AN IMPROVED ROOT LOCATION METHOD FOR FAST CONVERGENCE OF NON-LINEAR EQUATIONS

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ABSTRACT

In this paper an improved root location method has been suggested for nonlinear equations $f(x)=0$. The proposed improved root location method is very much effective for solving nonlinear equations and several numerical examples associated with algebraic and transcendental functions are present in this paper to investigate the new method. Throughout the study we have proved that proposed method is cubically convergent. All the results are executed on MATLAB 16 which has a machine precision of around 10^{-16}.

Key words: Newton's method, Iterative method, third order convergent, Root finding methods.

1. INTRODUCTION

We are familiar that extensive verity of problems originated in countless practices of mathematics and physical sciences are nonlinear in nature. In mathematics and physical sciences, Newton's method that approximates the root of a non-linear equations $f(x)=0$ in one variable using the value of the function and its derivative [1]. Numerous techniques have been investigated for the importance of nonlinear equations, as one most useful technique is Quadrature Rule, which is used in various numerical observations. Quadrature Rule is not only for the solutions of numerical integrations but also help to drive some techniques to solve nonlinear equations with batter results [4]. Newton's Method (NR) is the most common and easy numerical method. This method is used to approximate solution nearer to its exact root. This method is fast converging numerical techniques but not dependable because keeping a kind of drawback. Method will fail in cases where the derivative is zero. However, it is most convenient and useful numerical techniques. We have studied many such techniques are developed form the modifications of Regula Falsi Position method and Newton's Raphson method with cubic order of convergence. [4]. Throughout this study we are makes an improvement in the iteration of Newton's method, and Regula falsi position method, (Umair Khalid Qureshi, et al 2018) The researcher has used quadrature formula to proposed tow step scheme for non-linear equations such that f(x) = 0. He has checked several types of equations and test the results with classical Newton Raphson method and Hellay method. His proposed method is cubically convergent [7].

2. RESEARCH METHEDOLOGY

The new method is proposed by the use of 3-pont open Newton-Cotes formula (Milne's rule) and the well-known fundamental theorem of calculus, such as

$$\int_a^b f(x)dx = \frac{4h}{3}\left[2f(x) - f(x_1) + 2f(x_2)\right] \quad (1)$$

put $h = \frac{b-a}{n+2} = \frac{b-a}{4} = \frac{\beta-\psi}{4}$ in (1) and β be the root of $f(x)=0$ while ψ is initial root.

$$f(\beta) \approx f(\psi) + \frac{\beta-\psi}{3}\left[2f^{'}(\psi) - f^{'}(\frac{\psi+\beta}{2}) + 2f^{'}(\beta)\right]$$

Here we are taking $f(\beta) = f(x)$ such that $\beta = x$

$$f(x) \approx f(\psi) + \frac{x-\psi}{3}\left[2f^{'}(\psi) - f^{'}(\frac{\psi+x}{2}) + 2f^{'}(x)\right] \quad (2)$$

Put $f(x)=0$ in equation (2), we have

$$(x-\psi)\left[2f^{'}(\psi) - f^{'}(\frac{\psi+x}{2}) + 2f^{'}(x)\right] = -3f(\psi) \quad (3)$$

$f^{'}\left(\frac{x_x+y_n}{2}\right)$, is replaced with $\frac{f(y_n)-f(x_n)}{y_n-x_n}$ and y_n defined as $y_n = x_n - \frac{f(x_n)}{f^{'}(x_n)}$, [11]

then we will have,

$$(x-\psi)\left[2f^{'}(\psi) - \left\{\frac{f(y_n)-f(x_n)}{y_n-x_n}\right\} + 2f^{'}(x)\right] = -3f(\psi) \quad (4)$$

Putting $\psi = x_n$ and $x = x_{n+1}$ in equation (16)

$$(x_{n+1}-x_n)\left[2f^{'}(x_n) - \left\{\frac{f(y_n)-f(x_n)}{y_n-x_n}\right\} + 2f^{'}(x_{n+1})\right] = -3f(x_n)$$

$$(x_{n+1}-x_n)\left[2f^{'}(x_n) - \left\{\frac{f(y_n)-f(x_n)}{y_n-x_n}\right\} + 2f^{'}(\mathrm{y}_n)\right] = -3f(x_n)$$

$$x_{n+1} = x_n - \frac{3f(x_n)}{2\left[f^{'}(\mathrm{y}_n) + f^{'}(x_n)\right] - \left\{\frac{f(y_n)-f(x_n)}{y_n-x_n}\right\}} \quad (5)$$

for $n = 0,1,2,3,....$ $\quad y_n = x_n - \frac{f(x_n)}{f^{'}(x_n)}$ The relationship (5) is a proposed iterative method

3. RATE OF CONVERGENCE

The procedure will show that the new method has third order convergence.

