The Reduced Differential Transform Method for Initial Value Problem of One Dimensional Time Fractional Airy's and Airy's Type Partial Differential Equation

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Abstract
The Fractional Calculus is the theory of integrals and derivatives of arbitrary order which unifies and generalizes the concepts of integer-order differentiation and n-fold integration. Time fractional partial differential equation is one of the topics in the analysis of fractional calculus theory which can be obtained from the standard partial differential equations by replacing the integer order time derivative by a fractional derivative.

In this study a recent and reliable method, namely the reduced differential transform method which is introduced recently by Keskin and Oturanc (Keskin Y. and Oturan G. 2009, 2010)was applied to find analytical solutions of one dimensional time-fractional Airy's and Airy's type partial differential equations subjected to initial condition. The fractional derivative involved here is in the sense of Caputo definition, for its advantage that the initial conditions for fractional differential equations take the traditional form as for integer-order differential equations.

In order to show the reliability of the solutions examples are constructed and 3D figures for some of the solutions are also depicted.
Keywords: One dimensional; Time-fractional

1. INTRODUCTION

Fractional Calculus is the field of mathematical analysis which deals with the investigation and application of integrals and derivatives of arbitrary order. It is the theory of integrals and derivatives of arbitrary order which unifies and generalizes the concepts of integer-order differentiation and n-fold integration as in [29].Today the theory of fractional differential has gained much more attention as the fractional order system response ultimately converges to the integer order equations. Even though the beginning of the fractional calculus is considered to be the Leibniz's letter which raised a question: "Can the meaning of derivatives with integer order be generalized to derivatives with non-integer orders?" to L'Hopital in 1695,no analytical solution method was available for such type of equations before the nineteenth century as explained in[2].

In recent past, the glorious developments have been envisaged in the field of fractional calculus and fractional differential equations. Differential equations involving fractional order derivatives are used to model a variety of systems of real world physical problem, of which the important applications lie in field of viscoelasticity, electrode-electrolyte polarization, heat conduction, electromagnetic waves, diffusion equation and so on [5]. And also several real phenomena emerging in engineering and science fields can be demonstrated successfully by developing model using the fractional calculus theory.

Time fractional partial differential equation (TFPDE) is one of the topics in the analysis of fractional calculus theory. And they are differential equations which can be obtained from the standard partial differential equations by replacing the integer order time derivative by a fractional derivative [2]. Some of these are time fractional heat equations, time fractional heat-like equations, time fractional wave equations time fractional telegraphic equation and so on. Which are represented by linear and nonlinear PDEs and solving such fractional differential equations is very important. The Airy's partial differential equation is one of the linear partial differential equation used in many real world physical applications and, as in [25] Airy's equation is one the first model of water waves: a small wave exist-traveling "wave trains" in deep water. As in [17] in the early day of mathematical modeling of water waves, it was assumed that the wave height was small compared to the water depth which leads to linear dispersive equations a representative model of which is Airy's partial differential equation. Such equations are somewhat satisfying in this regard because they have solutions that resemble wave traveling along with constant speed and fixed profile along the water surface, just like one sees in nature [17, 25].

Fractional calculus involves different definitions of the fractional integral and derivatives such as the Riemann–Liouville fractional derivatives, Caputo fractional derivatives, Riesz fractional derivatives and Grunwald–Letnikov fractional derivative [22]. Among this the first to give definition is due to Riemann–Liouville. But, in this study we considered the Caputo's definition of fractional derivatives for its certain advantages when trying to model real world phenomena with traditional differential equations. That is, as in [5, 6] the alternative definition given by Caputo over the Riemann–Liouville for fractional derivatives thus incorporates the initial values of the functions and fractional derivatives for a constant is still zero.

In particular, $D_*^{\alpha} 1 \equiv 0, \alpha > 0$

A mathematical model is a simplified description of physical reality expressed in mathematical terms. Thus, the investigation of the exact or approximation solution helps us to understand the means of these mathematical models. Many authors applied numerical and analytic methods to solve linear and non-linear fractional differential equations. A few of these methods are the Differential Transform Method (DTM) [30], the A domian Decomposition Method (ADM) [28], the Variational Iteration Method (VIM) [18], and the Homotopy Perturbation Method (HPM) [18].Recently, Keskin Y. and Oturanc G. [14, 15, 16] developed the reduced differential transform method (RDTM) for the fractional differential equations and showed that RDTM is the easily useable semi analytical method and gives the exact solution for both the linear and nonlinear differential equations. Using Reduced Differential Transform Method (RDTM), it is possible to find exact solution or closed approximate solution of a differential equation, as in [26] .It is an iterative procedure for obtaining Taylor series solution of differential equations, as in [27].

In the last several years other authors [1, 7, 13, and 19]have discussed about the analysis of the solution of Airy's and Airy's type equationusing the reduced differential method (DTM), particle method, and variational iteration method (VIM).And Jonathan G. and Walter C. [13]

shown the existence, uniqueness and regularity result of the solution to the Airy's and Airy's type equation based on the energy estimates using weighted Sobolev norms.

The new Fractional Reduced Differential Transform method (FRDTM) introduced recently by, Keskin and Oturanc in [14, 15, 16] is used to solve fractional partial differential equations. RDTM successfully applied to solve time-fractional heat equations, time-fractional wave equation, time fractional telegraphic equations and so on. But, nothing was discussed about time fractional Airy's and Airy's type equations by applying the RDTM in the existing literature. However the Mathematical result of this study was in part applied the RDTM to find the analytical solution for the time fractional Airy's and Airy's type equations defined as:

1) $\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} = \beta \frac{\partial^{3}u(x,t)}{\partial x^{3}}, x\epsilon\mathbb{R}, t > 0, 0 < \alpha \leq 1$, where$\beta = \pm 1$
   subjected to initial condition: $u(x,t) = \phi(x)$.
2) $\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} = \frac{u(x,t)\partial^{3}u(x,t)}{\partial x^{3}}, x\epsilon\mathbb{R}, t > 0, 0 < \alpha \leq 1,$

   subjected to initial condition: $u(x,t) = \varphi(x,t)$.

2. LITERATURE REVIEW

Fractional Calculus is a tool of Mathematical Analysis applied to the study of integrals and derivatives of arbitrary order, not only fractional but also real. Commonly this fractional integrals and derivatives are not known for many scientists and up to recent years have been used only in a pure mathematical context. But during the last decade this integrals and derivatives have been applied in many contexts of sciences.

Before the nineteenth century, no analytical method was available for fractional order differential equations. In 1998 the first analytical method the variation iteration method (VIM), was proposed by Noorani M.S. et al [18] to solve fractional differential equations (seepage flow with fractional derivatives in porous media) and then after it also used to solve more complex fractional differential equations such as linear and nonlinear viscoelastic models with fractional derivatives, nonlinear differential equations of fractional order, linear fractional partial differential equations arising in fluid mechanics and the fractional heat and wave-like equations with variable coefficients.

The classical Taylor series method has been one of the earlier methods for solving the differential equations. With an advent of high-speed computers there has been an increasing trend towards exploring new ideas out of traditional techniques for the last couple of decades. In 1986 an updated version of Taylor series method, called the differential transform method (DTM) was introduced by Zhou Jk [30], and then applied DTM in order to solve electric circuit.

