Exponentiated Rama Distribution: Properties and Application

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ABSTRACT

In this study, a new distribution known as the Exponentiated Rama distribution has been proposed. The aim was to generalize the one parameter Rama distribution using the exponentiation technique. Some properties of proposed distribution are derived. The maximum likelihood method was used for the estimation of model parameters. The proposed distribution was subjected to real life application using a set of lifetime data and compared to Rama distribution, Exponentiated Akash distribution and Exponentiated Exponential distribution and it was found to provide the best fit than other competing distributions.

Keywords: Rama distribution, Exponentiated distributions, Order statistics, Moments

1. Introduction

There are many statistical distributions for modelling datasets occurring in applied sciences, finance, engineering and insurance, among others. A certain statistical distribution may be useful for a particular dataset but for a different dataset it may not be useful due to the features of the data being analyzed. In view of this challenge, researchers have, over the years, channelled efforts towards developing methods of generating new probability distributions to enhance its capability to fit datasets that have a high degree of skewness and kurtosis. Such extended distributions have been found to provide greater flexibility in modelling several kinds of datasets.

Recently, a new distribution called the Rama distribution was proposed and studied by Shanker (2017). Suppose X follows a Rama distribution with parameter θ. Then, the probability density function (PDF) and cumulative density function (CDF) of the Rama distribution are, respectively given by

$$f\left(x;\theta\right)=\frac{\theta^4}{\theta^3+6}\left(1+x^3\right)e^{-\theta x};\ \ x,\theta>0 \qquad (1)$$

and

$$F\left(x;\theta\right)=1-\left[1+\frac{\theta^3x^3+3\theta^2x^2+6\theta x}{\theta^3+6}\right]e^{-\theta x};x,\theta>0 \qquad (2)$$

However, the Rama distribution contains only a scale parameter θ and for this reason is not flexible for statistical modelling. In an attempt to improve the flexibility of the Rama distribution in modelling datasets with varieties of shapes, some extensions of the Rama distribution have been given. Notable among them are articles by Vijayakumar *et al*. (2020), Abebe *et al.* (2019), Umeh *et al*. (2019) and Chrisogonus *et al*. (2020).

A very useful method for generating new statistical distribution is the exponentiation method proposed by Mudholkar and Srivastava (1993). This method generalizes a baseline distribution by introducing a shape parameter, which makes the baseline distribution to provide better fit and greater flexibility. The work by Gupta and Kundu (1999) applied the exponentiation method to generate a new distribution called the generalized

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exponential distribution and the applications of the generalized exponential showed that it outperformed the Weibull and Gamma distributions. In the same vein, Pat *et al*. (2006) adopted the exponentiation method to develop a new distribution called exponentiated Weibull distribution, which also performed better than two-parameter Weibull and Gamma distributions in modelling real life data. Other distributions generated using the exponentiated method include Exponentiated Frechet distribution by Nadarajah and Kotz (2006), proposed exponentiated exponential distribution by Zakaria and Jammal (2008), exponentiated Pareto distribution by Shawky and Abu-Zinadah (2009), generalized Lindley distribution due to Nadarajah *et al*. (2011), exponentiated power Lindley distribution by Ashour and Eltehiwy (2014), exponentiated quasi Lindley distribution by Elbatal *et al.* (2016), Exponentiated Sushila distribution by Elgarhy and Shawki (2017), exponentiated Akash distribution by Okereke and Uwaeme (2018) and so on.

The aim of this paper is to introduce an exponentiated Rama distribution, which generalizes the Rama distribution. The content of this article is organized as follows. Section 2 contains the proposed distribution. In Section 3, we provide the properties of the proposed distribution. Section 4 deals with the maximum likelihood estimation of parameters of the proposed distribution. Section 5 illustrates an application by using real life data. Finally, in Section 6, the conclusion of the article is given.

2. Proposed distribution

A random variable X is said to follow an Exponentiated distribution if the cumulative density function CDF and probability density function PDF are respectively given by

$$G(x,\alpha)=\left[F(x,\alpha)\right]^{\alpha};\square \in \Re^{|},\alpha\geq 0 \qquad (3)$$

and

$$g(x,\alpha)=\alpha\left[F(x,\alpha)\right]^{\alpha-1} f(x,\alpha),\square \in \Re^{|},\alpha\geq 0 \qquad (4)$$

Substituting (1.2) into (1.3), we obtain CDF of the Exponentiated Rama distribution as

$$G_{Ra}(x,\alpha,\theta)=\left\{1-\left[1+\frac{\theta^3x^3+3\theta^2x^2+6\theta x}{\theta^3+6}\right]e^{-\theta x}\right\}^{\alpha};x,\alpha,\theta>0 \qquad (5)$$

Substituting (1) and (2) into (4), we obtain PDF and cdf of the Exponentiated Rama distribution as

$$g_{Ra}(x,\alpha,\theta)=\frac{\alpha\theta^4}{\theta^3+6}\left(1+x^3\right)\left\{1-\left[1+\frac{\theta^3x^3+3\theta^2x^2+6\theta x}{\theta^3+6}\right]e^{-\theta x}\right\}^{\alpha-1}e^{-\theta x};x,\alpha,\theta>0 \qquad (6)$$

For $\alpha=1$, (5) and (6) reduces to the Rama distribution. Another motivation for the new distribution in (5) and (6) can be described as follows. Suppose $X_1,X_2,...,X_\alpha$ are independent random variables distributed according to (2) and represent the failure time of components of a series system, assumed to be independent. Then the probability that the system will fail before x is given by

$$\Pr(\max(X_1,X_2,...,X_\alpha)\leq x)=\Pr(X_1\leq x)\Pr(X_2\leq x)..\Pr(X_\alpha\leq x)$$

$$=\left[1-\left(1+\frac{\theta^3x^3+3\theta^2x^2+6\theta x}{\theta^3+6}\right)e^{-\theta x}\right]^{\alpha}$$

So, (1.2) gives the distribution of a series system with independent components.

