Subordination Properties for Analytic Functions Defined By Convolution

Ezekiel A. Oyekan[1*] and Samuel O. Ayinde[2]

1. Faculty of Science and Science Education, Department of Mathematics and Statistics, Bowen University, P.M.B 284, Iwo, Osun State, Nigeria
2. Department of Mathematical Sciences, Ekiti State University, Ado-Ekiti, Nigeria

* E-mail of the corresponding author: shalomfa@yahoo.com

Abstract

In this paper, we derive several interesting subordination results for certain class of analytic functions defined by the Salagean Operator which was introduced and studied by Oyekan et. Al.[1].

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1. INTRODUCTION AND DEFINITIONS

Let A denote the class functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1,1}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$. For two functions $f(z)$ and $g(z)$ given by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \text{ and } g(z) = \sum_{k=2}^{\infty} c_k z^k \tag{1.2}$$

Their Hadamard product (or Convolution) is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k c_k z^k \tag{1.3}$$

Let D^n be the Salagean operator (see [2]),

$D^n: A \to A, n \in \mathbb{N},$ defined as

$D^0 f(z) = f(z)$

$D^1 f(z) = Df(z) = zf'(z)$

…

$D^n f(z) = D\left(D^{n-1} f(z)\right).$

By using the above Salagean operator, Oyekan et. al.[1] introduced and investigated certain properties of the class $E_{m,n}(\gamma, \eta; A, B, \alpha, \beta)$. This class is due to the class $E_{m,n}(\Phi, \psi; A, B, \alpha)$ earlier introduced and studied by Eker and Seker [3]. The class $E_{m,n}(\gamma, \eta; A, B, \alpha, \beta)$ is defined as the class of all functions $f(z) \in A$ which satisfies the following condition:

$$\frac{D^n\gamma(z)}{D^m\eta(z)} \prec (1-\alpha)\left(\frac{1+Az}{1-Bz}+\alpha\right) \tag{1.4}$$

where $\prec$ denotes subordination, A and B are arbitrary fixed numbers $-1 \le B < A \le 1, -1 \le B < 0, 0 \le \alpha < 1, m \in \mathbb{N}, n \in \mathbb{N}_0$ $(m > n)$, and $\eta(z) \neq 0$.

We note that

$$\gamma(z) = (f * \Phi)^\beta(z) = z^\beta + \sum_{k=2}^{\infty} a_k(\beta)\lambda_k(\beta)z^{k+\beta-1} \tag{1.5}$$

$$\eta(z) = (f * \psi)^\beta(z) = z^\beta + \sum_{k=2}^{\infty} a_k(\beta)\mu_k(\beta)z^{k+\beta-1}, \tag{1.6}$$

and $\beta \in \mathbb{N}$ (see [1]).

Also,

$$\Phi(z) = z + \sum_{k=2}^{\infty} \lambda_k z^k \ \ \textit{and} \ \ \psi(z) = z + \sum_{k=2}^{\infty} \mu_k z^k \tag{1.7}$$

(see [2]).

Furthermore, for functions $f(z) \in E_{m,n}(\gamma, \eta; A, B, \alpha, \beta)$, Oyekan et. al.[1] proved the following inequality:

Lemma 1.1 ([1]). If $f(z) \in A$ satisfies

$$\sum_{k=2}^{\infty}\Big[(1-B)\Big(\beta^m - \beta^n + (k+\beta-1)^m\lambda_k(\beta) - (k+\beta-1)^n\mu_k(\beta)\Big) + (A-B)(1-\alpha)(k+\beta-1)^n\mu_k(\beta)\Big]|a_k(\beta)| \le (A-B)(1-\alpha)\beta^n \tag{1.8}$$

For some $\lambda_k(\beta) \ge 0, \mu_k(\beta) \ge 0, \lambda_k(\beta) \ge \mu_k(\beta), \alpha(0 \le \alpha < 1), \ \beta, m \in \mathbb{N} \ and \ n \in \mathbb{N}_0$, then

$f(z) \in E_{m,n}(\gamma, \eta; A, B, \alpha, \beta)$.

Let $E^*_{m,n}(\gamma, \eta; A, B, \alpha, \beta)$ denote the class of functions $f(z) \in A$ whose Taylor-Maclaurin coefficients $a_{k,}(\beta)$ satisfy the condition (1.8).