Proof:

Let α be a simple zero of f. By expanding $f(x_n)$ and $f'(x_n)$ in the Taylor series.

$$f(x_n) = f'(\alpha)\left\{e_n + c_2 e_n^{\ 2} + c_3 e_n^{\ 3} + c_4 e_n^{\ 4} + o(e_n^{\ 5})\right\} \tag{i}$$

$$f'(x_n)=f'(\alpha)\left\{1+2c_2 e_n + 3c_3 e_n^{\ 2} + 4c_4 e_n^{\ 3} + 5c_5 e_n^{\ 4} + o(e_n^{\ 5})\right\} \tag{ii}$$

If $c_k = \dfrac{1}{k!}\dfrac{f^k}{f'(\alpha)}, k = 0,1,2,3,4,....$

From (i) , (ii) and $e_n = x_n - \alpha$, $x_n = e_n + \alpha$,we have

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} = \alpha + c_2 e_n^{\ 2} + 2(c_3 - c_2^{\ 2})e_n^{\ 3} - (7c_2c_3 - 4c_2^{\ 4} - 3c_4)e_n^{\ 4} + o(e_n^{\ 5}) \tag{iii}$$

$$f(y_n) = f'(\alpha)\left[c_2 e_n^{\ 2} + 2(c_3 - c_2^{\ 2})e_n^{\ 3} - (7c_2c_3 - 4c_2^{\ 3} - 3c_4 - c_2^{\ 2})\,e_n^{\ 4} + o(e_n^{\ 5})\right] \tag{iv}$$

$$f'(y_n) = f'(\alpha)\left[1 + 2c_2^{\ 2}e_n^{\ 2} + 4(c_2c_3 - c_2^{\ 3})e_n^{\ 3} + (-11c_2^{\ 2}c_3 + 6c_2^{\ 2}c_3 + 8c_2^{\ 4})e_n^{\ 4} + o(e_n^{\ 5})\right] \tag{v}$$

From (i), (ii) (iii) and (iv)

$$2\left[f'(y_n) + f'(x_n)\right] - \left[\frac{f(y_n) - f(x_n)}{y_n - x_n}\right]$$
$$= f'(\alpha)\left\{3 + c_n e_n + (5c_3 + 3c_2^{\ 2})e_n^{\ 2} + (7c_4 + 5c_2c_3 - 11c_2^{\ 3} + c_2^{\ 2})e_n^{\ 3} + o(e_n^{\ 4})\right\} \tag{iv}$$

$$\frac{3f(x_n)}{2\left[f'(y_n) + f'(x_n)\right] - \left[\dfrac{f(y_n) - f(x_n)}{y_n - x_n}\right]}$$

$$= \frac{f'(\alpha)\left\{3e_n + 3c_2 e_n^{\ 2} + 3c_3 e_n^{\ 3} + 3c_4 e_n^{\ 4} + o(e_n^{\ 5})\right\}}{f'(\alpha)\left\{3 + c_n e_n + (5c_3 + 3c_2^{\ 2})e_n^{\ 2} + (7c_4 + 5c_2c_3 - 11c_2^{\ 3} + c_2^{\ 2})e_n^{\ 3} + o(e_n^{\ 4})\right\}}$$

$$= e_n - \frac{1}{3}(2c_3 + 3c_2^{\ 3})e_n^{\ 3} + (-4c_4 - 3c_2c_3 - c_2^{\ 2} + 11c_2^{\ 3} + 3c_2^{\ 4})e_n^{\ 4} + o(e_n^{\ 5}) \tag{vii}$$

$$x_{n+1} = x_n - \left\{e_n - \frac{1}{3}(2c_3 + 3c_2^{\ 3})e_n^{\ 3} + (-4c_4 - 3c_2c_3 - c_2^{\ 2} + 11c_2^{\ 3} + 3c_2^{\ 4})e_n^{\ 4} + o(e_n^{\ 5})\right\} \tag{viii}$$

Substitute $e_{n+1} = x_{n+1} - \alpha$ in (vii) for the above discussion , we obtain

$$e_{n+1} = (\frac{2}{3}c_3 + c_2^{\ 3})e_n^{\ 3} + (-4c_4 - 3c_2c_3 - c_2^{\ 2} + 11c_2^{\ 3} + 3c_2^{\ 4})e_n^{\ 4} + o(e_n^{\ 5}) \tag{ix}$$

Error in equation (ix) will represents that proposed method is cubically convergent.

4. RESULTS AND DISCUSSIONS

In this section we have tested some examples such as algebraic and transcendental nonlinear equations. The following examples are taken from the literature. We have used MATLAB programming in 'format long' format to execute the results. The proposed method is compared with the Bisection method, Regula Falsi method (RF) and Newton Raphson method (NR). From the results as shown in Table 1-5, it is clear that the proposed method is best as compare to the existing methods.

Example 01:Consider the following equation. $f(x)=5x^2-2x-9$, $x_0=2$ interval (1,2).

