

Jordan Derivations on Lie Ideals of Prime Γ-Rings

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Abstract

Let M be a 2-torsion free prime Γ-ring satisfying the condition $a\alpha b\beta c = a\beta b\alpha c, \forall a, b, c \in M$ and $\alpha, \beta \in \Gamma$, U be a Lie ideal of M and d be a Jordan derivation of U into M. Then we prove the following results:

- $d(u\alpha v) = d(u)\alpha v + u\alpha d(v), \forall u, v \in U, \alpha \in \Gamma$, if U is an admissible Lie ideal of M.
- Every Jordan derivation on U is a derivation on U, if U is a commutative Lie ideal of M.

Keywords: Derivation, Jordan derivation, Lie ideal, admissible Lie ideal, square closed Lie ideal, prime Γ-ring.

2010 AMS Subject Classification: Primary 13N15; Secondary 16W10,17C50.

1 Introduction

The notion of a Γ-ring has been developed by Nobusawa [12], as a generalization of a ring. Following Barnes [3] generalized the concept of Nobusawa's Γ-ring as a more general nature. Now a days, Γ-ring theory is a showpiece of mathematical unification, bringing together several branches of the subject. It is the best research area for the Mathematicians and during 40 years, many classical ring theories have been generalized in Γ-rings by many authors.

The notions of derivation and Jordan derivation in Γ-rings have been introduced by Sapanci and Nakajima [13]. Afterwards, in the light of some significant results due to Jordan left derivation of a classical ring obtained by Jun and Kim in [10], some extensive results of left derivation and Jordan left derivation of a Γ-ring were determined by Ceven in [4]. In [7], Halder and Paul extended the results of [4] in Lie ideals.

In [8], Herstein proved a well-known result in prime rings that every Jordan derivation is a derivation. Afterwards many Mathematicians studied extensively the derivations in prime rings. In [2], Awtar extended this result in Lie ideals. (U, R)-derivations in rings have been introduced by Faraj, Haetinger and Majeed [5], as a generalization of Jordan derivations on a Lie ideals of a ring. The notion of a (U,R)-derivation extends the concept given in [2]. In the paper [5], they proved that if R is a prime ring, char$(R) \neq 2$, U a square closed Lie ideal of R and d a (U, R)- derivation of R, then $d(ur) = d(u)r + ud(r), \forall, u \in U, r \in R$. This result is a generalization of a result in Awtar [2, Theorem in section 3].

In this article, we prove if M be a 2-torsion free prime Γ-ring satisfying the condition $a\alpha b\beta c = a\beta b\alpha c, \forall a, b, c \in M$, $\alpha, \beta \in \Gamma$ and d be a Jordan derivation of U into M,where U is an admissible Lie ideal of M, then $d(u\alpha v) = d(u)\alpha v + u\alpha d(v), \forall u, v \in U, \alpha \in \Gamma$ and if $u\alpha u \in U, \forall u \in U, \alpha \in \Gamma$ and U is commutative, then $d(u\alpha v) = d(u)\alpha v + u\alpha d(v), \forall u, v \in U$ and $\alpha \in \Gamma$.

Let M and Γ be additive abelian groups. If there is a mapping $M \times \Gamma \times M \to M$ (sending (x, α, y) into $x\alpha y$) such that

- $(x + y)\alpha z = x\alpha z + y\alpha z, x(\alpha + \beta)y = x\alpha y + x\beta y, x\alpha(y + z) = x\alpha y + x\alpha z$
- $(x\alpha y)\beta z = x\alpha(y\beta z), \forall x, y, z \in M$ and $\alpha, \beta \in \Gamma$

then M is called a Γ-*ring*. This concept is more general than a ring and was introduced by Barnes [3]. A Γ-ring M is called a *prime* Γ-*ring* if $\forall a, b \in M, a\Gamma M\Gamma b = 0$ implies $a = 0$ or $b = 0$ and M is called *semiprime* if $a\Gamma M\Gamma a = 0$ (with $a \in M$) implies $a = 0$. A Γ-ring M is *2-torsion free* if $2a = 0$ implies $a = 0, \forall a \in M$.