In 2009 another improved approach for solving initial-value problem for partial differential equation, known as the reduced differential transform (RDT) method, has recently been used by the Turkish mathematician Keskin Y. and Otura G. [14].And they developed the reduced differential transform method (RDTM) for the fractional differential equations and showed that RDTM is the easily useable semi analytical method and gives the exact solution for both the linear and nonlinear differential equations.

In 2007, the Homotopy Perturbation Method (HPM) was applied to both non-linear and linear fractional differential equations and it was showed that HPM is an alternative analytical

method for fractional differential equations. HPM also used to solve the fractional heat- and wave-like equations with variable coefficients, in Noorani M. S. M, et al [18].
To solve the third-order dispersive equations in 1991 Djidjeli k. and Twizell EH [9] develop a family of numerical method in a single space-variable with time-dependent boundary conditions.
In addition to the work of Djidjeli k. and Twizell EH [9] in 2003 Wazwaz [28] demonstrated how exact solutions to third-order dispersive partial differential equations are derived through the A domian decomposition method in an analytic study of the third-order dispersive partial differential equations. And also in 2009 Batiha B. [3] found an approximate solution of the dispersive equations by variational iteration method.
In the last several years authors have discussed about solution of Aiy's equation. For example in 2001 Alina C. and Doron L. [1] were discussed about Airy's equation in using Particle Methods for approximating solutions of linear and nonlinear dispersive equation. The Airy's is one of the linear partial differential equation used in many real world physical applications and, As Russel J. S [25, 17] in 1884 in his report on wave shown that Airy's equation is one the first model of water waves: a small waves in a deep water and wave-like solutions exist-traveling "wave trains"
In 2013 Naseem T. and Tahir M. [19] use RDT method for solving dispersive partial differential equations and applied RDTM on One-dimensional linear third-order dispersive partial differential equation and shown the reliability and efficiency of the methods.
The new Fractional Reduced Differential Transform method (FRDTM) introduced recently by, Keskin and Oturanc in [14, 15, 16] used to solve fractional partial differential equations. RDTM successfully applied to solve time-fractional heat equations, time-fractional wave equation and time fractional telegraphic equations and so on. But, nothing has been discussed about time fractional Airy's and Airy's type equations by applying RDTM in the existing literature. However the Mathematical result of this study was in part applied the RDTM to find analytical solution for the time fractional Airy's equations.

3. RELIMINARY RESULT AND DISCUSSION

3.1 The Gamma Function

Definition 3.1.1 $\Gamma(Z)$ represents the Gamma function which is an extension of the fractional function to complex and real number arguments as in Hanna Ray J. and Rowland John H., (1990) defined by:

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt, \mathcal{R}e(z) > 0 \qquad (1)$$

For all $z > 0$ with $\mathcal{R}e(z) > 0$ and$\forall n \epsilon \mathbb{N}$ then, the following holds:

i. $\Gamma(z+1)=z\Gamma(z)$
ii. $\Gamma(n) = (n-1)!$
iii. $\Gamma(1) = 1, n = 1$

3.2 Fractional Calculus Theorems

The Riemann-Liouville and fractional derivative, the Caputo derivative and the modification versions plays important roles in many areas of science, engineering, and mathematics. Some definitions of fractional derivatives and their properties are given as follows.
Definition 3.2.1 As in Diethelm. K and Luchko. Y, (2004), Podiubny I. (1999), Rida S. Z, (2010) a real function $f(x), x > o$ is said to be in the space $C_\mu, \mu \epsilon \mathbb{R}$ if there exists a real number $(p > \mu)$ such that $f(x) = x^p f_1(x)$ where $f_1(x) \epsilon [0, \infty)$ and it is said to be in the space C_μ^m if $f^m \epsilon C_\mu, m \epsilon \mathbb{N}$.

Definition 3.2.3 The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f \epsilon C_{\mu}, \mu \geq -1$ as in Boss B. and Millar KS. (1993) Podiubny I. (1999), Rida S. Z, (2010) is defined by:

$$\{ J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-a)^{\alpha-1} f(t)dt, \alpha > 0, J^0 f(x) = f(x) \} \qquad (2)$$

Properties of the operator J^{α} can be found in Diethelm. K and Luchko. Y, (2004), Podiubny I. (1999), Rida S. Z, (2010) are the following:

For $f \epsilon C_{\mu}, \mu \geq -1, \alpha, \beta \geq 0$ and $\alpha > 0$;

1. $J^{\alpha} J^{\beta} f(x) = J^{\alpha+\beta} f(x)$ (3)
2. $J^{\alpha} J^{\beta} f(x) = J^{\beta} J^{\alpha} f(x)$ (4)
3. $J^{\alpha} x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$ (5)

The Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D^{α} proposed by M.Caputo in his work of the theory of viscoelasticity which allows the utilization of initial and boundary conditions integer order derivatives, which have clear physical interpretations.

Definition 3.2.4 The fractional derivative of $f(x)$ in the Caputo sense as in(1967, 1871) and Podiubny I. (1999) is defined as:

$$D_*^{\alpha} f(x) = \begin{cases} J^{m-\alpha} D^m f(x) \\ \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f(t) dt for m-1 < \alpha \leq m, m \epsilon \mathbb{N}, x > 0, f \epsilon C_{-1}^m \end{cases} \qquad (6)$$

The unknown function $f = f(x,t)$ is assumed to be a casual function derivative (i. e vanishing for $\alpha < 0$) in Caputo sense as follows.

Definition 3.2.5 For m as the smallest integer that exceeds α the Caputo time fractional derivative operator of order $\alpha > 0$is defined as:

$$D^{\alpha} f(x,t) = \frac{\partial^{\alpha} f(x,t)}{\partial t^{\alpha}} = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{\partial^m f(x,\tau)}{\partial \tau^m} d\tau, m-1 < \alpha < m \\ \frac{\partial^m f(x,t)}{\partial t^m}, \alpha = m \end{cases} \qquad (7)$$

The fundamental basic properties of the Caputo fractional derivative as in Ishteva M, et al. (2003) and Podiubny I. (1999) are given as:

Lemma: If $m-1 < \alpha \leq m, m \epsilon \mathbb{N}$ and $f(x) \epsilon C_{\mu}^m, \mu \geq -1$. Then

1. $D^{\alpha} J^{\alpha} f(x) = f(x), x > 0.$ (8)
2. $D^{\alpha} J^{\alpha} f(x) = f(x) - \sum_{k=0}^{m} f^{(k)}(0^+) \frac{x^k}{k!}, x > 0$. (9)
3. $(J_a^{\alpha} D^{\alpha} f)(x) = (J_a^{\alpha} D_a^m f)(x) = f(x) - \sum_{k=0}^{m-1} f_{(a)}^{(k)} \frac{(x-a)^k}{k!}, x > a.$ (10)

3.3 Reduced Differential Transform Method (RDTM)

The reduced deferential transform method was first proposed by the Turkish mathematician Keskin and Oturance in 2009. It has received much attention since it has applied to solve a wide verity of problems by many authors.

