}{2}\lambda u_i\right)\lambda, \qquad k_3=\left(u_i+\frac{h}{2}\left(u_i+\frac{h}{2}\lambda u_i\right)\lambda\right)\lambda;$$

Therefore, $u_{i+1}=u_i+\frac{h}{2}\left(k_2+k_3\right)$

$$=u_i+\frac{h}{2}\left(h\lambda+\frac{h^2}{2}\lambda^2+\frac{h^3}{8}\lambda^3\right)u_i$$

$$=\left\{1+h\lambda+\frac{1}{2}(h\lambda)^2+\frac{1}{8}(h\lambda)^3\right\}u_i$$

$$\therefore u_i=\left\{1+h\lambda+\frac{1}{2}(h\lambda)^2+\frac{1}{8}(h\lambda)^3\right\}^i u_0$$

$$=\left\{1+h\lambda+\frac{1}{2}(h\lambda)^2+\frac{1}{8}(h\lambda)^3\right\}^i \because u_0=u(0)=1$$

$$= \left\{1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \frac{1}{8}(h\lambda)^3\right\}^{\frac{t_i}{h}}$$

$$\ln u_i = \frac{t_i}{h} \ln \left\{1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \frac{1}{8}(h\lambda)^3\right\}$$

$$\lim_{h\to 0} \ln u_i = t_i \lim_{h\to 0} \frac{1}{h} \ln \left\{1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \frac{1}{8}(h\lambda)^3\right\}$$

Applying L-Hopital's rule, gets

$$\lim_{h\to 0} \ln u_i = t_i \ \lambda$$

It indicates that $\ln \lim_{h\to 0} u_i = t_i \ \lambda$, and thus, $\lim_{h\to 0} u_i = e^{t_i \ \lambda}$

Whereas, the approximate solution of u_i goes to be analytical solution of model problem $h \to 0$. The ratio u_i / u_{i+1} is knows as stability function *R(z)* ,

Whereas the stability function *R(z)* of Accelerated Hybrid Euler's method is defined as,

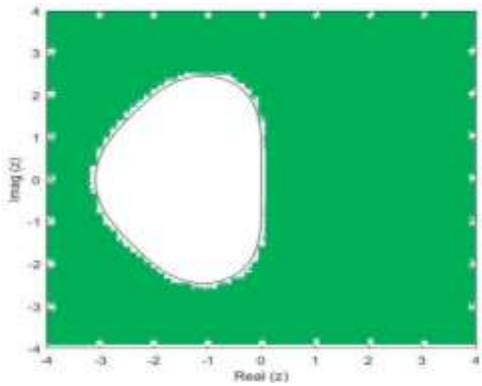
$$R(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{8}$$

While that stability function *R(z)* for Third Order Euler's method is defined as,

$$R(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6}$$

The stability region of proposed method discussed above, as shown in figure 1. The unshaded area defines the stability region for Accelerated Hybrid Euler's method, as it falls in the interval $(-4\ ,\ 0)$ in the complex plane. Hence on these grounds proposed method is stable.

Fig 1



6. Numerical Computation

In this part we consider some examples from literature to compute the absolute error of numerical results of $u(t)$ for the $IVP's$ and compare the accuracy of proposed hybrid Euler's method with that third order Euler's and modified improved modified Euler's methods. These computations are carried out using varies step sizes of h.

Example 1. The Algebraic initial value problem is under consideration as given:

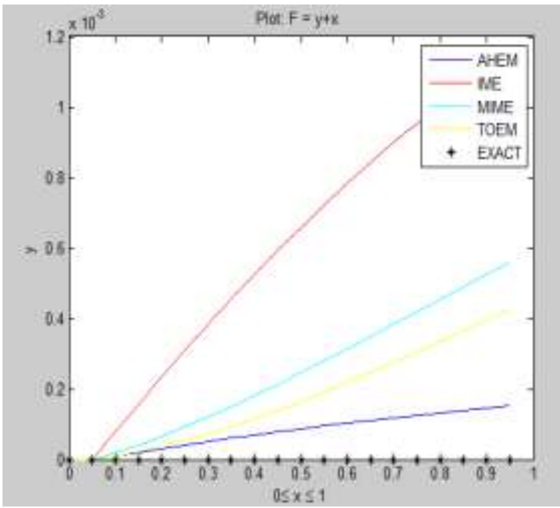
$$\frac{dy}{dx} = y + x, \quad y(0) = 1 \quad \text{Therefore, the exact solution is given as:} \quad y(x) = 2e^{x} - x - 1$$

It is displayed in table 1. As the number of integration steps are increasing, then the solution moving towards the exact solution at every step of integration on the interval $[0,1]$ where $h = 0.05$ as well the developed AHEM has a smallest error among all, it can be analysed by the blue line in the following figure 2.

