NUMERICAL SOLUTION OF NON-LINEAR COUPLED SYSTEM OF HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS BY θ- FINITE DIFFERENCE METHOD

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Abstract

We constructed a θ-finite difference method to get the numerical solution for nonlinear couple system of hyperbolic partial differential equations. Von-Neumann stability analysis was enforced to explain the stability of the present method. Toward the end, one illustrative example has been introduced to comparing the numerical and exact solutions to the problem. The results obtained indicate that the proposed method is very effective and highly accurate for such treatment problems.

Keywords: Numerical solution, Theta method, coupled system, Finite Difference, Hyperbolic equation, Nonlinear equation, initial condition, boundary condition, stability.

1. Introduction

The finite difference method (FDM) is a numerical method has widely used for solving differential equations. It has been used to solve many problems such as linear and non-linear partial differential equations. This method can be applied to problems with different boundary shapes, different types of boundary conditions, and for a region containing a number of different materials [2], [12], [8]. The hyperbolic partial differential equations are one of these problems that attract many scientists, especially mathematicians and physics scientists, where the hyperbolic partial differential equations occur in many applications. Many authors in servable fields such as biological, physics, fluid flows, electrical networks, viscoelasticity, try to modeling of these phenomena as a coupled system [1], [7], [9], [6]. θ-method is one of finite difference methods. θ-method was used to get numerical solution for many partial differential equations, such as wave equation, burger's equation. For more details see [13], [15], [11]. In this work, we develop a numerical method using θ-finite difference method for the solution of non-linear coupled system of hyperbolic partial differential equation.

$$\begin{aligned} &u_{tt} - u_{xx} - \frac{1}{x} u_x - v u_x = f(x,t), t \in [0,T], x \in [a,b], \\ &v_{tt} - v_{xx} - \frac{1}{x} v_x - u v_x = g(x,t), t \in [0,T], x \in [a,b], \end{aligned} \tag{1.1}$$

with initial and boundary conditions:

$$\begin{aligned} &u(x,0) = f_1(x) \quad v(x,0) = g_1(x), x \in [a,b], \\ &u_t(x,0) = f_2(x) \quad v_t(x,0) = g_2(x), x \in [a,b], \end{aligned} \tag{1.2}$$

$$\begin{aligned} &u(a,t) = f_3(t) \qquad u(b,t) = f_4(t), x \in [a,b], \\ &v(a,t) = g_3(t) \qquad v(b,t) = g_4(t), x \in [a,b], \end{aligned} \tag{1.3}$$

where $u(x,t), v(x,t)$ are unknown functions. Laplace decomposition method [5] was listed to solve the proposed problem.

2. The Method for Nonlinear Coupled Systems

In this section, we apply the θ-method to solve Eq. (1.1). Let's consider that the solution domain of our problem is $0 \le x \le 1, 0 \le t \le 1$ is divided into intervals having equal lengths h in the x direction and having equal time intervals k in time t such that $x_j = jh, t_n = nk,\ j = 0,1,\dots,M$ and u_j^n is given by $u(x_j, t_n)$ and the finite differences approximations for terms u_{tt}, u_t, u_{xx}, u_x are defined as follows:

$$u_{tt}(x_j,t_n)=\frac{u_j^{n-1}-2u_j^n+u_j^{n+1}}{\triangle t^2}, \qquad (2.1a)$$
$$v_{tt}(x_j,t_n)=\frac{v_j^{n-1}-2v_j^n+v_j^{n+1}}{\triangle t^2}, \qquad (2.1b)$$
$$u_{xx}(x_j,t_n)=\frac{u_{j-1}^n-2u_j^n+u_{j-1}^n}{\triangle x^2} \qquad (2.1c)$$
$$v_{xx}(x_j,t_n)=\frac{v_{j-1}^n-2v_j^n+v_{j-1}^n}{\triangle x^2}, \qquad (2.1d)$$
$$u_x(x_j,t_n)=\frac{u_{j+1}^n-u_{j-1}^n}{2\triangle x}, \qquad (2.1e)$$
$$v_x(x_j,t_n)=\frac{v_{j+1}^n-v_{j-1}^n}{2\triangle x}, \qquad (2.1f)$$
$$u_t(x_j,t_n)=\frac{u_j^{n+1}-u_j^n}{\triangle t}, \qquad (2.1g)$$
$$v_t(x_j,t_n)=\frac{v_j^{n+1}-v_j^n}{\triangle t}, \qquad (2.1h)$$

Firstly, to linearize the nonlinear term uu_x we use the following equation, for more details see [14]

$$(vu_x)_j^{n+1}=v_j^{n+1}(u_x)_j^n+v_j^n(u_x)_j^{n+1}-v_j^n(u_x)_j^n,$$
$$(uv_x)_j^{n+1}=u_j^n(v_x)_j^{n-1}+u_j^{n-1}(v_x)_j^n-u_j^{n-1}(v_x)_j^{n-1}. \qquad (2.2)$$