We note that

$$E^*_{m,n}(\gamma, \eta; A, B, \alpha, \beta) \subseteq E_{m,n}(\gamma, \eta; A, B, \alpha, \beta) \tag{1.9}$$

and that $a_k(\beta)$, $\lambda_k(\beta)$, and $\mu_k(\beta)$ are the coefficients a_k, λ_k, and μ_k depending on β.

In this paper, we obtain a sharp subordination result associated with the class $E^*_{m,n}(\gamma, \eta; A, B, \alpha, \beta)$ by using the same techniques as in [4] (see also [5-7]).

However, before we state and prove our main result we need the following definitions and lemmas.

Definition 1.1 (Subordination Principle**).** Let $g(z)$ be analytic and univalent in U. If $f(z)$ is analytic in U, $f(0) = g(0)$ and $f(U) \subset g(U)$, then we see that the function $f(z)$ is subordinate to $g(z)$, and we write $f(z) \prec g(z)$.

Definition 1.2 (Subordinating Factor Sequence). A sequence $\{b_k\}_{k=1}^{\infty}$ of complex numbers is called a subordinating factor sequence if, whenever $f(z)$ is analytic, univalent and convex in U, we have the subordination given by

$$\sum_{k=2}^{\infty} b_k a_k z^k \prec f(z) \qquad (z \in U, \ a_1 = 1) \tag{1.10}$$

Lemma 1.2 ([8]). The sequence $\{b_k\}_{k=1}^{\infty}$ is a subordinating factor sequence if and only if

$$\text{Re}\{1+2\sum_{k=1}^{\infty} b_k z^k\}>0 \qquad (z\in U). \tag{1.11}$$

MAIN THEOREM

Theorem 2.1. Let the function $f(z)$ defined by (1.1) be in the class $E^{*}_{m,n}(\gamma,\eta;A,B,\alpha,\beta)$ where $-1\le B<A\le 1, -1\le B<0, 0\le\alpha<1, m\in\mathbb{N}, n\in\mathbb{N}_0\ (m>n)$, and $\lambda_k(\beta)\ge\mu_k(\beta)\ge 0$. Also let κ denote the familiar class of functions $f(z)\in A$ which are also univalent and convex in U. Then,

$$\frac{(1-B)[\beta^m-\beta^n+(1+\beta)^m\lambda_2(\beta)-(1+\beta)^n\mu_2(\beta)]+(A-B)(1-\alpha)(1+\beta)^n\mu_2(\beta)}{2\{(A-B)(1-\alpha)\beta^n+(1-B)[\beta^m-\beta^n+(1+\beta)^m\lambda_2(\beta)-(1+\beta)^n\mu_2(\beta)]+(A-B)(1-\alpha)(1+\beta)^n\mu_2(\beta)\}}(f*g)(z)$$
$$\prec g(z) \qquad (z\in U, g\in K), \tag{2.1}$$

and

$$\operatorname{Re}(f(z))>\frac{\{(A-B)(1-\alpha)\beta^n+(1-B)[\beta^m-\beta^n+(1+\beta)^m\lambda_2(\beta)-(1+\beta)^n\mu_2(\beta)]+(A-B)(1-\alpha)(1+\beta)^n\mu_2(\beta)}{(1-B)[\beta^m-\beta^n+(1+\beta)^m\lambda_2(\beta)-(1+\beta)^n\mu_2(\beta)]+(A-B)(1-\alpha)(1+\beta)^n\mu_2(\beta)\}},$$
$$(z\in U). \tag{2.2}$$

The constant

$$\frac{(1-B)[\beta^m-\beta^n+(1+\beta)^m\lambda_2(\beta)-(1+\beta)^n\mu_2(\beta)]+(A-B)(1-\alpha)(1+\beta)^n\mu_2(\beta)}{2\{(A-B)(1-\alpha)\beta^n+(1-B)[\beta^m-\beta^n+(1+\beta)^m\lambda_2(\beta)-(1+\beta)^n\mu_2(\beta)]+(A-B)(1-\alpha)(1+\beta)^n\mu_2(\beta)\}}$$

is the best estimate.