In this section the basic definitions of the reduced deferential transform method (RDTM) and differential inverse transform in Keskin Y. and Oturan G. (2009, 2010) were discussed as follows:

Consider a function of two variables$u(x,t)$ and suppose that it can be represented as a product of two single-variable functions, i.e.$u(x,t) = f(x)g(t)$. Based on the properties of one-dimensional differential transform, the function$u(x,t)$can be represented as:

$$u(x,t) = \left(\sum_{i=0}^{\infty} F(i)\chi^i\right)\left(\sum_{j=0}^{\infty} G(j)t^j\right) = \sum_{k=0}^{\infty} U_k\,(x)t^k \qquad (11)$$

where $U_k(x)$ is called t-dimensional spectrum function of $u(x,t)$.

The basic definition of fractional RDTM as introduced by Batiha B. (2993) Srivasttava VK, et al(2014) and Sohail M. and Mohyud-Din S. T(2012) is given bellow:

Definition 3.3.1 If$u(x,t)$is analytic and continuously differentiable with respect to space variable$x$ and time variable$t$ in the domain of interest, then the t-dimensional spectrum function.

$$R_D[u(x,t)] = U_k(x) = \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}u(x,t)\right]_{t=t_0} \qquad (12)$$

is the reduced transformed function; where$\alpha$is a parameter which describes the order of time-fractional derivative in a Caputo sense and $U_k(x)$is the transformed function of $u(x,t)$.

Definition 3.3.2. the differential inverse transforms of $U_k(x)$is defined as:

$$R_{D^{-1}}[U_k(x)] = u(x,t) = \sum_{k=0}^{\infty} U_k\,(x)t^{k\alpha} \qquad (13)$$

Combining (12) and (13), we find that

$$u(x,t) = \sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}u(x,t)\right]_{t=t_0} t^{k\alpha} \qquad (14)$$

Notation: $\boldsymbol{R_D}$ Denoted the reduced differential transformed operator and $\boldsymbol{R_{D^{-1}}}$ denoted the inverse reduced differential transform operator.

Some basic theorems of the reduced differential transform method explained in Keskin Y. and Oturan G. (2009, 2010) were given bellow.

Theorem 3.3.1 If $w(x,t) = u(x,t)$ then $W_k(x) = U_k(x)$.

Theorem 3.3.2 If$(x,t) = u(x,t) \pm v(x,t)$ then $W_k(x) = U_k(x) \pm V_k(x)$.

Theorem 3.3.3 If $w(x,t) = \alpha u(x,t)$ then $W_k(x) = \alpha U_k(x)$.

Theorem 3.3.4 If $w(x,t) = u(x,t)v(x,t)$ then

$$W_k(x) = \sum_{n=0}^{k} U_n(x)\,V_{k-n}(x) = \sum_{n=0}^{k} V_n(x)U_{k-n}\,(x).$$

Theorem 3.3.5 If $w(x,t) = \frac{\partial^n}{\partial t^n}u(x,t)$ then

$$W_k(x) = (k+1)(k+2)\ldots\ldots..(k+n)U_{k+n}(x).$$

Theorem 3.3.6 If $w(x,t) = \frac{\partial}{\partial x}u(x,t)$ then $W_k(x) = \frac{\partial}{\partial x}U_k(x)$.

Theorem 3.3.7 If $w(x,t) = \frac{\partial^n}{\partial x^n}u(x,t)$ then $W_k(x) = \frac{\partial^n}{\partial x^n}U_k(x) k = 0,1,2,\ldots$

Theorem 3.3.8 If $w(x,t) = \frac{\partial^{N\alpha}}{\partial t^{N\alpha}}u(x,t)$ then $k = 0,1,2,\ldots$and $N\epsilon\mathbb{N}$.

3.4 Main Results

The new fractional Reduced Differential Transform method (FRDTM) introduced recently by Keskin and Oturanc (2009, 2010) is used to solve fractional partial differential equations. RDTM successfully applied to solve time-fractional heat and heat-like equations, time-fractional wave and wave-like equation and time fractional telegraphic equations and so on. But, nothing was discussed about time fractional Airy's and Airy's type equations by applying the RDTM in the existing literature. Therefore, this study presents the solution of time fractional Ariy's and Ariy's type equation by using RDTM.

Theorem 3.4.1 If $w(x,t) = v(x,t)\frac{\partial^n}{\partial x^n}u(x,t)$ then

$$\sum_{r=0}^{k} V_r(x)\frac{\partial^n}{\partial x^n}U_{k-r}(x) = \sum_{r=0}^{k} V_{k-r}(x)\frac{\partial^n}{\partial x^n}U_r(x)$$

Proof:

Let $w(x,t)$, $u(x,t)$ and $v(x,t)$ be analytic and continuously differentiable functions with respect to the variable x and time t in the domain of interest and $t > 0$ such that;

$w(x,t) = v(x,t)\frac{\partial^n}{\partial x^n}u(x,t)$, where $n = 0,1,2,\ldots$

and let $W_k(x), U_k(x)$ and V_k be t-dimensional spectrum function of $w(x,t)$, $u(x,t)$ and $v(x,t)$ respectively.

Appling definition 3.3.1

$$R_D[w(x,t)] = W_k(x) = \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}w(x,t)\right]_{t=t_0} = \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial^{k\alpha}}\left(v(x,t)\frac{\partial^n}{\partial x^n}u(x,t)\right)\right]_{t=t_0},$$

since $w(x,t) = v(x,t)\frac{\partial^n}{\partial x^n}u(x,t)$

Now let $r(x,t)$ be analytic and continuously differentiable functions with respect to variable x and time t in the domain of interest and assume that $r(x,t) = \frac{\partial^n u(x,t)}{\partial x^n}, n = 0,1,2,\ldots$

and let $R(x,t)$ be t-dimensional spectrum function of $r(x,t)$, then $R_k(x,t) = \frac{\partial^n U_k(x)}{\partial x^n}$ by theorem 3.3.7

Then $W_k(x) = \left(\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}(v(x,t)r(x,t))\right]_{t=t_0}\right)$, since $r(x,t) = \frac{\partial^n u(x,t)}{\partial x^n}$

But, from theorem 3.3.7 we can obtain that

$$W_k = \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}(v(x,t).r(x,t))\right]_{t=t_0} = \sum_{r=0}^{k} V_r(x)R_{k-r}(x) = \sum_{r=0}^{k} V_{k-r}(x)R_r(x)$$

$$W_k(x) = \sum_{r=0}^{k} V_r(x)\frac{\partial^n U_{k-r}(x)}{\partial x^n} = \sum_{r=0}^{k} V_{k-r}(x)\frac{\partial^n U_r(x)}{\partial x^n}$$

Therefore,

If $w(x,t) = v(x,t)\frac{\partial^n}{\partial x^n}u(x,t)$ then

$$\sum_{r=0}^{k} V_r(x)\frac{\partial^n}{\partial x^n}U_{k-r}(x) = \sum_{r=0}^{k} V_{k-r}(x)\frac{\partial^n}{\partial x^n}U_r(x)$$