Now let us defined the well-known θ-method with second-order central differencing to E.q (1.1) such as:

$$u_{tt}(x_j,t_n)=\theta H_j^{n+1}+(1-\theta)H_j^n, \qquad (2.3a)$$
$$v_{tt}(x_j,t_n)=\theta I_j^{n+1}+(1-\theta)I_j^n, \qquad (2.3b)$$

Where

$$H_j^n(x_j,t_n)=u_{xx}(x_j,t_n)+\frac{1}{x}u_x(x_j,t_n)+v(x_j,t_n)u_x(x_j,t_n)-f(x_j,t_{n+1}),$$
$$I_j^n(x_j,t_n)=v_{xx}(x_j,t_n)+\frac{1}{x}v_x(x_j,t_n)+u(x_j,t_n)v_x(x_j,t_n)-g(x_j,t_{n+1}).$$

Then after substituting Eqs. (2.1a)-(2.1h) in Eq. 1.1 and using the formula of θ-method (2.3a) and (2.3b) respectively then we get:

$$u_j^{n+1}-2u_j^n+u_j^{n-1}-\theta \triangle t^2\left[\frac{1}{\triangle x^2}\left(u_{j-1}^{n+1}-2u_j^{n+1}+u_{j+1}^{n+1}\right)+\frac{1}{2x\triangle x}\left(u_{j+1}^{n+1}-u_{j-1}^{n+1}\right)+(vu_x)_j^{n+1}-f(x_j,t_{n+1})\right]$$
$$=(1-\theta)\triangle t^2\left[\frac{1}{\triangle \mathrm{x}^2}\left(u_{j-1}^n-2u_j^n+u_{j+1}^n\right)+\frac{1}{2\triangle x}\left(u_{j+1}^n-u_{j-1}^n\right)+(vu_x)_j^n-f(x_j,t_n)\right], \qquad (2.4)$$

and

$$v_j^{n+1}-2v_j^n+v_j^{n+1}-\theta \triangle t^2\left[\frac{1}{\triangle x^2}\left(v_{j-1}^{n+1}-2v_j^{n+1}+v_{j+1}^{n+1}\right)+\frac{1}{2\triangle x}\left(v_{j+1}^n-v_{j-1}^n\right)+(uv_x)_j^n-g(x_j,t_n)\right], \qquad (2.5)$$

by simplifying above equation we get:

$$\gamma_j^- u_{j-1}^{n+1}+\gamma_j^c u_j^{n+1}+\gamma_j^+ u_{j+1}^{n+1}=\beta_j^- u_{j-1}^n+\beta_j^c u_j^n+\beta_j^+ u_{j+1}^n-(1-\theta)\frac{\triangle t^2}{2\triangle x}u_j^{n-1}\left(u_{j+1}^{n-1}-u_{j-1}^{n-1}\right)+\theta\triangle t^2 f(x_j,t_{n+1})+(1-\theta)\triangle t^2 f(x_j,t_n)-u_j^{n-1}, \qquad (2.6)$$

and

$$\zeta_j^- v_{j-1}^{n+1} + \zeta c_j v_j^{n+1} + \zeta_j^+ v_{j+1}^{n+1} = \eta_j^- v_{j-1}^n + \eta_j^c v_j^n + \eta_j^+ v_{j+1}^n - (1-\theta)\frac{\triangle t^2}{2\triangle x} v_j^{n-1}\left(v_{j+1}^{n-1} - v_{j-1}^{n-1}\right) + \theta \triangle t^2 g\left(x_j, t_{n+1}\right) + (1-\theta)\triangle t^2 g\left(x_j, t_n\right) - v_j^{n-1}, \quad (2.7)$$

Where

$$\gamma_j^- = (-r_1 + r_2 + r_2 u_j^n)\theta \qquad \gamma_j^c = 1 + \left(2r_1 - r_2(u_{j+1}^n - u_{j-1}^n)\right)\theta,$$

$$\gamma_j^+ = (-r_1 - r_2 - r_2 u_j^n)\theta \quad \beta_j^- = \left(r_1 - r_2 - r_2\left(u_j^{n-1}\right)\right)(1-\theta) + r_2\left(u_j^n\right)\theta \quad (2.8)$$

$$\beta_j^c = 2 + \left(-2r_1 + r_2\left(u_{j+1}^{n-1} - u_{j-1}^{n-1}\right)\right)(1-\theta),$$

$$\beta_j^+ = \left(r_1 + r_2 + r_2\left(u_j^{n-1}\right)\right)(1-\theta) - r_2\left(u_j^n\right)\theta,$$