Table:01

n	R.F method		N.R		Proposed method	
	x_n	A.E%	x_n	A.E%	x_n	A.E%
1	2.000000000000000		2.000000000000000		2.000000000000000	
2	1.461538461538462	9.492753508659192	1.611111111111111	38.888888888888886	1.564013840830450	43.598615916955019
3	1.542713567839196	1.375242878585770	1.557524059492564	5.358705161854771	1.556466054564076	0.754778626637376
4	1.554525103933483	0.194089269157094	1.556466408955511	0.105765053705276	1.556465996625054	0.000005793902225
5	1.556193102329637	0.027289429541710	1.556465996625116	0.000041233039449	1.556465996625054	0.000000000000022
6	1.556427647292354	0.003834933269919	1.556465996625054	0.000000000006262		

We have obtained the solution in given table:01, RF method takes 19 iterations to reach at x=1.556465996625054, when NR method takes 06 iterations and the proposed method only take 05 iterations, with less A.E%. if we compare the third iteration of proposed method it converges up to 15 decimal places.

Example 02**:** Let the nonlinear equation [2,]. $f(x)=\cos x-xe^x$ **, at** $x_0=1$ interval(0,1).

Table:02

n	R.F method		N.R		Proposed method	
	x_n	A.E%	x_n	A.E%	x_n	A.E%
1	1.000000000000000		1.000000000000000		1.000000000000000	
2	0.314665337800771	20.309202588168741	0.653079403526177	34.692059647382337	0.569391169671682	43.060883032831818
3	0.655376891806630	13.761952812417121	0.531343367606581	12.173603591959580	0.517871648253503	5.151952141817839
4	0.493721242608680	2.403612107377839	0.517909913135675	1.343345447090605	0.517757363683799	0.011428456970397
5	0.515133882643510	0.262348103894849	0.517757383164834	0.015252997084103	0.517757363682458	0.000000000134115

In table 02 the R.F method locate the root x=0.517757363682458 after 10 number of iterations, when NR method takes 05 number of iterations and our proposed method gives the same root on fourth number of iterations with less error.

Example 03: Let the nonlinear equation [7]. $f(x)=e^x-4x$, at $x_0=0$ interval(0,1)

Table:03

n	R.F method		N.R method		Proposed method	
	x_n	A.E%	x_n	A.E%	x_n	A.E%
1	0.000000000000000		0.000000000000000		0.000000000000000	
2	0.438266220812298	8.086326463090870	0.333333333333333	33.333333333333329	0.357329501185167	35.732950118516669
3	0.332526290172162	2.487666600922700	0.357246476043982	2.391314271064854	0.357402956181383	0.007345499621603
4	0.357982355959812	0.057939977842320	0.357402949373307	0.015647332932534	0.357402956181389	0.000000000000622
5	0.357406905749349	0.000394956796018	0.357402956181389	0.000000680808171		

In table 03 the R.F method locate the root x=0.357402956181389 after 19 number of iterations, when NR method takes 05 number of iterations and our method gives the same root on fourth number of iterations.

Example 04: Let the nonlinear equation. $f(x) = x^5 - x - 15$, at $x_0 = 2$ interval (1,2)

Table 04:

n	R.F method		N.R method		Proposed method	
	x_n	A.E%	x_n	A.E%	x_n	A.E%
1	2.000000000000000		2.000000000000000		2.000000000000000	
2	1.500000000000000	25.727892140240694	1.810126582278481	18.987341772151911	1.772926007592272	22.707399240772787
3	1.686274509803922	7.100441159848536	1.760332862906794	4.979371937168664	1.757285569872875	1.564043771939705
4	1.739310945636316	1.796797576609066	1.757289724399713	0.304313850708149	1.757278921402407	0.000664847046772
5	1.752845099990162	0.443382141224524	1.757278921538075	0.001080286163768	1.757278921402407	0.000000000000044
1	1.756191854510288	0.108706689211946	1.757278921402407	0.000000013566814		

In table 04 the R.F method locate the root x=1.757278921402407 after 26 number of iterations, when NR method takes 06 number of iterations and our method gives the same root on 05 number of iterations with less error.

Example 05: Let the nonlinear equation $f(x) = \sin x + x - 1$, at $x_0 = 1$ interval (0,1)

Table 05:

n	R.F method		N.R method		Proposed method	
	x_n	A.E%	x_n	A.E%	x_n	A.E%
1	1.000000000000000		1.000000000000000		1.000000000000000	
2	0.543044125185779	3.207069579721034	0.453697510156210	54.630248984379051	0.508893505154802	49.110649484519840
3	0.512407920114436	0.143449072586654	0.510580102574971	5.688259241876182	0.510973429701471	0.207992454666950
4	0.511035662227546	0.006223283897644	0.510973409195908	0.039330662093651	0.510973429388569	0.000000031290204
5	0.510976125506415	0.000269611784542	0.510973429388569	0.000002019266121		

The following results shown in table 05. Regula falsi method will give the root x=0.510973429388569 up to 13 number of iterations checked by the software, the same root by Newton's Raphson method performs in 5 number of iterations while the present proposed method gives in 04 number of iterations with less absolute percentile error.

CONCLUSION

The root locating problems are occurring in scientific work. In this research paper, the proposed method for fast convergence of non-linear equations has been developed to solve the non-linear equations. The proposed method is third order of convergence. Proposed method has compared with Regula Falsi method, well known Newton's Raphson method and found that composed method reduces the number of iterations with less absolute percentage error. All the results are executed on MATLAB 16 which has a machine precision of around 10^{-16} .

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