For any $x, y \in M$ and $\alpha \in \Gamma$, we induce a new product , the *Lie product* by $[x, y]_\alpha = x\alpha y - y\alpha x$.
An additive subgroup $U \subset M$ is said to be a *Lie ideal* of M if whenever $u \in U, m \in M$ and $\alpha \in \Gamma$, then $[u, m]_\alpha \in U$.
In the main results of this article we assume that the Lie ideal U verifies $u\alpha u \in U, \forall u \in U$. A Lie ideal of this type is called a *square closed Lie ideal*.
Furthermore, if the Lie ideal U is square closed and U is not contained in $Z(M)$,where $Z(M)$denotes the center of M,then U is called an *admissible Lie ideal* of M.

Let M be a Γ-ring. An additive mapping $d : M \to M$ is called a *derivation* if $d(a\alpha b) = d(a)\alpha b + a\alpha d(b), \forall a, b \in M$ and $\alpha \in \Gamma$.
An additive mapping $d : M \to M$ is called a *Jordan derivation* if
$d(a\alpha a) = d(a)\alpha a + a\alpha d(a), \forall a \in M$ and $\alpha \in \Gamma$.

Throughout the article, we use the condition $a\alpha b\beta c = a\beta b\alpha c, \forall a, b, c \in M$ and $\alpha, \beta \in \Gamma$ and this is represented by (*).

We make the basic commutator identities:

- $[x\alpha y, z]_\beta = [x, z]_\beta \alpha y + x[\alpha, \beta]_z y + x\alpha[y, z]_\beta$ and
- $[x, y\alpha z]_\beta = [x, y]_\beta \alpha z + y[\alpha, \beta]_x z + y\alpha[x, z]_\beta$, $\forall a, b, c \in M$ and $\forall \alpha, \beta \in \Gamma$.

According to the condition (*), the above two identities reduces to:

- $[x\alpha y, z]_\beta = [x, z]_\beta \alpha y + x\alpha[y, z]_\beta$ and
- $[x, y\alpha z]_\beta = [x, y]_\beta \alpha z + y\alpha[x, z]_\beta, \forall a, b, c \in M$ and $\forall \alpha, \beta \in \Gamma$.

2 Jordan Derivation on Lie Ideal

2.1 Definition: Let M be a Γ-ring and U be a Lie ideal of M. An additive mapping $d : U \to M$ is said to be a *Jordan derivation on Lie ideal* of M if $d(u\alpha u) = d(u)\alpha u + u\alpha d(u), \forall u \in U$ and $\alpha \in \Gamma$.

2.2 Example: Let R be a ring of characteristic 2 having a unity element 1.
Let $M = M_{1,2}(R)$ and $\Gamma = \{\begin{pmatrix} n.1 \\ n.1 \end{pmatrix} : n \in \mathbf{Z}$, n is not divisible by 2$\}$.
Then M is a Γ-ring.
Let $N = \{(x, x) : x \in R\} \subseteq M$.
Now $\forall (x, x) \in N, (a, b) \in M$ and $\begin{pmatrix} n \\ n \end{pmatrix} \in \Gamma$, we have

$(x, x)\begin{pmatrix} n \\ n \end{pmatrix}(a, b) - (a, b)\begin{pmatrix} n \\ n \end{pmatrix}(x, x)$
$= (xna - bnx, xnb - anx)$
$= (xna - 2bnx + bnx, bnx - 2anx + xna)$
$= (xna + bnx, bnx + xna) \in N$.
Therefore, N is a Lie ideal of M.

2.3 Example: Let M be a Γ-ring satisfying the condition (*) and let U be a Lie ideal of M. Let $a \in M$ and $\alpha \in \Gamma$ be fixed elements.
Define $d : U \to M$ by $d(x) = a\alpha x - x\alpha a, \forall x \in U$.
Now $\forall y \in U$ and $\beta \in \Gamma$, we have
$d(x\beta y) = a\alpha x\beta y - x\beta y\alpha a$
$= a\alpha x\beta y - x\alpha a\beta y + x\alpha a\beta y - x\beta y\alpha a$
$= (a\alpha x - x\alpha a)\beta y + x\beta a\alpha y - x\beta y\alpha a$, by using (*).
$= (a\alpha x - x\alpha a)\beta y + x\beta(a\alpha y - y\alpha a)$
$= d(x)\beta y + x\beta d(y)$, for every $x, y \in U$ and $\beta \in \Gamma$).
Therefore d is a derivation on U.