$$\zeta_j^- = (-r_1 + r_2 + r_2 u_j^n)\theta \quad \zeta_j^c = 1 + \left(2r_1 - r_2\left(u^{n(j+1)} - u_{j-1}^n\right)\right)\theta,$$

$$\zeta_j^+ = (-r_1 - r_2 - r_2 u_j^n)\theta \quad \eta_j^- = \left(r_1 - r_2 - r_2\left(u_j^{n-1}\right)\right)(1-\theta) + r_2\left(u_j^n\right)\theta,$$

$$\eta_j^c = 2 + \left(-2r_1 + r_2\left(u_{j+1}^{n-1} - u_{j-1}^{n-1}\right)\right)(1-\theta),$$

$$\eta_j^+ = \left(r_1 + r_2 + r_2\left(u_j^{n-1}\right)\right)(1-\theta) - r_2\left(u_j^n\right)\theta,$$

$$r_1 = \frac{\triangle t^2}{\triangle x^2} \quad r_2 = \frac{\triangle t^2}{2x \triangle x}$$

Now, we determine Eqs. (2.6) and (2.7) at $j = 1$ and $j = N - 1$ respectively

$$\gamma_1^c u_1^{n+1} + \gamma_1^+ u_2^{n+1} = \beta_1^c u_1^n + \beta_1^n + \beta_1^+ u_2 - \gamma_1^- g_1(t^n + 1) + \beta_1^- g_1(t^n) + \theta \triangle t^2 f(x_1, t_{n+1}) + (1-\theta) \triangle t^2 f\left(t^2 f(x_1, t_n) - u_1^{n-1} + (u_2^{n-1} - u_0^{n-1})\right)(1-\theta), \quad (2.9)$$

$$\zeta_1^c v_1^{n+1} + \zeta_1^+ v_2^{n+1} = \eta_1^c v_1^n + \eta_1^+ v_2^n - \zeta_1^- g_1(t^n + 1) + \eta_1^- g_1(t^n) + \theta \triangle t^2 g(x_1, t_{n+1}) + (1-\theta) \triangle t^2 g(x_1, t_n) - v_1^{n-1} + (v_2^{n-1} - v_0^{n-1})(1-\theta), \quad (2.10)$$

$$\gamma_{N-1}^- u_{N-2}^{n+1} + \gamma_{N-1}^c u_{N-1}^{n+1} = \beta_{N-1}^- u_{N-2}^n + \beta_{N-1}^c u_{N-1}^n - \gamma_{N-1}^+ g_2(t^n + 1) + \beta_{N-1}^+ g_2(t^n) + \theta \triangle t^2 f(x_{N-1}, t_{n+1}) + (1-\theta) \triangle t^2 f(x_{N-1}, t_n) - u_{N-1}^{n-1} + (u_N^{n-1} - u_{N-2}^{n-1})(1-\theta), \quad (2.11)$$

$$\gamma_{N-1}^- v_{N-2}^{n+1} + \zeta_{N-1}^c v_{N-1}^{n+1} = \eta_{N-1}^- v_{N-2}^n + \eta_{N-1}^c u_{N-1}^n - \zeta_{N-1}^+ g_2(t^n + 1) + \zeta_{N-1}^+ g_2(t^n) + \theta \triangle t^2 f(x_{N-1}, t_{n+1}) + (1-\theta) \triangle t^2 f(x_{N-1}, t_n) - v_{N-1}^{n-1} + (v_N^{n-1} - v_{N-2}^{n-1})(1-\theta). \quad (2.12)$$

To start the method, we needs to compute u_j^{n+1} and v_j^{n+1} at $j = 1, n = 0$, respectively and this leads to calculate terms u_j^{-1}, v_j^{-1}, to do this we can uses central difference to second initial conditions in (1.2) such as:

$$\frac{u_j^n - u_j^{n-1}}{\triangle x} = f_2\left(x_j, t_n\right), \quad (2.13a)$$

$$\frac{v^n - v_j^{n-1}}{\triangle x} = g_2\left(x_j, t_n\right), \quad (2.13b)$$

at $n = 0$

$$u_j^{-1} = u_j^1 - (\triangle t^2) f_2(x) \quad or, \quad u_j^1 = u_j^{-1} + (\triangle t^2) f_2(x), \quad (2.14a)$$

$$v_j^{-1} = v_j^1 - (\triangle t^2) g_2(x) \quad or, \quad v_j^1 = v_j^{-1} + (\triangle t^2) g_2(x) \quad (2.14b)$$