Corollary 3.4.2 If $w(x,t) = u(x,t)\frac{\partial^n}{\partial x^n}u(x,t)$ then

$$W_k(x) = \sum_{r=0}^{k} U_r(x)\frac{\partial^n}{\partial x^n}U_{k-r}(x) = \sum_{r=0}^{k} U_{k-r}(x)\frac{\partial^n}{\partial x^n}U_r(x), \qquad n = 0,1,2\,...$$

And by using theorem 3.3.8,

$$W_k(x) = U_{k+N}(x) = \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+N\alpha+1)}\left[\sum_{r=0}^{k} U_r(x)\frac{\partial^n}{\partial x^n}U_{k-r}(x)\right]$$

Then, for N=1 and n=0, 1, 2…we get the following iterative relation

$$U_{k+1}(x) = \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+\alpha+1)}\left[\sum_{r=0}^{k} U_r(x)\frac{\partial^n}{\partial x^n}U_{k-r}(x)\right]$$

3.4.1 Reduced Differential Transform Method for Solving One Dimensional Time Fractional Airy's and Airy's Type Partial Differential Equations

I. TIME FRACTIONAL AIRY'S EQUATION:

Consider one-dimensional time-fractional Airy's equation Alina C. and Doron L. (2001) Chandradeepa D. and Dhaigude D.B. (2012) in Caputo sense

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t) = \beta\frac{\partial^3}{\partial x^3}u(x,t), x\epsilon\mathbb{R}, t > 0, 0 < \alpha \le 1 \qquad (15)$$

where $\beta = \pm 1$ and

Subjected to the initial condition

$$u(x,0) = \phi(x), x\epsilon\mathbb{R} \qquad (16)$$

Step-1) Applying RDTM to both side of equation (15) and (16)

i.e, $$R_D\left[\frac{\partial^\alpha}{\partial t^\alpha}u(x,t)\right] = R_D\left[\beta\frac{\partial^3}{\partial x^3}u(x,t)\right] \qquad (17)$$

and,

$$R_D[u(x,0)] = R_D[\phi(x)] \qquad (18)$$

We get respectively the following iterative relations,

$$U_{k+1}(x) = \beta\frac{\Gamma(k\alpha+1)}{\Gamma((k+1)\alpha+1)}\left[\frac{\partial^3}{\partial x^3}U_k(x)\right], x\epsilon\mathbb{R} \text{ and } k = 0,1,2,... \qquad (19)$$

where we have used theorem 3.3.8, on the left hand side of (17) for $N = 1$theorem 3.3.7on the right hand side of (17) for $n = 3$and from (18) we have

$$U_0(x) = \phi(x), x\epsilon\mathbb{R} \qquad (20)$$

Step-2) Substituting (20) in to (19), yields the following iterated values.

That is,

For k=0, $$U_1(x) = \frac{\beta}{\Gamma(\alpha+1)}\left[\frac{\partial^3}{\partial x^3}U_o(x)\right] = \frac{\beta}{\Gamma(\alpha+1)}\left[\frac{\partial^3}{\partial x^3}\phi(x)\right]$$

For k=1, $U_2(x) = \beta \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\left[\beta \frac{\partial^3}{\partial x^3} U_1(x)\right] = \frac{\beta^2}{\Gamma(2\alpha+1)}\left[\frac{\partial^6}{\partial x^6}\phi(x)\right]$

Fork=2, $U_3(x) = \beta \frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\left[\beta^2 \frac{\partial^3}{\partial x^3} U_2(x)\right] = \frac{\beta^3}{\Gamma(3\alpha+1)}\left[\frac{\partial^9}{\partial x^9}\phi(x)\right], \ldots$

Step-3) Using definition 3.3.2, the differential inverse transforms of $U_k(x)$ gives

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x) t^{k\alpha}, t > 0$$

II. TIME FRACTIONAL AIRY'S TYPE EQUATION:

Consider one-dimensional time-fractional Airy's type equation [13] described in Caputo sense.

$$\frac{\partial^\alpha}{\partial t^\alpha} u(x,t) = u(x,t)\frac{\partial^3}{\partial x^3} u(x,t), x \epsilon \mathbb{R}, t > 0, 0 < \alpha \leq 1 \quad (21)$$

Subjected to the initial condition

$$u(x,0) = \varphi(x), x\epsilon\mathbb{R}, \quad (22)$$

Step-1) Applying RDTM to both side of equation (21) and (22),
i.e,

$$R_D\left[\frac{\partial^\alpha}{\partial t^\alpha} u(x,t)\right] = R_D\left[u(x,t)\frac{\partial^3}{\partial x^3} u(x,t)\right] \quad (23)$$

and

$$R_D[u(x,0)] = [\varphi(x)], \quad (24)$$

we get respectively the following iterative relations

$$U_{k+1}(x) = \frac{\Gamma(k\alpha+1)}{\Gamma((k+1)\alpha+1)}\left[\sum_{r=0}^{k} U_r(x)\frac{\partial^3}{\partial x^3} U_{k-r}(x)\right], \quad (25)$$

and

$$U_0(x) = \varphi(x), x\epsilon\mathbb{R} \quad (26)$$

where, we have used theorem 3.3.8 on the left hand side of (23) and theorem 3.3.1, corollary 3.4.2 for $n = 3$ on the right hand side of (23).

Setp-2) Substituting (24) in to (23), we get the following iterative values

For k=0, $U_1(x) = \frac{1}{\Gamma(\alpha+1)}\left[U_0(x)\frac{\partial^3}{\partial x^3} U_0(x)\right]$,

For k=1, $U_2(x) = \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\left[U_0(x)\frac{\partial^3}{\partial x^3} U_1(x) + U_1(x)\frac{\partial^3}{\partial x^3} U_0(x)\right]$

For k=2, $U_3(x) = \frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\left[U_0(x)\frac{\partial^3}{\partial x^3} U_2(x) + U_1(x)\frac{\partial^3}{\partial x^3} U_1(x) + U_2(x)\frac{\partial^3}{\partial x^3} U_0(x)\right], \ldots$

Step-3) Using definition 4.1.3.2 the differential inverses transform of $U_k(x)$ gives us:

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x) t^{k\alpha}, t > 0$$

3.5 Application

In this section, we describe the application of the method explained in section 3.3 and section 3.4.1 by considering test examples of Airy's and Airy's type partial differential equation to show the efficiency and accuracy of the fractional reduced differential transform method.