Proof. Let $f(z)\in E^{*}_{\min}(\mu,\eta;A,B,\alpha,\beta)$ and let $g(z)=z+\sum_{k=2}^{\infty}c_k z^k\in K$, then

$$\frac{(1-B)[\beta^m-\beta^n+(1+\beta)^m\lambda_2(\beta)-(1+\beta)^n\mu_2(\beta)]+(A-B)(1-\alpha)(1+\beta)^n\mu_2(\beta)}{2\{(A-B)(1-\alpha)\beta^n+(1-B)[\beta^m-\beta^n+(1+\beta)^m\lambda_2(\beta)-(1+\beta)^n\mu_2(\beta)]+(A-B)(1-\alpha)(1+\beta)^n\mu_2(\beta)\}}(f*g)(z)$$

(2.3)

$$=\frac{(1-B)[\beta^m-\beta^n+(1+\beta)^m\lambda_2(\beta)-(1+\beta)^n\mu_2(\beta)]+(A-B)(1-\alpha)(1+\beta)^n\mu_2(\beta)}{2\{(A-B)(1-\alpha)\beta^n+(1-B)[\beta^m-\beta^n+(1+\beta)^m\lambda_2(\beta)-(1+\beta)^n\mu_2(\beta)]+(A-B)(1-\alpha)(1+\beta)^n\mu_2(\beta)\}}\times$$
$$\left(z+\sum_{k=2}^{\infty}a_k c_k z^k\right).$$

By invoking definition (1.2), the subordination (2.1) of our theorem will hold true if the sequence

$$\left\{\frac{(1-B)[\beta^m-\beta^n+(1+\beta)^m\lambda_2(\beta)-(1+\beta)^n\mu_2(\beta)]+(A-B)(1-\alpha)(1+\beta)^n\mu_2(\beta)}{2\{(A-B)(1-\alpha)\beta^n+(1-B)[\beta^m-\beta^n+(1+\beta)^m\lambda_2(\beta)-(1+\beta)^n\mu_2(\beta)]+(A-B)(1-\alpha)(1+\beta)^n\mu_2(\beta)\}}a_k\right\}_{k=1}^{\infty} \tag{2.4}$$

is a subordinating factor sequence. By virtue of Lemma 1.2, there is equivalent to the inequality

$$\operatorname{Re}\left\{1+2\sum_{k=1}^{\infty}\frac{(1-B)[\beta^m-\beta^n+(1+\beta)^m\lambda_2(\beta)-(1+\beta)^n\mu_2(\beta)]+(A-B)(1-\alpha)(1+\beta)^n\mu_2(\beta)}{2\{(A-B)(1-\alpha)\beta^n+(1-B)[\beta^m-\beta^n+(1+\beta)^m\lambda_2(\beta)-(1+\beta)^n\mu_2(\beta)]+(A-B)(1-\alpha)(1+\beta)^n\mu_2(\beta)\}}\times a_k z^k\right\}$$
$$>0 \qquad (z\in U). \tag{2.5}$$

Now, let us put

$$\Omega(m,n)=(1-B)[\beta^m-\beta^n+(1+\beta)^m\lambda_2(\beta)-(1+\beta)^n\mu_2(\beta)]+(A-B)(1-\alpha)(1+\beta)^n\mu_2(\beta)$$

and we write

$$\Omega(m,n)=(1+\beta)^n\left\{(1-B)\left[\frac{\beta^m-\beta^n}{(1+B)^n}+\frac{(1+\beta)^m}{(1+\beta)^n}\lambda_2(\beta)-\mu_2(\beta)\right]+(A-B)(1-\alpha)\mu_2(\beta)\right\}$$

It is observed that the sequence $\Omega(m,n)$ is a non-decreasing function of m, n under the condition (or constraints)

$$[\lambda_2\geq\mu_2>0;\ \ 0\leq\alpha<1;\ \ \beta,m\in\mathrm{N},\ \ n\in\mathrm{N}_0(m>n),\ \ -1\leq B<A\leq 1,-1\leq B<0].$$

In particular (under the same condition)

$$(1-B)[\beta^m-\beta^n+(1+\beta)^m\lambda_2(\beta)-(1+\beta)^n\mu_2(\beta)]+(A-B)(1-\alpha)(1+\beta)^n\mu_2(\beta)\leq$$
$$\{(1-B)[\beta^m-\beta^n+(1+\beta)^m\lambda_k(\beta)-(1+\beta)^n\mu_k(\beta)]+(A-B)(1-\alpha)(1+\beta)^n\mu_k(\beta)\}$$