By combining Eqs. (2.6),(2.7) and (2.9),(2.12) also (2.14a),(2.14b) respectively we obtain the following system:

$$\begin{bmatrix} \zeta^c & \zeta^+ & 0 & 0 & 0 & 0 \\ \zeta^- & \zeta^c & \zeta^+ & 0 & 0 & 0 \\ 0 & \zeta^- & \zeta^c & \zeta^+ & 0 & 0 \\ 0 & 0 & \zeta^- & \zeta^c & \zeta^+ & 0 \\ 0 & 0 & 0 & \zeta^- & \zeta^c & \zeta^+ \\ 0 & 0 & 0 & 0 & \zeta^- & \zeta^c \end{bmatrix} \begin{bmatrix} v_1^{n+1} \\ v_2^{n+2} \\ 0 \\ 0 \\ v_{N-2}^{n+1} \\ v_{N-1}^{n+1} \end{bmatrix} = \begin{bmatrix} \eta^c & \eta^+ & 0 & 0 & 0 & 0 \\ \eta^- & \eta^c & \eta^+ & 0 & 0 & 0 \\ 0 & \eta^- & \eta^c & \eta^+ & 0 & 0 \\ 0 & 0 & \eta^- & \eta^c & \eta^+ & 0 \\ 0 & 0 & 0 & \eta^- & \eta^c & \eta^+ \\ 0 & 0 & 0 & 0 & \eta^- & \eta^c \end{bmatrix} \begin{bmatrix} v_1^n \\ v_2^n \\ 0 \\ 0 \\ v_{N-2}^n \\ v_{N-1}^n \end{bmatrix} + \begin{bmatrix} -\gamma_1^- f_3(t^n+1) + \beta_1^- f_3(t^n) + \theta \triangle t^2 f(x_1, t_{n+1}) \\ +(1-\theta) \triangle t^2 f(x_1, t_n) \\ 0 \\ 0 \\ -\gamma_{N-1}^+ f_4(t^n+1) + \beta_{N-1}^+ f_4(t^n) + \theta \triangle t^2 f(x_{N-1}, t_{n+1}) \\ +(1-\theta) \triangle t^2 f(x_{N-1}, t_n) \end{bmatrix} + \begin{bmatrix} v_1^{n-1} \\ v_2^{n-1} \\ 0 \\ 0 \\ v_{N-2}^{n-1} \\ v_{N-1}^{n-1} \end{bmatrix}$$

And

$$\begin{bmatrix} \zeta^c & \zeta^+ & 0 & 0 & 0 & 0 \\ \zeta^- & \zeta^c & \zeta^+ & 0 & 0 & 0 \\ 0 & \zeta^- & \zeta^c & \zeta^+ & 0 & 0 \\ 0 & 0 & \zeta^- & \zeta^c & \zeta^+ & 0 \\ 0 & 0 & 0 & \zeta^- & \zeta^c & \zeta^+ \\ 0 & 0 & 0 & 0 & \zeta^- & \zeta^c \end{bmatrix} \begin{bmatrix} v_1^{n+1} \\ v_2^{n+2} \\ 0 \\ 0 \\ v_{N-2}^{n+1} \\ v_{N-1}^{n+1} \end{bmatrix} = \begin{bmatrix} \eta^c & \eta^+ & 0 & 0 & 0 & 0 \\ \eta^- & \eta^c & \eta^+ & 0 & 0 & 0 \\ 0 & \eta^- & \eta^c & \eta^+ & 0 & 0 \\ 0 & 0 & \eta^- & \eta^c & \eta^+ & 0 \\ 0 & 0 & 0 & \eta^- & \eta^c & \eta^+ \\ 0 & 0 & 0 & 0 & \eta^- & \eta^c \end{bmatrix} \begin{bmatrix} v_1^n \\ v_2^n \\ 0 \\ 0 \\ v_{N-2}^n \\ v_{N-1}^n \end{bmatrix} + \begin{bmatrix} -\gamma_1^- g_3(t^n+1) + \beta_1^- g_3(t^n) + \theta \triangle t^2 g(x_1, t_{n+1}) \\ +(1-\theta) \triangle t^2 g(x_1, t_n) \\ 0 \\ 0 \\ -\gamma_{N-1}^+ g_4(t^n+1) + \beta_{N-1}^+ g_4(t^n) + \theta \triangle t^2 g(x_{N-1}, t_{n+1}) \\ +(1-\theta) \triangle t^2 g(x_{N-1}, t_n) \end{bmatrix} + \begin{bmatrix} v_1^{n-1} \\ v_2^{n-1} \\ 0 \\ 0 \\ v_{N-2}^{n-1} \\ v_{N-1}^{n-1} \end{bmatrix}$$

3. Stability Analysis

In numerical analysis, one of the most important tasks is to guarantee the convergence of the sequence of the numerical solutions. Generally, consistency in itself is not enough for convergence. To guarantee this property we introduce the notion of stability.