Example 3.5.1 Consider one-dimensional time-fractional Airy's partial differential equation for

$$\beta = 1.$$

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^3 u}{\partial x^3}, x\epsilon\mathbb{R}, t > 0, 0 < \alpha \leq 1, \quad (27)$$

subjected to initial condition: $u(x,0) = \cos \pi x + e^{\pi x}$ (28)

Solution:

Appling the RDTM to both side of equation (27), we obtain the following iteration relation

$$U_{k+1}(x) = \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+\alpha+1)}\left[\left(\frac{\partial^3}{\partial x^3}U_k(x)\right)\right] \quad ,k = 0,1,2 \dots \qquad (29)$$

Using the RDTM to the initial conditions (28), we obtain

$$u(x,0) = U_0(x) = \cos\pi x + e^{\pi x} \qquad (30)$$

Using iteration equation (29) and (30), we obtain the following$U_k(x)$values successively.

$$U_1(x) = \frac{\pi^3(\sin\pi x + e^{\pi x})}{\Gamma(\alpha+1)}, U_2(x) = \frac{-\pi^6(\cos\pi x - e^{\pi x})}{\Gamma(2\alpha+1)}, U_3(x) = \frac{-\pi^9(\sin\pi x - e^{\pi x})}{\Gamma(3\alpha+1)}, \dots$$

Thus, the fractional differential inverse transform of $U_k(x)$ gives,

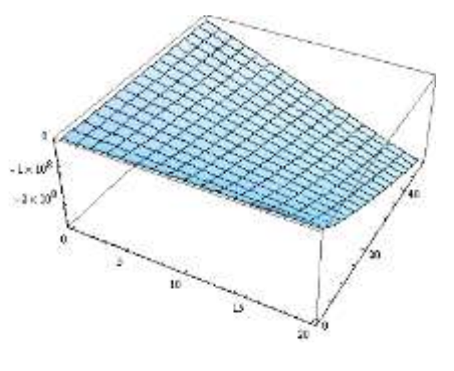
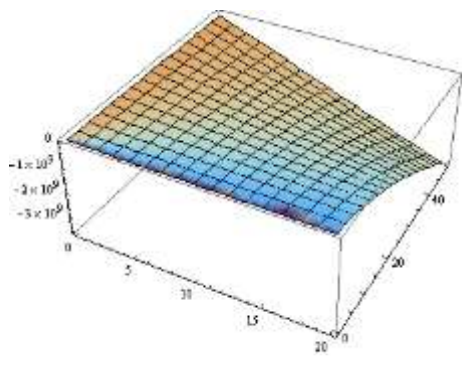
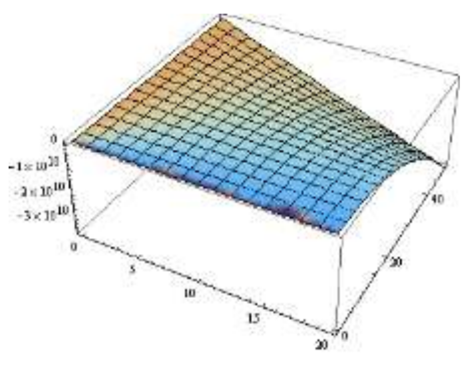
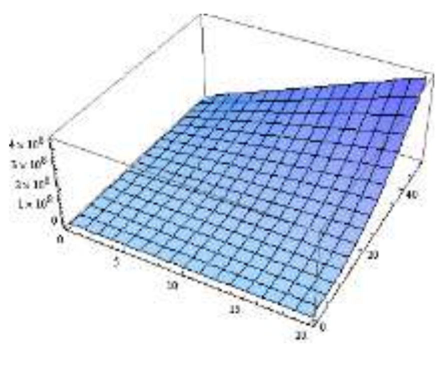
$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^{k\alpha} = U_0(x) + U_1(x)t^{\alpha} + U_2(x)t^{2\alpha} + U_3(x)t^{3\alpha} + \dots$$

$$u(x,t) = (\cos\pi x + e^{\pi x}) + \frac{\pi^3(\sin\pi x + e^{\pi x})t^{\alpha}}{\Gamma(\alpha+1)} - \frac{\pi^6(\cos\pi x - e^{\pi x})t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{\pi^9(\sin\pi x + e^{\pi x})t^{3\alpha}}{\Gamma(3\alpha+1)} + \dots$$

Specially, for $\alpha = 1, u(x,t)$ becomes

$$u(x,t) = (\cos\pi x + e^{\pi x}) + \frac{\pi^3(\sin\pi x + e^{\pi x})t}{1!} - \frac{\pi^6(\cos\pi x - e^{\pi x})t^2}{2!} - \frac{\pi^9(\sin\pi x + e^{\pi x})t^3}{3!} + \dots$$

The 3D plot of the solution of example 3.5.1in the domain $x\epsilon\mathbb{R}$ for $U_k, k = 0, 1, 2. 3$ $\alpha = 0.25, \alpha = 0.5, \alpha = 75$, and $\alpha = 1$are shown in fig.1.

	
For $\alpha = 0.25$	For $\alpha = 0.5$
	

For $\alpha = 0.75$	For $\alpha = 1$

Fig. 1: 3D plot of the solution of one dimensional time fractional Ariy's equation (example 3.5.1)

Example 3.5.2 Consider one-dimensional time-fractional Airy's type equation.

$$\frac{\partial^\alpha}{\partial t^\alpha} u(x,t) = u(x,t) \frac{\partial^3}{\partial x^3} u(x,t), x\epsilon\mathbb{R}, t > 0, 0 < \alpha \leq 1 \qquad (31)$$

subjected to the initial condition $u(x,0) = (\omega - 2\delta x)^{1/2}, x\epsilon\mathbb{R}$ (32)

whereω and δ are constants

Solution:

Applying (RDTM)to both side of equation (31), we get the iterative relation.

$$U_{k+1}(x) = \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+\alpha+1)} \left[\sum_{r=0}^{k} u_r(x) \frac{\partial^3}{\partial x^3} U_{k-r}(x)\right] \qquad (33)$$

and from the initial condition (32), we have

$$U_0(x) = u(x,0) = (\omega - 2\delta x)^{1/2}, x\epsilon\mathbb{R}, \qquad (34)$$

where, the t-dimensional spectrum function $U_k(x)$ is the transform function.

Using iteration equation (33) and (34), we obtain the following values of $U_k(x)$ successively.

$$U_1(x) = \frac{-(3\delta^3)}{\Gamma(\alpha+1)(\omega-2\delta x)^2},\ U_2(x) = \frac{-(3\delta^3)^2(63)}{\Gamma(2\alpha+1)(\omega-2\delta x)^{9/2}}, U_3(x) = \frac{-(3\delta^3)^3[26964\Gamma^2(\alpha+1) - 64\Gamma(2\alpha+1)]}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)(\omega-2\delta x)^7}, \ldots$$

Thus, the fractional differential inverse transform of $U_k(x)$ gives,

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x) t^{k\alpha} = U_0(x) + U_1(x)t^\alpha + U_2(x)t^{2\alpha} + U_3(x)t^{3\alpha} + \ \ldots$$

$$u(x,t) = (\omega - 2ax)^{1/2} + \frac{-(3\delta^3)t^\alpha}{\Gamma(\alpha+1)(\omega-2\delta x)^2} + \frac{-(3\delta^3)^2(63)t^{2\alpha}}{\Gamma(2\alpha+1)(\omega-2\delta x)^{9/2}} + \frac{-(3\delta^3)^3[26964\Gamma^2(\alpha+1) - 64\Gamma(2\alpha+1)]t^{3\alpha}}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)(\omega-2\delta x)^7} +, \ldots$$

Specially, for $\alpha = 1, u(x,t)$ becomes

$$u(x,t) = (\omega - 2\delta x)^{1/2} - \frac{3\delta^3 t}{1!}\left(\frac{1}{(\omega-2\delta x)^2}\right) - \frac{(3\delta^3)^2 t^2}{2!}\left(\frac{63}{(\omega-2\delta x)^{9/2}}\right) - \frac{(3\delta^3)^3 t^3}{3!}\left(\frac{26836}{(\omega-2\delta x)^7}\right) -, \ldots$$