Table 1.

T	AHEM	IME	MIME	TOEM	EXACT
0	1.0000	1.0000	1.0000	1.0000	1.0000
0.0500	1.0525	1.0526	1.0526	1.0526	1.0525
0.1000	1.1103	1.1105	1.1104	1.1104	1.1103
0.1500	1.1736	1.1739	1.1737	1.1737	1.1737
0.2000	1.2428	1.2432	1.2429	1.2429	1.2428
0.2500	1.3180	1.3186	1.3182	1.3181	1.3181
0.3000	1.3996	1.4004	1.3999	1.3998	1.3997
0.3500	1.4880	1.4889	1.4884	1.4883	1.4881
0.4000	1.5835	1.5846	1.5840	1.5839	1.5836
0.4500	1.6865	1.6877	1.6870	1.6869	1.6866
0.5000	1.7973	1.7987	1.7979	1.7978	1.7974
0.5500	1.9163	1.9180	1.9171	1.9169	1.9165
0.6000	2.0440	2.0460	2.0450	2.0447	2.0442
0.6500	2.1808	2.1830	2.1819	2.1817	2.1811
0.7000	2.3272	2.3297	2.3285	2.3282	2.3275
0.7500	2.4837	2.4865	2.4851	2.4848	2.4840
0.8000	2.6507	2.6539	2.6524	2.6520	2.6511
0.8500	2.8289	2.8324	2.8308	2.8304	2.8293
0.9000	3.0187	3.0227	3.0209	3.0205	3.0192
0.9500	3.2209	3.2253	3.2233	3.2229	0

Fig 2.



Example 2. The trigonometric initial value problem is under consideration as given:

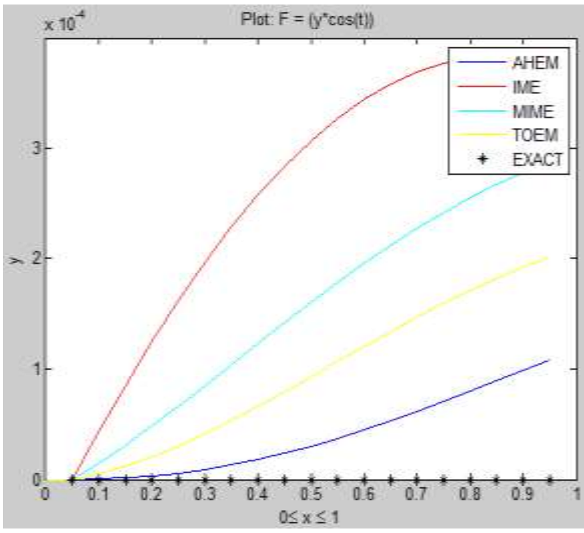
$\frac{dy}{dx} = y\cos t$ The exact solution of this function is $y(t) = e^{\sin t}$

It is observed in table 2 the error of proposed method is more accurate in comparison to other methods. Whenever the steps are decreased, then the solution moving toward the exact solution on the interval $[0,1]$ where $h = 0.05$ as well the developed AHEM has a smallest error among all, it is clear by blue line in the figure 3.

Table 2.

T	AHEM	IME	MIME	TOEM	EXACT
0	1.0000	1.0000	1.0000	1.0000	1.0000
0.0500	1.0512	1.0513	1.0513	1.0513	1.0512
0.1000	1.1050	1.1051	1.1050	1.1050	1.1050
0.1500	1.1612	1.1613	1.1612	1.1612	1.1612
0.2000	1.2198	1.2200	1.2199	1.2198	1.2198
0.2500	1.2807	1.2809	1.2808	1.2807	1.2807
0.3000	1.3438	1.3441	1.3440	1.3439	1.3438
0.3500	1.4091	1.4094	1.4092	1.4091	1.4090
0.4000	1.4762	1.4765	1.4763	1.4762	1.4761
0.4500	1.5450	1.5454	1.5452	1.5451	1.5449
0.5000	1.6152	1.6157	1.6154	1.6153	1.6151
0.5500	1.6866	1.6871	1.6869	1.6868	1.6866
0.6000	1.7589	1.7594	1.7592	1.7591	1.7588
0.6500	1.8317	1.8323	1.8320	1.8319	1.8316
0.7000	1.9046	1.9052	1.9050	1.9048	1.9045
0.7500	1.9773	1.9779	1.9776	1.9775	1.9771
0.8000	2.0492	2.0498	2.0496	2.0494	2.0490
0.8500	2.1199	2.1205	2.1203	2.1201	2.1197
0.9000	2.1890	2.1896	2.1894	2.1892	2.1887
0.9500	2.2559	2.2565	2.2563	2.2561	0

Fig 3.