In this section, we prove the stability estimates of the approximation obtained by the present scheme, we will use the (Von-Neumann) stability. Which this method of stability has widely uses in numerical analysis for more details see [3], [4], [10]. In which the growth factor of a typical Fourier mode is defined as:

$$u_j^n = e^{an\triangle t+ik1mj\triangle x} \qquad (3.1)$$

$$v_j^n = e^{an\triangle t+ik2mj\in x} \qquad (3.2)$$

wherei $= \sqrt{-1}$ investigate the stability of the numerical scheme, the non- linear term uu_x and vv_x in the pde have been linearized by making the quantities $((uu_x)_j^{n+1} = u_j^{n+1}(u_x)_j^n + u_j^n(u_x)_j^{n+1} - u_j^n(u_x)_j^n$ and $(vv_x)_j^{n+1} = v_j^{n+1}(v_x)_j^n + v_j^n(v_x)_j^{n+1} - v_j^n(v_x)_j^n$ a local constants. Thus the nonlinear terms in the equations converts into $\hat{u}u_x$ and $\hat{v}v_x$ then Eq.(1.1) becomes:

$$u_j^{n+1} - 2u_j^n + u_j^{n-1} - \theta \triangle t^2\left[\frac{1}{\triangle x^2}\left(u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}\right) + \frac{1}{2x\triangle x}\left(u_{j+1}^{n+1} - u_{j-1}^{n+1}\right) + (\hat{u}u_x)_j^{n+1} - f(x_j, t_{n+1})\right] = (1-\theta) \triangle t^2\left[\frac{1}{\triangle x^2}\left(u_{j-1}^n - 2u_j^n + u_{j+1}^n\right) = \frac{1}{2\triangle x}\left(u_{j+1}^n - u_{j-1}^n\right) + (\hat{u}u_x)_j^n - f(x_j, t_n)\right], \quad (3.3)$$

$$v_j^{n+1} - 2v_j^n + v_j^{n-1} - \theta \triangle t^2\left[\frac{1}{\triangle x^2}\left(v_{j-1}^{n+1} - 2v_j^{n+1} + v_{j+1}^{n+1}\right) + \frac{1}{2x\triangle x}\left(v_{j+1}^{n+1} - v_{j-1}^{n+1}\right) + (\hat{v}v_x)_j^{n+1} - g(x_j, t_{n+1})\right] = (1-\theta) \triangle t^2\left[\frac{1}{\triangle x^2}\left(v_{j-1}^n 2v_j^n + v_{j+1}^n\right) = \frac{1}{2\triangle x}\left(v_{j+1}^n - v_{j-1}^n\right) + (\hat{v}v_x)_j^n - g(x_j, t_n)\right], \quad (3.4)$$

The generalized m^{th} row for Eqs. (3.3), (3.4) takes the forms:

$$\Theta1_j^- u_{j-1}^{n+1} + \Theta1_j^c u_j^{n+1} + \Theta1_j^+ u_{j+1}^{n+1} = \Theta2_j^- u_{j-1}^n + \Theta2_j^c u_j^n + \Theta2_j^+ u_{j+1}^n + \theta \triangle t^2 f(x_j, t_{n+1}) + (1-\theta) \triangle t^2 f(x_j, t_n) - u_j^{n-1}, \quad (3.5)$$

$$\Gamma1_j^- u_{j-1}^{n+1} + \Gamma1_j^c u_j^{n+1} + \Gamma1_j^+ u_{j+1}^{n+1} = \Gamma2_j^- u_{j-1}^n + \Gamma2_j^c u_j^n + \Gamma2_j^+ u_{j+1}^n + \theta \triangle t^2 g(x_j, t_{n+1}) + (1-\theta) \triangle t^2 g(x_j, t_n) - u_j^{n-1}, \quad (3.6)$$

Where

$$\Theta1_j^- = (-r_1 + r_2 + r3\,\hat{u})\theta \quad \Theta1_j^c = 1 + (2r_1)\theta \quad \Theta1_j^+ = (-r_1 - r_2 - r3\ \hat{u})\theta,$$

$$\Theta2_j^- = (r_1 - r_2 - r3\,\hat{u})(1-\theta) \quad \Theta2_j^c = 2 - (2r_1)(1-\theta),$$

$$\Theta2_j^+ = (r_1 + r_2 + r3\,\hat{u})(1-\theta) \quad r_1 = \frac{\triangle t^2}{\triangle x^2} \qquad r_2 = \frac{\triangle t^2}{2x\ \triangle x}.$$