2.4 Example: Let M be a Γ-ring and let U be a Lie ideal of M.
Let $d : U \to M$ is a derivation.
Let $M_1 = \{(x, x) : x \in M\}$ and $\Gamma_1 = \{(\alpha, \alpha) : \alpha \in \Gamma\}$.
Define addition and multiplication on M_1 as follows:
$(x, x) + (y, y) = (x + y, x + y)$
and $(x, x)(\alpha, \alpha)(y, y) = (x\alpha y, x\alpha y)$.
Then M_1 is a Γ_1-ring.
Define $U_1 = \{(u, u) : u \in U\}$.
Now $(u, u)(\alpha, \alpha)(x, x) - (x, x)(\alpha, \alpha)(u, u)$
$= (u\alpha x, u\alpha x) - (x\alpha u, x\alpha u)$
$= (u\alpha x - x\alpha u, u\alpha x - x\alpha u) \in U_1$ for $u\alpha x - x\alpha u \in U$.
Hence U_1 is a Lie ideal of M_1.
Now define a mapping $D : U_1 \to M_1$ by $D((u, u)) = (d(u), d(u))$. Then it is clear that D is a Jordan derivation on U which is not a derivation on U.

2.5 Lemma: Let M be a Γ-ring and U be a Lie ideal of M such that $u\alpha u \in U, \forall u \in U$ and $\alpha \in \Gamma$. If d is a Jordan derivation of U into M, then $\forall a, b, c \in U$ and $\alpha, \beta \in \Gamma$, the following statements hold:

(i) $d(a\alpha b + b\alpha a) = d(a)\alpha b + d(b)\alpha a + a\alpha d(b) + b\alpha d(a)$.
(ii) $d(a\alpha b\beta a + a\beta b\alpha a) = d(a)\alpha b\beta a + d(a)\beta b\alpha a + a\alpha d(b)\beta a + a\beta d(b)\alpha a + a\alpha b\beta d(a) + a\beta b\beta d(a)$.

In particular,if M is 2-torsion free and if M satisfies the condition (*), then
(iii) $d(a\alpha b\beta a) = d(a)\alpha b\beta a + a\alpha d(b)\beta a + a\alpha b\beta d(a)$.
(iv) $d(a\alpha b\beta c + c\alpha b\beta a) = d(a)\alpha b\beta c + d(c)\alpha b\beta a + a\alpha d(b)\beta c + c\alpha d(b)\beta a + a\alpha b\beta d(c) + c\alpha b\beta d(a)$.

Proof: Since U is a Lie ideal satisfying the condition $a\alpha a \in U, \forall a \in U, \alpha \in \Gamma$. For $a, b \in U, \alpha \in \Gamma, (a\alpha b + b\alpha a) = (a + b)\alpha(a + b) - (a\alpha a + b\alpha b)$ and so $(a\alpha b + b\alpha a) \in U$.
Also, $[a, b]_\alpha = a\alpha b - b\alpha a \in U$ and it follows that $2a\alpha b \in U$.
Hence $4a\alpha b\beta c = 2(2a\alpha b)\beta c \in U, \forall a, b, c \in U, \alpha, \beta \in \Gamma$.

Thus $d(a\alpha b+b\alpha a) = d((a+b)\alpha(a+b)-(a\alpha a+b\alpha b)) = d(a+b)\alpha(a+b)+(a+b)\alpha d(a+b)-d(a)\alpha a - a\alpha d(a)-d(b)\alpha b-b\alpha d(b) = d(a)\alpha a+d(a)\alpha b+d(b)\alpha a+d(b)\alpha b+a\alpha d(a)+a\alpha d(b)+b\alpha d(a)+b\alpha d(b) - d(a)\alpha a - a\alpha d(a) - d(b)\alpha b - b\alpha d(b) = d(a)\alpha b + a\alpha d(b) + d(b)\alpha a + b\alpha d(a)$.

Replacing $a\beta b + b\beta a$ for b in (i) we get
$d(a\alpha(a\beta b+b\beta a)+(a\beta b+b\beta a)\alpha a) = d(a)\alpha(a\beta b+b\beta a)+a\alpha d(a\beta b+b\beta a)+d(a\beta b+b\beta a)\alpha a+(a\beta b+b\beta a)\alpha d(a)$.