Therefore, for |z| = r (r < 1), we obtain that

$$\mathrm{Re}\left\{1+\sum_{k=1}^{\infty}\frac{(1-B)[\beta^m-\beta^n+(1+\beta)^m\lambda_2(\beta)-(1+\beta)^n\mu_2(\beta)]+(A-B)(1-\alpha)(1+\beta)^n\mu_2(\beta)}{\{(A-B)(1-\alpha)\beta^n+(1-B)[\beta^m-\beta^n+(1+\beta)^m\lambda_2(\beta)-(1+\beta)^n\mu_2(\beta)]+(A-B)(1-\alpha)(1+\beta)^n\mu_2(\beta)\}}a_kz^k\right\}$$

$$=\mathrm{Re}\left\{\begin{array}{l}1+\dfrac{(1-B)[\beta^m-\beta^n+(1+\beta)^m\lambda_2(\beta)-(1+\beta)^n\mu_2(\beta)]+(A-B)(1-\alpha)(1+\beta)^n\mu_2(\beta)}{(A-B)(1-\alpha)\beta^n+(1-B)[\beta^m-\beta^n+(1+\beta)^m\lambda_2(\beta)-(1+\beta)^n\mu_2(\beta)]+(A-B)(1-\alpha)(1+\beta)^n\mu_2(\beta)}z+\\ \displaystyle\sum_{k=1}^{\infty}\frac{(1-B)[\beta^m-\beta^n+(1+\beta)^m\lambda_2(\beta)-(1+\beta)^n\mu_2(\beta)]+(A-B)(1-\alpha)(1+\beta)^n\mu_2(\beta)}{(A-B)(1-\alpha)\beta^n+(1-B)[\beta^m-\beta^n+(1+\beta)^m\lambda_2(\beta)-(1+\beta)^n\mu_2(\beta)]+(A-B)(1-\alpha)(1+\beta)^n\mu_2(\beta)}a_kz^k\end{array}\right\}$$

$$\geq 1-\frac{(1-B)[\beta^m-\beta^n+(1+\beta)^m\lambda_2(\beta)-(1+\beta)^n\mu_2(\beta)]+(A-B)(1-\alpha)(1+\beta)^n\mu_2(\beta)}{(A-B)(1-\alpha)\beta^n+(1-B)[\beta^m-\beta^n+(1+\beta)^m\lambda_2(\beta)-(1+\beta)^n\mu_2(\beta)]+(A-B)(1-\alpha)(1+\beta)^n\mu_2(\beta)}r$$

$$-\frac{1}{(A-B)(1-\alpha)\beta^n+(1-B)[\beta^m-\beta^n+(1+\beta)^m\lambda_2(\beta)-(1+\beta)^n\mu_2(\beta)]+(A-B)(1-\alpha)(1+\beta)^n\mu_2(\beta)}\times$$
$$\sum_{k=1}^{\infty}\{(1-B)[\beta^m-\beta^n+(k+\beta-1)^m\lambda_k(\beta)-(k+\beta-1)^n\mu_k(\beta)]+(A-B)(1-\alpha)(k+\beta-1)^n\mu_k(\beta)\}a_k(\beta)|r^k$$

$$>1-\frac{(1-B)[\beta^m-\beta^n+(1+\beta)^m\lambda_2(\beta)-(1+\beta)^n\mu_2(\beta)]+(A-B)(1-\alpha)(1+\beta)^n\mu_2(\beta)}{(A-B)(1-\alpha)\beta^n+(1-B)[\beta^m-\beta^n+(1+\beta)^m\lambda_2(\beta)-(1+\beta)^n\mu_2(\beta)]+(A-B)(1-\alpha)(1+\beta)^n\mu_2(\beta)}r$$
$$-\frac{(A-B)(1-\alpha)\beta^n}{(A-B)(1-\alpha)\beta^n+(1-B)[\beta^m-\beta^n+(1+\beta)^m\lambda_2(\beta)-(1+\beta)^n\mu_2(\beta)]+(A-B)(1-\alpha)(1+\beta)^n\mu_2(\beta)}r$$
$$=1-r>0;\ \ (|z|=r<1).$$

This evidently establishes the inequality (2.5), and consequently, the subordination relation (2.1) of our theorem 2.1, is proved.