And

$$\Gamma1_j^- = (-r_1 + r_2 + r3\,\hat{u})\theta \quad \Gamma1_j^c = 1 + (2r_1)\theta \quad \Gamma1_j^+ = (-r_1 - r_2 - r3\ \hat{u})\theta,$$

$$\Gamma2_j^- = (r_1 - r_2 - r3\,\hat{u})(1-\theta) \quad \Gamma2_j^c = 2 - (2r_1)(1-\theta),$$

$$\Gamma2_j^+ = (r_1 + r_2 + r3\,\hat{u})(1-\theta) \qquad (3.7)$$

Substituting Eqs. (3.1) and (3.2) into the last equations respectively then we get:

$$g_1 = \frac{\left(\left(\Theta2_j^- + \Theta2_j^+\right)\cos\phi_1 + \Theta2_j^c\right) + i\triangle t^2(1-\theta)\left(\Theta2_j^+ - \Theta2_j^-\right)\sin\phi_1}{\left(\left(\Theta1_j^- + \Theta_j^+\right)\cos\phi_1 + \Theta1_j^c\right) - i\triangle t^2\theta\left(\Theta1_j^+ - \Theta1_j^-\right)\sin\phi_1}$$

and

$$g_1 = \frac{\left((\Gamma 2_j^- + \Gamma 2_j^+)\cos\phi_1 + \Gamma 2_j^c\right) + i\triangle t^2(1-\theta)(\Gamma 2_j^+ - \Gamma 2_j^-)\sin\phi_1}{\left((\Gamma 1_j^- + \Gamma_j^+)\cos\phi_1 + \Gamma 1_j^c\right) - i\triangle t^2\theta(\Gamma 1_j^+ - \Gamma 1_j^-)\sin\phi_1}$$

The stability condition for the method is $|g_1| \le 1$ and $|g_2| \le 1$ so after take

$$y_1 = \left(\frac{1}{hx_1} + \frac{\hat{u}}{h}\right) \quad x_1 = (4\cos\phi_1^2 + 4\cos\phi_1^2 + 4)\ x_1^* = (4\cos\phi_1^2 - 8\cos\phi_1^2 + 16)$$

and

$$y_2 = \left(\frac{1}{hx_1} + \frac{\hat{v}}{h}\right) \quad x_2 = (4\cos\phi_2^2 + 4\cos\phi_2^2 + 4)\ x_2^* = (4\cos\phi_2^2 - 8\cos\phi_2^2 + 16)$$

And some arithmetic operations we find thet $|x-1| \le |x_1^*|$ and $|x_2| \le |x_2^*|$ so

$$|g_1| = \sqrt{\frac{\frac{\triangle t^4(1-\theta)^2}{\triangle x_1^4}(x_1 + y_1^2)}{\frac{\triangle t^4(\theta)^2}{\triangle x_1^4}(x_1^* + y_1^2)}} \le 1, \qquad |g_2| = \sqrt{\frac{\frac{\triangle t^4(1-\theta)^2}{\triangle x_1^4}(x_1 + y_2^2)}{\frac{\triangle t^4(\theta)^2}{\triangle x_2^4}(x_2^* + y_2^2)}} \le 1.$$

For $(0.5 \le \theta \le 1)$. There for the suggested method is unconditionally stable.

4. The Numerical Result

To illustrate the efficiency of the θ-finite difference method, we investigate the following example, consider the coupled system of hyperbolic partial differential equation.

$$u_{tt} - u_{xx} - \frac{1}{x}u_x - \mathrm{v}u_x = -x^2\sin(t) - 2x^3\sin(t)\cos(t) - 4\sin(t), \qquad t \in [0,1], \qquad x \in [0,1],$$

$$\mathrm{v_{tt}} - \mathrm{v_{xx}} - \frac{1}{\mathrm{x}}\mathrm{v_x} - \mathrm{uv_x} = -\mathrm{x}^2\cos(\mathrm{t}) - 2\mathrm{x}^3\sin(\mathrm{t})\cos(\mathrm{t}) - 4\cos(\mathrm{t}), \qquad \mathrm{t} \in [0,1], \qquad \mathrm{x} \in [0,1],$$

with initial and boundary conditions:

$$u(x,0) = 0 \quad v(x,0) = x^2, \quad x \in [0,1],$$
$$u_t(x,0) = x^2 \quad v_t(x,0) = 0, \quad x \in [0,1].$$
$$u(0,t) = 0 \quad u(1,t) = \sin(t), \quad x \in [0,1],$$
$$v(0,t) = 0 \quad v(1,t) = \cos(t), \quad x \in [0,1] \qquad (4.1)$$

and exact solution is $\mathrm{u(x,t) = x^2\sin(t)}$, $\mathrm{v(x,t) = x^2\cos(t)}$. We solve Eq. (4.1) by the θ-finite difference scheme, with step size $\mathrm{h} = 0.1, \mathrm{k} = 0.001$ and different values of Theta $(\theta = 0.5)$, $(\theta = 1)$ and $(\theta = 0.75)$. The results of exact and numerical solutions with approximations errors are shown in Table (1) and Figs. (1), (2).