This implies that
$d(a\alpha a)\beta b + (a\alpha a)\beta d(b) + d(b)\beta(a\alpha a) + b\beta d(a\alpha a) + d(a\alpha b\beta a + a\beta b\alpha a) = d(a)\alpha a\beta b + d(a)\alpha b\beta a + a\alpha d(a)\beta b + a\alpha a\beta d(b) + a\alpha d(b)\beta a + a\alpha b\beta d(a) + d(a)\beta b\alpha a + a\beta d(b)\alpha a + d(b)\beta a\alpha a + b\beta d(a)\alpha a + a\beta b\alpha d(a) + b\beta a\alpha d(a)$, by using (i).

This implies that
$d(a)\alpha a\beta b+a\alpha d(a)\beta b+a\alpha a\beta d(b)+d(b)\beta a\alpha a+b\beta d(a)\alpha a+b\beta a\alpha d(a)+d(a\alpha b\beta a+a\beta b\alpha a) = d(a)\alpha a\beta b+d(a)\alpha b\beta a + a\alpha d(a)\beta b + a\alpha a\beta d(b) + a\alpha d(b)\beta a + a\alpha b\beta d(a) + d(a)\beta b\alpha a + a\beta d(b)\alpha a + d(b)\beta a\alpha a + b\beta d(a)\alpha a + a\beta b\alpha d(a) + b\beta a\alpha d(a)$.

Now canceling the like terms from both sides we get the required result.

Using the condition (*) and since M is 2-torsion free, (iii) follows from (ii).

And finally (iv) is obtained by replacing $a + c$ for a in (iii).

2.6 Definition: Let M be a Γ-ring and U be a Lie ideal of M and let d be a Jordan derivation of U into M. We define
$\phi_\alpha(u, v) = d(u\alpha v) - d(u)\alpha v - u\alpha d(v), \forall u, v \in U$ and $\alpha \in \Gamma$.

2.7 Lemma: Let M be a Γ-ring and U be a Lie ideal of M and let d be a Jordan derivation of U into M, then
$\forall u, v, w \in U$ and $\alpha, \beta \in \Gamma$:
(i)$\phi_\alpha(u, v) = -\phi_\alpha(v, u)$
(ii) $\phi_\alpha(u + w, v) = \phi_\alpha(u, v) + \phi_\alpha(w, v)$
(iii) $\phi_\alpha(u, v + w) = \phi_\alpha(u, v) + \phi_\alpha(u, w)$
(iv) $\phi_{\alpha+\beta}(u, v) = \phi_\alpha(u, v) + \phi_\beta(u, v)$

The proofs are obvious by using the definition 2.6

Remark: It is clear that $\phi_\alpha(u,v)=0$ if and only if d is a derivation on U.

2.8 Lemma: Let M be a 2-torsion free Γ-ring satisfying the condition (*) and U be a Lie ideal of M. If d is a Jordan derivation on U then $\phi_\alpha(u,v)\beta w\gamma[u,v]_\alpha+[u,v]_\alpha\beta w\gamma\phi_\alpha(u,v)=0, \forall u,v,w\in U$ and $\alpha,\beta,\gamma\in\Gamma$.

Proof: Let $x=4(u\alpha v\beta w\gamma v\alpha u+v\alpha u\beta w\gamma u\alpha v)$.
Then by using Lemma 2.5(iv), we have
$d(x)=d((2u\alpha v)\beta w\gamma(2v\alpha u)+(2v\alpha u)\beta w\gamma(2u\alpha v))$
$=d(2u\alpha v)\beta w\gamma(2v\alpha u)+2u\alpha v d\beta d(w)\gamma 2v\alpha u+2u\alpha v\beta w\gamma d(2v\alpha u)+d(2v\alpha u)\beta w\gamma(2u\alpha v)+2v\alpha u\beta d(w)\gamma 2u\alpha v+$
$2v\alpha u\beta w\gamma d(2u\alpha v)$,