The assertion (2.2) follows readily from (2.1) when the function g(z) is selected as

$$g(z)=\frac{z}{1-z}=z+\sum_{k=2}^{\infty}z^{k}\in K.$$

The sharpness of the multiplying factor

$$\frac{(1-B)[\beta^{m}-\beta^{n}+(1+\beta)^{m}\lambda_2(\beta)-(1+\beta)^{n}\mu_2(\beta)]+(A-B)(1-\alpha)(1+\beta)^{n}\mu_2(\beta)}{2\{(A-B)(1-\alpha)\beta^{n}+(1-B)[\beta^{m}-\beta^{n}+(1+\beta)^{m}\lambda_2(\beta)-(1+\beta)^{n}\mu_2(\beta)]+(A-B)(1-\alpha)(1+\beta)^{n}\mu_2(\beta)\}}$$

by considering a function $h(z)\in E^{*}_{\min}(\mu,\eta;A,B,\alpha,\beta)$ given by

$$h(z)=z-\frac{(A-B)(1-\alpha)\beta^{n}}{(1-B)[\beta^{m}-\beta^{n}+(1+\beta)^{m}\lambda_2(\beta)-(1+\beta)^{n}\mu_2(\beta)]+(A-B)(1-\alpha)(1+\beta)^{n}\mu_2(\beta)}z^{2} \qquad (2.6)$$

$$\left(-1\le B<A\le 1,-1\le B<0,0\le\alpha<1,\beta,m\in\mathbb{N},n\in\mathbb{N}_0(m>n)\,and\,\lambda_2(\beta)\ge\mu_2(\beta)>0\right)$$

Thus from 2.1, we have

$$\frac{(1-B)[\beta^{m}-\beta^{n}+(1+\beta)^{m}\lambda_2(\beta)-(1+\beta)^{n}\mu_2(\beta)]+(A-B)(1-\alpha)(1+\beta)^{n}\mu_2(\beta)}{2\{(A-B)(1-\alpha)\beta^{n}+(1-B)[\beta^{m}-\beta^{n}+(1+\beta)^{m}\lambda_2(\beta)-(1+\beta)^{n}\mu_2(\beta)]+(A-B)(1-\alpha)(1+\beta)^{n}\mu_2(\beta)\}}h(z)\prec\frac{z}{1-z}.$$

(2.7)

It can easily be verified that,

$$\underset{|z|\le 1}{Min}\left\{\mathrm{Re}\left(\frac{(1-B)[\beta^{m}-\beta^{n}+(1+\beta)^{m}\lambda_2(\beta)-(1+\beta)^{n}\mu_2(\beta)]+(A-B)(1-\alpha)(1+\beta)^{n}\mu_2(\beta)}{2\{(A-B)(1-\alpha)\beta^{n}+(1-B)[\beta^{m}-\beta^{n}+(1+\beta)^{m}\lambda_2(\beta)-(1+\beta)^{n}\mu_2(\beta)]+(A-B)(1-\alpha)(1+\beta)^{n}\mu_2(\beta)\}}h(z)\right)\right\}$$

$$=-\frac{1}{2},\qquad (z\in U) \qquad (2.8)$$

This shows that the constant

$$\frac{(1-B)[\beta^{m}-\beta^{n}+(1+\beta)^{m}\lambda_2(\beta)-(1+\beta)^{n}\mu_2(\beta)]+(A-B)(1-\alpha)(1+\beta)^{n}\mu_2(\beta)}{2\{(A-B)(1-\alpha)\beta^{n}+(1-B)[\beta^{m}-\beta^{n}+(1+\beta)^{m}\lambda_2(\beta)-(1+\beta)^{n}\mu_2(\beta)]+(A-B)(1-\alpha)(1+\beta)^{n}\mu_2(\beta)\}}$$

is best possible. Which complete the proof of theorem (2.1)

Corollary 2.2. Let the function f(z) defined by (1.1) be in the class $E^{*}_{\min}(\mu,\eta;1,-1,\alpha,\beta)$ and satisfy the condition

$$\sum_{k=2}^{\infty}\left\{\beta^{m}-\beta^{n}+(k+\beta-1)^{m}\lambda_k(\beta)-(k+\beta-1)^{n}\mu_k(\beta)]+(1-\alpha)(k+\beta-1)^{n}\mu_k(\beta)\right\}|a_k(\beta)| \le(1-\alpha)\beta^{n} \qquad (2.9)$$

then,

$$\frac{\beta^{m}-\beta^{n}+(1+\beta)^{m}\lambda_2(\beta)-\alpha(1+\beta)^{n}\mu_2(\beta)}{2\{(1-\alpha)\beta^{n}+\beta^{m}-\beta^{n}+(1+\beta)^{m}\lambda_2(\beta)-\alpha(1+\beta)^{n}\mu_2(\beta)\}}(f*g)(z)\prec g(z) \qquad (2.10)$$

$$\left(0\le\alpha<1,m,\beta\in\mathbb{N},n\in\mathbb{N}_0,\lambda_2(\beta)\ge\mu_2(\beta)>0;z\in U;\quad g\in K\right).$$

and

$$\mathrm{Re}(f(z)) > -\frac{(1-\alpha)\beta^n + \beta^m - \beta^n + (1+\beta)^m \lambda_2(\beta) - \alpha(1+\beta)^n \mu_2(\beta)}{\beta^m - \beta^n + (1+\beta)^m \lambda_2(\beta) - \alpha(1+\beta)^n \mu_2(\beta)}, \qquad (z \in U). \qquad (2.11)$$