$u(x,t)$ values for $h = 0.1, k = 0.001$

	Implicit method	Crack-Nicolson method	Theta=0.75 method	Exact solution
$\mathbf{x_4}$	$(\theta = 1)$	$(\theta = 0.5)$	$(\theta = 0.75)$	
0	0	0	0	0
0.1	0.0010	0.0010	0.0010	0.0010
0.2	0.0040	0.0040	0.0040	0.0040
0.3	0.0090	0.0090	0.0090	0.0090
0.4	0.0160	0.0160	0.0160	0.0160
0.5	0.0249	0.0250	0.0250	0.0250
0.6	0.0359	0.0359	0.0359	0.0359
0.7	0.0489	0.0489	0.0489	0.0489
0.8	0.0639	0.0639	0.0639	0.0639
0.9	0.0808	0.0809	0.0809	0.0809
1	0.0998	0.0998	0.0998	0.0998

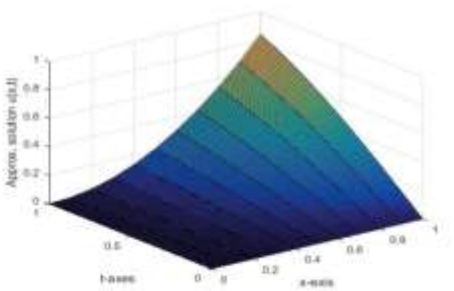
$v(x,t)$ values for $h = 0.1$, $k = 0.001$

	Implicit method	Crack-Nicolson method	Theta=0.75 method	Exact solution
x_4	$(\theta = 1)$	$(\theta = 0.5)$	$(\theta = 0.75)$	
0	0	0	0	0
0.1	0.0100	0.0100	0.0100	0.0100
0.2	0.0398	0.0336	0.0398	0.0398
0.3	0.0896	0.0896	0.0896	0.0896
0.4	0.1593	0.1345	0.1592	0.1592
0.5	0.2489	0.2489	0.2488	0.2488
0.6	0.3584	0.3584	0.3582	0.3582
0.7	0.4878	0.4878	0.4876	0.4876
0.8	0.6371	0.6371	0.6368	0.6368
0.9	0.8062	0.8062	0.8060	0.8060
1	0.9950	0.9950	0.9950	0.9950

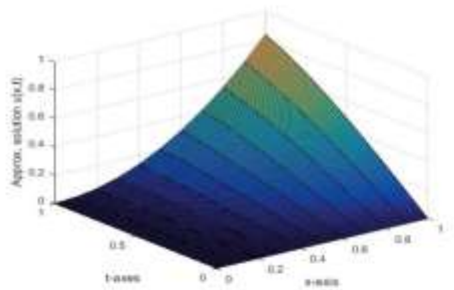
Table 1. Comparison between the exact solution and θ-method for some values of θ.

By using numerical results in Table (1), We have compared implicit, Crank-Nicholson finite difference methods by the θ-method for arbitrary value between $0.5 \leq \theta \leq 1$) such as ($\theta = 0.75$), Figs. 1, 2 shows a comparison between numerical and $u(x,t), v(x,t)$ at different values of θ with approximation errors. We conclude that the numerical solution using θ-finite difference is in a good agreement exact solution.

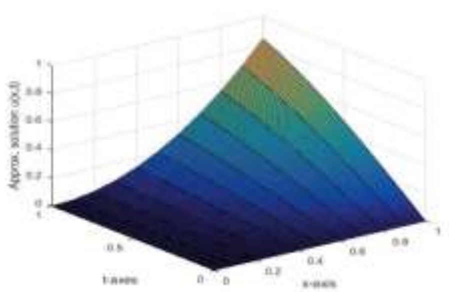
Numerical Solution By Theta-Method



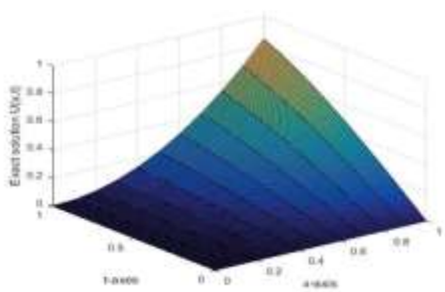
(a) numerical solution with $\theta = 1$.



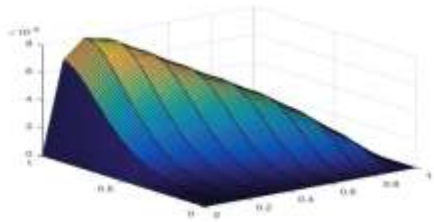
(B) numerical solution with $\theta = 0.5$.