Example 3.5.3 Consider example 4.3.2 above, if the constant$\omega = 1$and $\delta = 1/2$

$$\frac{\partial^\alpha u}{\partial t^\alpha} = u\frac{\partial^3 u}{\partial x^3}, x\epsilon\mathbb{R}, t > 0, 0 < \alpha \leq 1 \qquad (35)$$

then the initial condition: $u(x,0) = (1-x)^{1/2},$ (36)

Solution: We obtain the following values of $U_k(x)$ successively.

$$U_1(x) = \frac{-3/8}{\Gamma(\alpha+1)(1-x)^2},$$

$$U_2(x) = \frac{-\left(3/8\right)^2(63)}{\Gamma(2\alpha+1)(1-x)^{9/2}},\ U_3(x) = \frac{-\left(3/8\right)^3[26964\Gamma^2(\alpha+1) - 64\Gamma(2\alpha+1)]}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)(1-x)^7}, \ldots$$

Thus, the fractional differential inverse transform of $U_k(x)$ gives,

$$u(x,t) = (1-x)^{1/2} + \left(\frac{-3/8\, t^{\alpha}}{\Gamma(\alpha+1)}\right)\left(\frac{1}{(1-x)^2}\right) + \left(\frac{-\left(3/8\right)^2 t^{2\alpha}}{\Gamma(2\alpha+1)}\right)\left(\frac{63}{(1-x)^{9/2}}\right)$$

$$+\left(\frac{-\left(3/8\right)^3 t^{3\alpha}}{\Gamma(3\alpha+1)}\right)\left(\frac{26964\Gamma^2(\alpha+1) - 64\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)(1-x)^7}\right) + \cdots$$

Specially for $\alpha = 1, u(x,t)$ be comes;

$$u(x,t) = (1-x)^{1/2} + \left(\frac{-3/8}{1!}t\right)\left(\frac{1}{(1-x)^2}\right) + \left(\frac{-\left(3/8\right)^2 t^2}{2!}\right)\left(\frac{63}{(1-x)^{9/2}}\right)$$

$$+\left(\frac{-\left(3/8\right)^3 t^3}{3!}\right)\left(\frac{26836}{(1-x)^7}\right) + \ldots$$

Example 3.5.4 Consider the one dimensional time-fractional Airy's equation of $\beta = -1$.

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = -\frac{\partial^3 u}{\partial x^3}, x\epsilon\mathbb{R}, t > 0, 0 < \alpha \leq 1 \qquad (37)$$

subjected to initial condition: $u(x,0) = 4 - e^{-x}$ (38)

Solution:

Appling the RDTM to both side of equation (37), we obtain

$$U_{k+1}(x) = -\frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+\alpha+1)}\left[\left(\frac{\partial^{3(k+1)}}{\partial x^{3(k+1)}}U_k(x)\right)\right] \qquad (39)$$

and using the RDTM to the initial conditions (38), we obtain

$$u(x,0) = U_0(x) = 4 - e^{-x} \qquad (40)$$

Where, the t-dimensional spectrum function $U_k(x)$ is the transform function.
Using iteration equation (39) and (40), we obtain the following values of $U_k(x)$ successively.

$$U_1(x) = \frac{-e^{-x}}{\Gamma(\alpha+1)}, U_2(x) = \frac{-e^{-x}}{\Gamma(2\alpha+1)}, U_3(x) = \frac{-e^{-x}}{\Gamma(3\alpha+1)}, \ldots$$

Thus, the fractional differential inverse transform of $U_k(x)$ gives,

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^{k\alpha} = u_0(x) + u_1(x)t^{\alpha} + U_2(x)t^{2\alpha} + u_3(x)t^{3\alpha} + \ \ldots$$

$$u(x,t) = 4 - e^{-x} - \frac{e^{-x}}{\Gamma(\alpha+1)}t^{\alpha} - \frac{e^{-x}}{\Gamma(2\alpha+1)}t^{2\alpha} - \frac{e^{-x}}{\Gamma(3\alpha+1)}t^{3\alpha} - \ \ldots$$

Specially for$\alpha = 1, u(x,t)$ be comes

$$u(x,t) = 4 - e^{-x}\left[1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \ \dots\right]$$

Example 3.5.5 Consider time-fractional Airy's type equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = u(x,t)\frac{\partial^3 u}{\partial x^3}, x\epsilon\mathbb{R}, t > 0, 0 < \alpha \leq 1 \qquad (41)$$

Subjected to initial condition: $u(x,0) = e^{-x/3}$ (42)

Solution:

Appling RDTM on both side of equation (41), we obtain the iterative relation.

$$U_{k+1}(x) = \frac{\Gamma(\mathrm{k}\alpha+1)}{\Gamma(\mathrm{k}\alpha+\alpha+1)}\left[\sum_{r=0}^{k} U_r(x)\frac{\partial^3}{\partial x^3}U_{k-r}(x)\right] \qquad (43)$$

and using RDTM on initial condition (42), we obtain

$$u(x,t) = U_0(x) = e^{-x/3} \qquad (44)$$

where, the t-dimensional spectrum function $U_k(x)$ is the transform function.
Using iteration equation (43) and (44), we obtain the following values of $U_k(x)$ successively

$$U_1(x) = -\frac{e^{-2x/3}}{27\Gamma(\alpha+1)}, U_2(x) = \frac{e^{-x}}{81\Gamma(2\alpha+1)}, U_3(x) = \frac{-e^{-4x/3}}{19683}\frac{[252\Gamma^2(\alpha+1) + 8\Gamma(2\alpha+1)]}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}, \dots$$

Thus, the inverse transform of $U_k(x)$ gives

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^{k\alpha} = u_0(x) + u_1(x)t^\alpha + U_2(x)t^{2\alpha} + u_3(x)t^{3\alpha} + \ \dots$$

$$u(x,t) = u_0(x) + u_1(x)t^\alpha + U_2(x)t^{2\alpha} + u_3(x)t^{3\alpha} + \ \dots$$

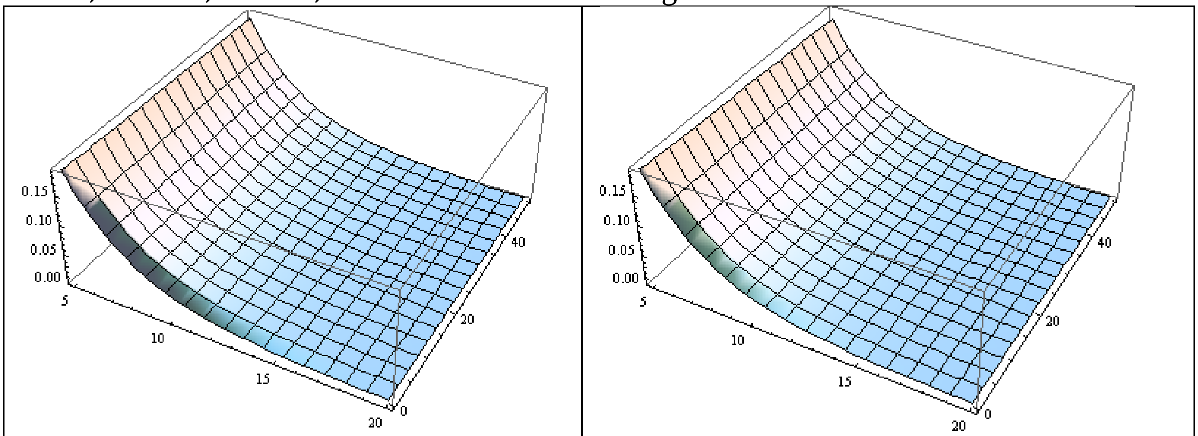
$$u(x,t) = \frac{1}{e^{x/3}} - \frac{{}^{1}/_{27}\, t^\alpha}{\Gamma(\alpha+1)e^{2x/3}} + \frac{{}^{1}/_{81}\, t^{2\alpha}}{\Gamma(2\alpha+1)e^{x}} - \frac{{}^{1}/_{19683}\, t^{3\alpha}[252\Gamma^2(\alpha+1) + 8\Gamma(2\alpha+1)]}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)e^{4x/3}} + \cdots$$