The constant $\dfrac{\beta^m - \beta^n + (1+\beta)^m \lambda_2(\beta) - \alpha(1+\beta)^n \mu_2(\beta)}{2\{(1-\alpha)\beta^n + \beta^m - \beta^n + (1+\beta)^m \lambda_2(\beta) - \alpha(1+\beta)^n \mu_2(\beta)\}}$ is the best estimate

Putting $\lambda_2(\beta) = \mu_2(\beta) = 1$ in corollary 2.2, we obtain

Corollary 2.3. Let the function f(z) defined by (1.1) be in $E^*_{\min}(\mu,\eta;1,-1,\alpha,\beta)$ and satisfy the condition (2.9), then

$$\frac{2\beta-\alpha}{2\{2\beta-2\alpha+1\}}(f*g)(z) \prec g(z) \qquad (z \in U, g \in K, 0 \le \alpha < 1, \beta \in \mathrm{N}) \qquad (2.12)$$

and

$$\mathrm{Re}(f(z)) > -\left(\frac{2\beta-2\alpha+1}{2\beta-\alpha}\right) \qquad (z \in U). \qquad (2.13)$$

The constant factor $\dfrac{2\beta-\alpha}{2\{2\beta-2\alpha+1\}}$ is the best estimate.

Putting $\beta = 1$ in corollary 2.3, we obtain

Corollary 2.4. Let the function f(z) defined by 1.1 be in $E^*_{\min}(\mu,\eta;1,-1,\alpha,1)$ and satisfies the condition (2.9) when $\lambda_2(\beta) = \mu_2(\beta) = 1$, m=1 and n=0, then

$$\frac{2-\alpha}{2(3-2\alpha)}(f*g)(z) \prec g(z) \qquad (z \in U, g \in K, 0 \le \alpha < 1) \qquad (2.14)$$

and

$$\mathrm{Re}(f(z)) > -\left(\frac{3-2\alpha}{2-\alpha}\right) \qquad (z \in U). \qquad (2.15)$$

The constant factor $\dfrac{2-\alpha}{2(3-2\alpha)}$ is the best estimate.

Remark 1: The result in corollary 2.4 was obtained by Selvaraj and Karthikeyan [9], Rosihan et.al. [5] and Frasin [7]. Putting $\alpha = 0$ in corollary 2.4, we obtain corollary 2.5. Let the function f(z) defined by 1.1 be in

$E^*_{\min}(\mu,\eta;1,-1,0,1)$ and satisfies the condition (2.9) when $\lambda_2(\beta)=\mu_2(\beta)=1$, m=1 and n=0, then

$$\frac{1}{3}(f*g)(z)\prec g(z) \qquad (z\in U, g\in K,) \tag{2.16}$$

and

$$\mathrm{Re}(f(z))>-\frac{3}{2} \qquad (z\in U). \tag{2.17}$$

The constant $\frac{1}{3}$ is the best estimate.

Remark 2: The result in corollary 2.5 was obtained by Oyekan and Opoola [10], Sukhjit [11], Selvaraj and Karthikeyan [9] and Frasin [7].

Finally, a simple computation shows that when $\alpha=\frac{2\beta^2+\beta+1}{1+2\beta}$ in corollary 2.3, we obtain the following:

Corollary 2.6.

. Let f (z) defined in 1.1 be in $E^*_{\min}(\mu,\eta;1,-1,\alpha,\beta)$, then

$$\frac{1}{2}(1+\beta)(f*g)(z)\prec g(z) \tag{2.18}$$

and

$$\mathrm{Re}(f(z))>-\left(\frac{1}{1+\beta}\right) \tag{2.19}$$

The constant factor $\frac{1}{2}(1+\beta)$ is the best estimate.

Remark 3: The result in corollary 2.6 is due to Ghanim and Darus [12].

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