On the other hand, by using Lemma 2.5(iii),we have
$d(x)=d(u\alpha(4v\beta w\gamma v)\alpha u+v\alpha(4u\beta w\gamma u)\alpha v)$
$=d(u)\alpha 4v\beta w\gamma v\alpha u+u\alpha d(4v\beta w\gamma v)\alpha u+u\alpha 4v\beta w\gamma v\alpha d(u)+$
$d(v)\alpha 4u\beta w\gamma u\alpha v+v\alpha d(4u\beta w\gamma u)\alpha v+v\alpha 4u\beta w\gamma u\alpha d(v)$
$=4d(u)\alpha v\beta w\gamma v\alpha u+4u\alpha d(v)\beta w\gamma v\alpha u+4u\alpha v\beta d(w)\gamma v\alpha u+$
$4u\alpha v\beta w\gamma d(v)\alpha u+4u\alpha v\beta w\gamma v\alpha d(u)+4d(v)\alpha u\beta w\gamma u\alpha v+$
$4v\alpha d(u)\beta w\gamma u\alpha v+4v\alpha u\beta d(w)\gamma u\alpha v+4v\alpha u\beta w\gamma d(u)\alpha v+$
$4v\alpha u\beta w\gamma u\alpha d(v)$.

Comparing the right side of $d(x)$ and using the definition 2.6,we obtain

$4(\phi_\alpha(u,v)\beta w\gamma v\alpha u+\phi_\alpha(v,u)\beta w\gamma u\alpha v+$
$u\alpha v\beta w\gamma\phi_\alpha(v,u)+v\alpha u\beta w\gamma\phi_\alpha(u,v))=0$

Using Lemma 2.7(i),we have

$4(\phi_\alpha(u,v)\beta[u,v]_\alpha\gamma v\alpha u-\phi_\alpha(u,v)\beta w\gamma u\alpha v-$
$u\alpha v\beta w\gamma\phi_\alpha(u,v)+v\alpha u\beta w\gamma\phi_\alpha(u,v))=0$
$=-4(\phi_\alpha(u,v)\beta w\gamma(u\alpha v-v\alpha u)+(u\alpha v-v\alpha u)\beta w\gamma\phi_\alpha(u,v))=0$
$=4(\phi_\alpha(u,v)\beta w\gamma[u,v]_\alpha+[u,v]_\alpha\beta w\gamma\phi_\alpha(u,v))=0$

Since M is 2-torsion free and by using (*), we have
$\phi_\alpha(u,v)\beta w\gamma[u,v]_\alpha+[u,v]_\alpha\beta w\gamma\phi_\alpha(u,v)=0, \forall u,v,w\in U, \alpha,\beta,\gamma\in\Gamma$

2.9 Lemma: Let U be a Lie ideal of a 2-torsion free prime Γ-ring M and U is not contained in $Z(M)$. Then there exists an ideal I of M such that $[I,M]_\Gamma\subseteq U$ but $[I,M]_\Gamma$ is not contained in $Z(M)$.

Proof: Since M is 2-torsion free and U is not contained in $Z(M)$, it follows from the result in [1] that $[U,U]_\Gamma\neq 0$ and $[I,M]_\Gamma\subseteq U$,where $I=I\Gamma[U,U]_\Gamma\Gamma M\neq 0$ is an ideal of M generated by $[U,U]_\Gamma$.

Now U is not contained in $Z(M)$ implies that $[I,M]_\Gamma$ is not contained in $Z(M)$; for if $[I,M]_\Gamma\subseteq Z(M)$, then $[I,[I,M]_\Gamma]_\Gamma=0$, which implies that $I\subseteq Z(M)$ and hence $I\neq 0$ is an ideal of M, so $M=Z(M)$.

2.10 Lemma: Let U be a Lie ideal of a 2-torsion free prime Γ-ring M satisfying the condition (*) and U is not contained in $Z(M)$. If $a, b \in M$ (resp.$b \in U$ and $a \in M$) such that $a\alpha U\beta b = 0, \forall\alpha, \beta \in \Gamma$, then $a = 0$ or $b = 0$.