Specially, for $\alpha = 1$

$$u(x,t) = \frac{1}{e^{x/3}} - \frac{{}^{1}/_{27}\, t}{1!\, e^{2x/3}} + \frac{{}^{1}/_{81}\, t^2}{2!\, e^{x}} - \frac{{}^{1}/_{19683}\, t^3[268]}{3!\, e^{4x/3}} + \cdots$$

The 3D plot of solution of example 3.5.5 in the domain$x\epsilon\mathbb{R}$forU_k, $k = 0, 1, 2, 3$when

$\alpha = 0.25, \alpha = 0.5, \alpha = 75,$ and $\alpha = 1$are shown in fig.2.



For $\alpha = 0.25$	For $\alpha = 0.5$

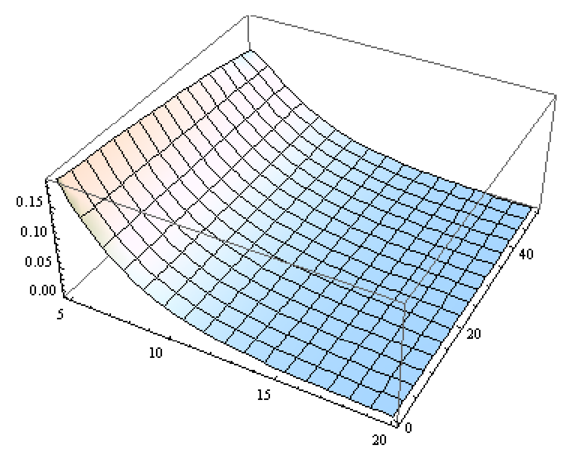
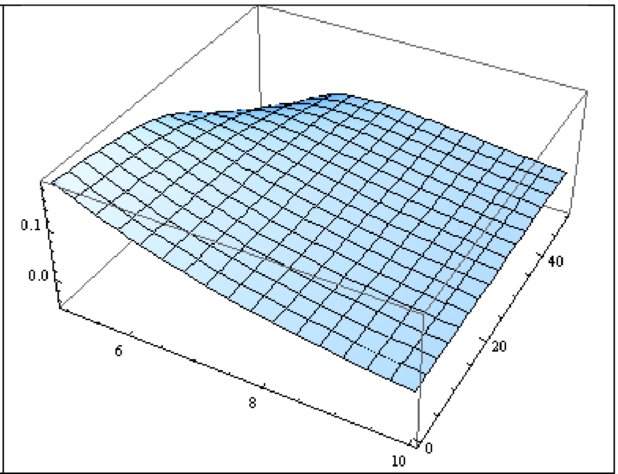
	
For $\alpha = 0.75$	For $\alpha = 1$

Fig. 2: 3D plot of the solution of one dimensional time fractional Ariy's type equation (Example 3.5.5)

4. CONCLUSION

In this study we have carried out the reduced differential transform method to find the solution of one dimensional time fractional Airy's and Airy's type partial differential equation based on the basic Caputo's definition of fractional derivatives. A Theorem is constructed, and its reliability is justified by constructing and presenting sufficient examples. The results show that the RDTM technique is highly accurate, elegant and easy to implement.

The techniques used in this work can also be applied to solve linear and non-linear time fractional partial differential equation and multi-dimensional physical problems emerging in various fields of engineering and applied sciences.

Appendix

In the Appendix we present the proof of generalized Taylor's formula that involves Caputo fractional derivatives. This generalization is presented in Odibat Z. M and Shawagteh N. T.(2007). We begin by introducing the generalized mean value theorem.

Theorem A.1 (Generalized Mean Value Theorem) Suppose that $f(x)\epsilon[a,b]$ and $D_a^\alpha f(x)\epsilon(a,b]$, for $0<\alpha\leq 1$, then we have in Caputo, M. (1967)

$$f(x)=f(a)+\frac{1}{\Gamma(\alpha)}(D_a^\alpha f)(\tau)(x-a)^\alpha \qquad (1)$$

With $a<\tau\leq x, \forall x\epsilon(a,b]$ and D^α is the Caputo fractional derivative of order $\alpha>0$.
In case of α=1, the generalized mean value theorem reduces to the classical mean value theorem. Before we present the generalized Taylor's formula in the Caputo sense, we need the following relation:

Theorem A.2 Suppose that $(D_a^\alpha)^n f(x).(D_a^\alpha)^{n+1}f(x)\epsilon C(a,b]\ for\ 0<\alpha\leq 1$ then as in Caputo, M. (1967)

we have.

$$(J_a^{n\alpha}(D_a^\alpha)^n f)(x)-(J_a^{(n+1)^\alpha}(D_a^\alpha)^{n+1}f)(x)=\frac{(x-a)^{n\alpha}}{\Gamma(n\alpha+1)}((D_a^\alpha)^n f)(a) \qquad (2)$$

where$(D_a^\alpha)^n=D_a^\alpha.D_a^\alpha\ldots D_a^\alpha$(n-times).

Proof: Using (8) of section 3.2 we have,

$$(J_a^{n\alpha}(D_a^\alpha)^n f)(x)-\left(J_a^{(n+1)\alpha}(D_a^\alpha)^{n+1}f\right)(x)=J_a^{n\alpha}\left(((D_a^\alpha)^n f)(x)-(J_a^\alpha(D_a^\alpha)^{n+1}f)(x)\right)$$

$$=J_a^{n\alpha}\left(((D_a^\alpha)^n f)(x)-(J_a^\alpha D_a^\alpha(D_a^\alpha)^n f)(x)\right)=J_a^{n\alpha}\left(((D_a^\alpha)^n f)(x)-(J_a^\alpha D_a^\alpha)((D_a^\alpha)^n f)(x)\right)$$

$$=J_a^{n\alpha}\left(((D_a^\alpha)^n f)(a)\right), using \qquad (10)$$

$$=\frac{(x-a)}{(n\alpha+1)}\left(((D_a^\alpha)^n f)(a)\right), using\ (9)$$

Hence,

$$(J_a^{n\alpha}(D_a^\alpha)^n f)(x)-\left(J_a^{(n+1)\alpha}(D_a^\alpha)^{n+1}f\right)(x)=\frac{(x-a)}{(n\alpha+1)}\left(((D_a^\alpha)^n f)(a)\right)$$