Example 3. The another trigonometric initial value problem is selected as defined:

$\frac{dy}{dx} = y\sin(2t) - 20y$ the exact solution of this function is

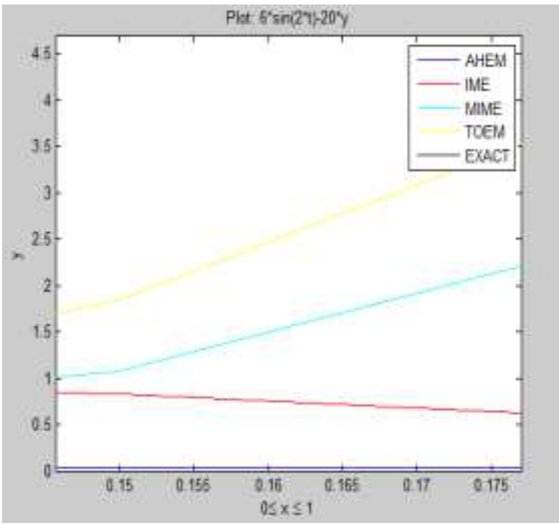
$$y(t) = \left(\frac{-3}{101}\right)\cos(2t) + \left(\frac{30}{101}\right)\sin(2t) + \left(\frac{104}{101}\right)e^{(-20t)}.$$

From the displayed results in table, it is clear that the accuracy is better than other methods, as the number steps of integration increasing, the absolute error decreasing on the interval $[0,1]$, with step size $h = 0.025$, figured in fig 4.

Table 3.

T	AHEM	IME	MIME	TOEM	EXACT
0	1.0000	1.0000	1.0000	1.0000	1.0000
0.0500	0.3862	0.0000	0.2575	0.3408	0.3789
0.1000	0.1747	0.0300	0.3518	0.4818	0.1693
0.1500	0.1137	0.0597	0.4617	0.6595	0.1107
0.2000	0.1087	0.0888	0.5910	0.8858	0.1072
0.2500	0.1240	0.1170	0.7445	1.1770	0.1233
0.3000	0.1461	0.1440	0.9279	1.5548	0.1458
0.3500	0.1698	0.1696	1.1490	2.0480	0.1696
0.4000	0.1929	0.1935	1.4172	2.6955	0.1927
0.4500	0.2145	0.2155	1.7445	3.5490	0.2143
0.5000	0.2341	0.2353	2.1459	4.6777	0.2339
0.5500	0.2514	0.2528	2.6403	6.1738	0.2513
0.6000	0.2662	0.2677	3.2514	8.1605	0.2661
0.6500	0.2784	0.2800	4.0087	10.8019	0.2783
0.7000	0.2878	0.2894	4.9496	14.3171	0.2877
0.7500	0.2943	0.2960	6.1204	18.9982	0.2942
0.8000	0.2979	0.2996	7.5792	25.2348	0.2978
0.8500	0.2985	0.3002	9.3989	33.5463	0.2984
0.9000	0.2962	0.2979	11.6704	44.6253	0.2960
0.9500	0.2908	0.2925	14.5073	59.3955	0

Fig 4.



7. Results and Discussion

The proposed accelerated hybrid Euler's method is proficient to solve initial value problems of ODEs. In this scope the number of *(IVPs)* are under consideration from literature to test performance of *(AHEM)*. Standard numerical schemes are selected for comparison having same order of accuracy. It is observed that the proposed method is stable under circumstance of region and the region is drawn in figure 1. As concern to accuracy, from table 1 to 3, presenting the computed results for various types of initial value problems along with various standard numerical methods. It is concluded that computed results by accelerated hybrid Euler's method are closer to exact solution as compared to other methods, hence in this regards best accuracy observed. Whereas, another computation is made in terms of absolute error for the addressed problems related with defined and existing numerical schemes. From figs: 2 to 4, it is observed that the defined scheme has least error in comparison to existing methods.

8. Conclusion

The focused area of this paper is to modify the efficiency of TOEM by introducing the theme of hybrid existing Euler's methods. The achievement of this method is due to least error and stable under the defined region, by the comparison the computed results with the existing methods. Hence, it is concluded that the overall performance of proposed method is efficient in sense of existing methods.

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