Proof: By Lemma 2.9, there exists an ideal I of M such that $[I, M]_\Gamma \subseteq U$ and $[I, M]_\Gamma$ is not contained in $Z(M)$. Now take $u \in U, c \in I, m \in M$ and $\alpha, \beta, \gamma \in \Gamma$, we have $[c\alpha a\beta u, m]_\Gamma \in [I, M]_\Gamma \subseteq U$ and so

$0 = a\delta[c\alpha a\beta u, m]_\gamma \mu b, \forall\delta, \mu \in \Gamma$.
$= a\delta[c\alpha a, m]_\gamma \beta u\mu b + a\delta c\alpha a\beta[u, m]_\gamma \mu b$, by using (*)
$= a\delta[c\alpha a, m]_\gamma \beta u\mu b$ since $a\beta[u, m]_\gamma \mu b \in a\beta U\mu b = 0$
$= a\delta(c\alpha a\gamma m - m\gamma c\alpha a)\beta u\mu b$
$= a\delta c\alpha a\gamma m\beta u\mu b - a\delta m\gamma c\alpha a\beta u\mu b$
$= a\delta c\alpha a\gamma m\beta u\mu b$, by using assumption $a\beta u\mu b = 0$

Thus $a\delta I\alpha a\gamma M\beta U\mu b = 0$. If $a \neq 0$, then by the primeness of $M, U\mu b = 0$.
Now if $u \in U$ and $m \in M$, then $[u, m]_\alpha \in U, \forall\alpha \in \Gamma$.
Hence $[u, m]_\alpha \beta b = 0, \forall\beta \in \Gamma$. Since $m\alpha u\beta b = 0, u\alpha m\beta b = 0$.
Since $U \neq 0$, we must have $b = 0$.
In the similar manner, it can be shown that if $b \neq 0$, then $a = 0$.

2.11 Lemma: Let M be a 2-torsion free prime Γ-ring and U an admissible Lie ideal of M. If $a, b \in M$ (resp. $a \in M$ and $b \in U$) such that $a\alpha x\beta b + b\alpha x\beta a = 0, \forall x \in U$ and $\alpha, \beta \in \Gamma$, then $a\alpha x\beta b = b\alpha x\beta a = 0$.

Proof: For $x, y \in U$ and using the relation
$a\alpha x\beta b = -b\alpha x\beta a$ three times, we obtain
$a\alpha x\beta b\gamma y\delta a\alpha x\beta b = -4b\alpha x\beta a\gamma y\delta a\alpha x\beta b = -b\alpha(4x\beta a\gamma y)\delta a\alpha(x\beta b)$
$= 4a\alpha x\beta a\gamma y\delta b\alpha x\beta b = 4a\alpha x\beta b\gamma y\delta a\alpha x\beta b$.
Thus $8a\alpha x\beta b\gamma y\delta a\alpha x\beta b = 0$.
By the 2-torsion freeness of M, we have
$(a\alpha x\beta b)\gamma y\delta(a\alpha x\beta b) = 0$.
By Lemma 2.10, we have $a\alpha x\beta b = 0$.
Similarly, it can be shown that $b\alpha x\beta a = 0$.

2.12 Lemma: Let M be a 2-torsion free prime Γ-ring and U an admissible Lie ideal of M. Let $G_1, G_2, ..., G_n$ be additive groups, $S : G_1 \times G_2 \times ... \times G_n \to M$ and $T : G_1 \times G_2 \times ... \times G_n \to M$ be mappings which are additive in each argument. If $S_\alpha(a_1, ..., a_n)\beta x\gamma T_\alpha(a_1, ..., a_n) = 0$, for every $x \in U, a_i \in G, i = 1, 2, ..., n, \alpha, \beta, \gamma \in \Gamma$, then $S_\alpha(a_1, ..., a_n)\beta x\gamma T_\delta(b_1, ..., b_n) = 0$

Proof: It suffices to prove the case $n = 1$.
The general proof is obtained by induction on n.
If $S_\alpha(a)\beta x\gamma T_\alpha(a) = 0$, for every $u \in U, a \in G_1$, we get
$(T_\alpha(a)\beta x\gamma S_\alpha(a))\mu y\nu(T_\alpha(a)\beta x\gamma S_\alpha(a)) = 0$, for all $x, y \in U$ and $\mu, \nu \in \Gamma$.
Then by Lemma 2.10, $T_\alpha(a)\beta x\gamma S_\alpha(a) = 0$, for every $x \in U, a \in G_1$ and $\beta, \gamma \in \Gamma$.
Now linearizing $T_\alpha(a)\beta x\gamma S_\alpha(a) = 0$ we obtain
$S_\alpha(a)\beta x\gamma T_\alpha(b) + S_\alpha(b)\beta x\gamma T_\alpha(a) = 0$, for every $x \in U, a, b \in G_1$.