Theorem A.3 (Generalized Taylor Formula) Suppose that
$D^{k\alpha}f(x)\epsilon C(a,b]\ for\ k=0,1,2,\ldots\ldots\ldots\ldots\ldots..n+1\ where\ 0<\alpha\leq 1$. Then as in Odibat Z. M and Shawagteh N. T. (2007) we have;

$$f(x)=\sum_{i=0}^{n}\frac{(x-a)}{\Gamma(i\alpha+1)}^{i\alpha}[D_a^{k\alpha}f(a)]+\frac{D_a^{(n+1)\alpha}f(\tau)}{\Gamma((n+1)\alpha+1)}(x-a)^{(n+1)\alpha}$$

$a\leq\tau\leq x, \forall x\epsilon(a,b]\ where\ D^{k\alpha}=D_{a.}^\alpha\ D_a^\alpha\ldots D_a^\alpha(k-times)$ (3)

Proof: From (12), we have

$$\sum_{i=0}^{n}(J_a^{i\alpha}\left((D_a^\alpha)^i f\right)(x)-J_a^{(i+1)\alpha}\left((D_a^\alpha)^{i+1}f\right)(x)=\sum_{i=1}^{n}\frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)}\left((D_a^\alpha)^i f\right)(a) \qquad (4)$$

That is;

$$f(x)-\left(J_a^{(n+1)\alpha}((D_a^\alpha)^{n+1})f\right)(x)=\sum_{i=0}^{n}\frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)}\left((D_a^\alpha)^i f\right)(a) \qquad (5)$$

Applying integral mean value theorem yields

$$(J_a^{(n+1)\alpha}(D_a^{\alpha})^{n+1}f)(x) = \frac{1}{\Gamma\big((n+1)\alpha+1\big)}\int_a^x (x-t)^{(n+1)\alpha}\,((D_a^{\alpha})^{n+1}f)(t)dt$$
$$= \frac{((D_a^{\alpha})^{n+1}f)(\tau)}{((n+1)\alpha+1)}(x-a)^{(n+1)\alpha} \quad (6)$$

From (5) and (6), the generalized Taylor's formula is obtained.

That is,

$$f(x) = \left(J_a^{(n+1)\alpha}((D_a^{\alpha})^{n+1})f\right)(x) + \sum_{i=0}^{n}\frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)}\big((D_a^{\alpha})^i f\big)(a)$$

In case of $\alpha = 1$, the Caputo generalized Taylor's formula (3) reduces to the classical Taylor's formula.

$$f(x) = \sum_{i=i}^{n}\frac{(x-a)^i}{i!}f^{(i)}(a) + \frac{f^{(n+1)}(\tau)}{(n+1)!}(x-a)^{n+1}$$

The radius of convergence, R for the generalized Taylor series

$$\sum_{i=1}^{\infty}\frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)}\big((D_a^{\alpha})^i f\big)(a) \quad (7)$$

depends on $f(x)$ and a and is given by:

$$R = |x-a|^{\alpha}\lim_{n\to\infty}\left|\frac{\Gamma(n\alpha+1)}{\Gamma\big((n+1)\alpha+1\big)}\frac{((D_a^{\alpha})^{n+1}f)(a)}{((D_a^{\alpha})^n f)(a)}\right| \quad (8)$$

Theorem A.4 Odibat Z. M and Shawagteh N. T. (2007) Suppose $((D_a^{\alpha})^k f)(x)\epsilon C(a,b], for\ k = 0,1,2,3, \dots\dots, n+1\ where$ $0 < \alpha \le 1$ If $x\epsilon[a,b]$ then

$$f(x) \cong P_N^{\alpha}(x) = \sum_{i=1}^{N}\frac{((D_a^{\alpha})^{i\alpha}f)(a)}{\Gamma(i\alpha+1)}(x-a)^{i\alpha} \quad (9)$$

In addition, there is a value τ with $a \le \tau \le x$, So that the error term

$$R_N^{\alpha}(x) = \frac{((D_a^{\alpha})^{N+1}f)(\tau)}{\Gamma\big((N+1)\alpha+1\big)}(x-a)^{(N+1)\alpha} \quad (10)$$

The accuracy of $P_N^{\alpha}(x)$ increases when we choose large N and it decreases as the value of x moves away from a. Hence we must choose N large enough so that the error does not exceed a specified abound. In the following theorem, we find precise condition under which the exponents hold for arbitrary fractional operators. This result is very useful on our approach for solving differential equations of fractional order.

Theorem A.5 Suppose that $f(x) = x^{\lambda}g(x) where\ \lambda > -1\ and\ g(x)$ has the generalized Taylor's series $g(x) = \sum_{n=0}^{\infty} a_n\,(x-a)^{n\alpha}$ with radius of convergence, R>0, 0<α≤1. Then as in Odibat Z. M and Shawagteh N. T. (2007)

$$D_a^{\gamma}D_a^{\beta}f(x) = D_a^{\gamma+\beta}f(x)\ for\ x\epsilon(0,R) \quad (11)$$

if

i. $\beta < \lambda + 1\ and\ \alpha$ is arbitrary or
ii. $\beta > \lambda + 1\ and\ \gamma$, is arbitrary and a_k for k $= 0, 1, 2, 3 \dots\ m-1 < \beta \le m.$

Proof: In case for $\beta < \lambda + 1$, the definition of Caputo fractional differential operator (3) and (10) of section 4.1.2, we have

$$D_a^{\beta}f(x) = \sum_{n=0}^{\infty} a_n D_a^{\beta}(x-x_0)^{n\alpha+\lambda} = \sum_{n=0}^{\infty} a_n\frac{\Gamma(n\alpha+\lambda+1)}{\Gamma(n\alpha+\lambda-\beta+1)}(x-a)^{(n\alpha+\lambda-\beta)} \quad (12)$$

, since $\lambda - \beta > -1$ and

$$D_a^{\gamma} D_a^{\beta} f(x) = \sum_{n=0}^{\infty} a_n \frac{\Gamma(n\alpha+\lambda+1)}{\Gamma(n\alpha+\lambda-\beta+1)} D_a^{\gamma}(x-a)^{(n\alpha+\lambda-\beta)}$$
$$= \sum_{n=0}^{\infty} a_n \frac{\Gamma(n\alpha+\lambda+1)}{\Gamma(n\alpha+\lambda-\beta+1)} \frac{\Gamma(n\alpha+\lambda-\beta+1)}{\Gamma(n\alpha+\lambda-\beta-\gamma+1)} D_a^{\gamma}(x-a)^{(n\alpha+\lambda-\beta-\gamma)}$$
$$= \sum_{n=0}^{\infty} a_n \frac{\Gamma(n\alpha+\lambda+1)}{\Gamma(n\alpha+\lambda-\beta-\gamma+1)} D_a^{\gamma}(x-a)^{(n\alpha+\lambda-\beta-\gamma)}$$

which is precisely $D_a^{\beta+\gamma} f(x)$ for the anther case (ii) $\beta > \lambda + 1$, in a similar way we can prove.

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