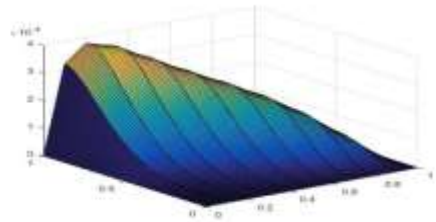
(C) numerical solution with $\theta = 0.75$.



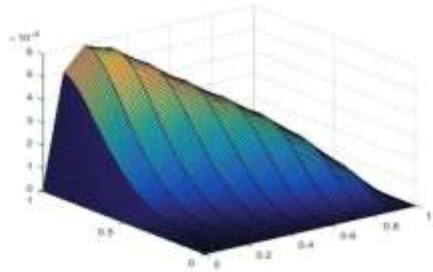
(D) Exact solution.



(E) Approximation error $\theta = 1$.

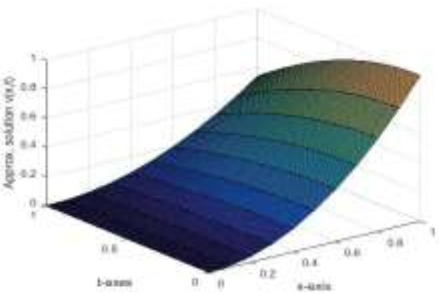


(f) Approximation error $\theta = 0.5$.

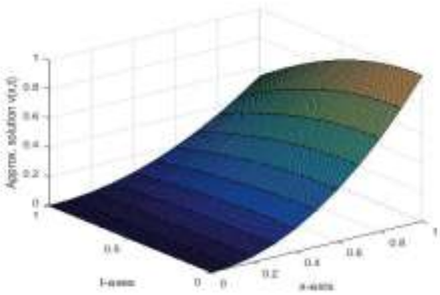


(g) Approximation error $\theta = 0.75$.

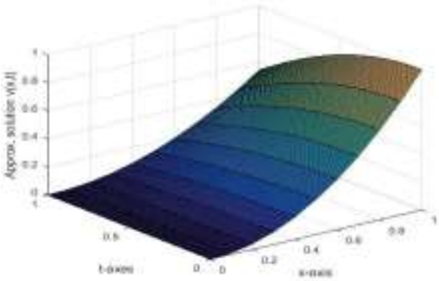
Figure1. Graphs of the example for $u(x, t)$



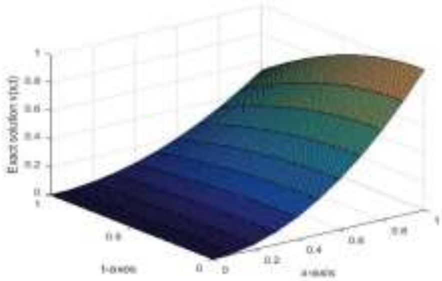
(a) numerical solution with $\theta = 1$.



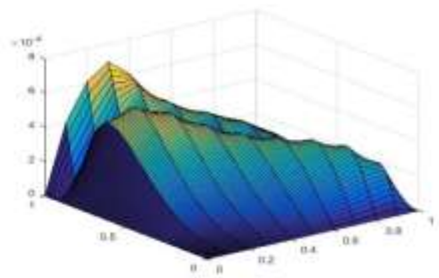
(B) numerical solution with $\theta = 0.5$.



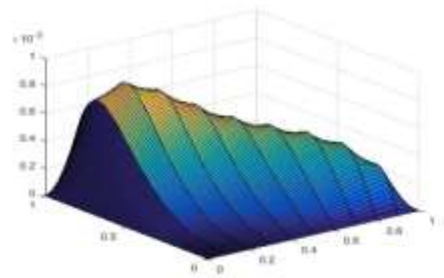
(c) numerical solution with $\theta = 0.75$.



(D) Exact solution.



(E) Approximation error θ = 1.



(f) Approximation error θ = 0.5.

(g) Approximation error θ = 0.75.

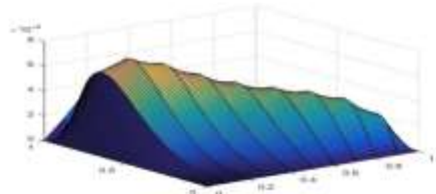


Figure 2. Graphs of example for $v(x, t)$

5. Conclusions

We proposed a θ-finite difference method for the solution of non-linear coupled system of hyperbolic partial differential equations. We applied the θ-method for different values of θ, ($\theta = 1, \theta = 0.5, \theta = 0.75$). The (Von-Neumann) stability showed that the θ- method is unconditionally stable. To verifying its validity, we considered a numerical example for different values of θ. Numerical results showed that the θ- method gives accurate results compared with exact solution of the proposed problem.

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