Hence $(S_\alpha(a)\beta x\gamma T_\alpha(b))\mu y\nu((S_\alpha(a)\beta x\gamma T_\alpha(b))$
$= -S_\alpha(a)\beta x\gamma T_\alpha(b)\mu y\nu S_\alpha(b)\beta x\gamma T_\alpha(a) = 0, \forall x, y \in U.$
By Lemma 2.10, $S_\alpha(a)\beta x\gamma T_\alpha(b) = 0$
Similarly we can prove that $T_\alpha(b)\beta x\gamma S_\alpha(a) = 0, \forall a, b \in G_1$ and $\alpha, \beta, \gamma \in \Gamma$.
Putting $\alpha + \delta$ for α in the equation $S_\alpha(a)\beta x\gamma T_\alpha(b) = 0$ and using Lemma 2.7(iv), we have
$S_\alpha(a)\beta x\gamma T_\delta(b) + S_\delta(a)\beta x\gamma T_\alpha(b) = 0.$
Therefore, we have $(S_\alpha(a)\beta x\gamma T_\delta(b))\mu y\nu(S_\alpha(a)\beta x\gamma T_\delta(b))$
$= -S_\alpha(a)\beta x\gamma T_\delta(b)\mu y\nu(S_\delta(a)\beta x\gamma T_\alpha(b) = 0$
Hence by Lemma 2.10, $S_\alpha(a)\beta x\gamma T_\delta(b) = 0.$

2.13 Theorem: Let M be a 2-torsion free prime Γ-ring satisfying the condition (*) and U an admissible Lie ideal of M. If $d : U \to M$ is a Jordan derivation, then $d(u\alpha v) = d(u)\alpha v + u\alpha d(v), \forall u, v \in U, \alpha \in \Gamma$.

Proof: By Lemma 2.8, we have
$\phi_\alpha(u, v)\beta w\gamma[u, v]_\alpha + [u, v]_\alpha\beta w\gamma\phi_\alpha(u, v) = 0, \forall u, v, w \in U$ and $\alpha, \beta, \gamma \in \Gamma$.
Using Lemmas 2.11 and 2.12, we have
$\phi_\alpha(u, v)\beta w\gamma[x, y]_\delta = 0, \forall u, v, w, x, y \in U$ and $\alpha, \beta, \gamma, \delta \in \Gamma$.
Since U is an admissible Lie ideal of M, $[x, y]_\delta$ is not contained in $Z(M)$.
Therefore, by Lemma 2.10, we get $\phi_\alpha(u, v) = 0$.

2.14 Theorem: Let M be a 2-torsion free prime Γ-ring satisfying the condition (*) and U a commutative Lie ideal of M such that $u\alpha u \in U, \forall u \in U$ and $\alpha \in \Gamma$. Then every Jordan derivation on U is a derivation on U.

Proof: Suppose U is a commutative Lie ideal of M.
Let $a \in U$ and $x \in M$.
Then $[a, x]_\alpha \in U, \forall \alpha \in \Gamma$ and so commutes with a.
Now for $x, y \in M$, we have $a\beta[a, x\gamma y]_\alpha = [a, x\gamma y]_\alpha\beta a, \forall \alpha, \beta, \gamma \in \Gamma$.
Expanding $[a, x\gamma y]_\alpha$ as $[a, x]_\alpha\gamma x + x\gamma[a, y]_\alpha$ and using the fact that
a commutes with this, with $[a, x]_\alpha$ and $[a, y]_\alpha$, we have $2[a, x]_\alpha\gamma[a, y]_\alpha = 0$ and so $[a, x]_\alpha\gamma[a, y]_\alpha = 0$, as M is 2-torsion free.

Replacing y by $a\beta x$ in $[a, x]_\alpha\gamma[a, y]_\alpha = 0$ and then using (*)
we have $[a, x]_\alpha\gamma M\beta[a, x]_\alpha = 0, \forall x \in M$ and $\alpha, \beta \in \Gamma$.
Since M is prime, $[a, x]_\alpha = 0$ and so $U \subseteq Z(M)$.
Hence by Lemma 2.5(i),we have $2d(a\alpha b) = 2(d(a)\alpha b + a\alpha d(b))$.
By the 2-torsion freeness of M, we get $d(a\alpha b) = d(a)\alpha b + a\alpha d(b)$.

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