Figures 1a, 1b, 1c and 1d show the PDF and CDF plots of the Exponentiated Rama distribution

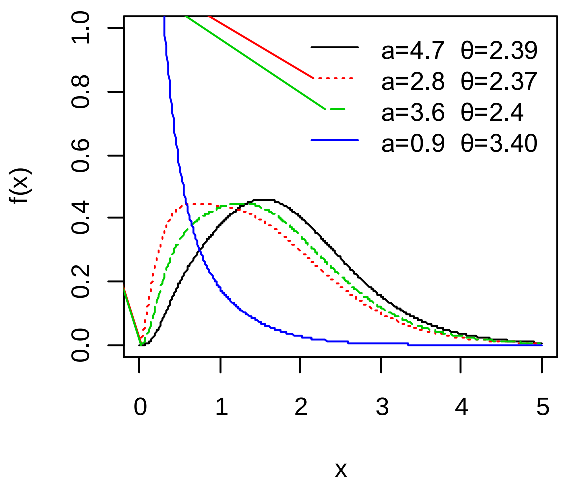

Fig 1a:pdf plot of ERD

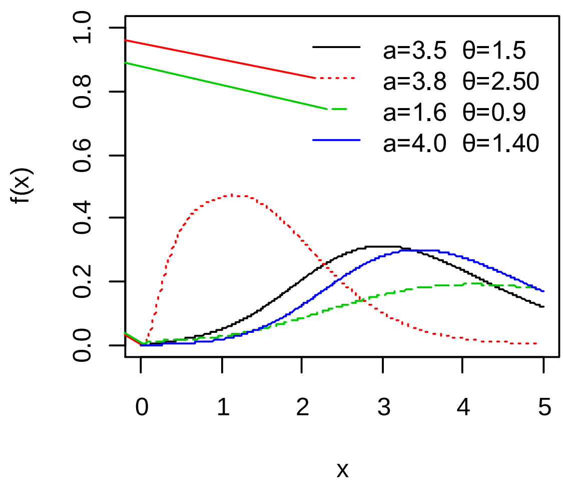

Fig 1b:pdf plot of ERD

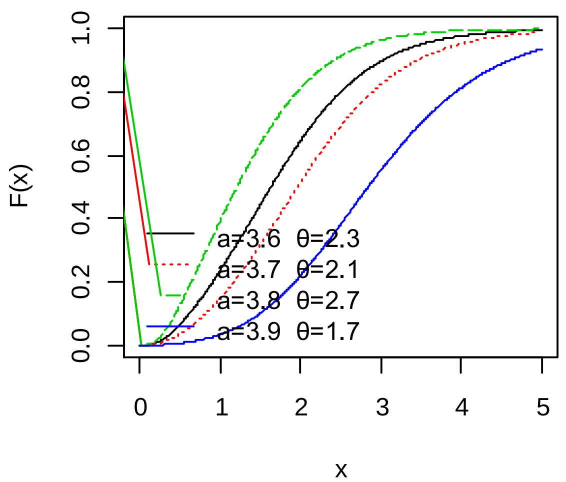

Fig 1c:cdf plot of ERD

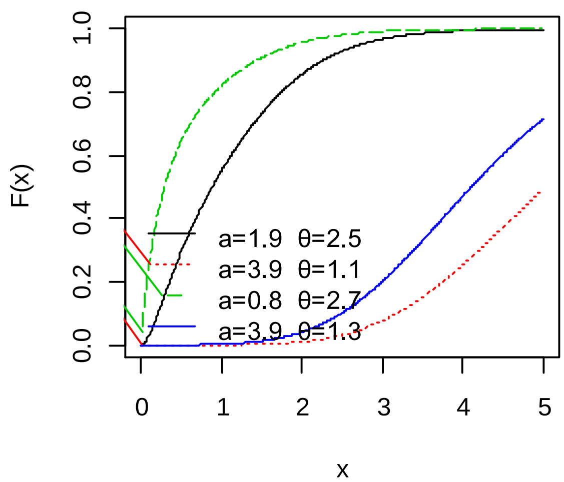

Fig 1d:cdf plot of ERD

Corollary 1: The Exponentiated Rama distribution defined by (1.6) is a valid density function.

Proof: We wish to show that

$$\int_0^\infty g(x,\alpha,\theta)\,dx = 1 \text{ or } \lim_{x\to\infty} G(x,\alpha,\theta) = 1$$

For convenience, we shall employ $\lim_{x\to\infty} G(x,\alpha,\theta) = 1$

$$\lim_{x\to\infty} G(x,\alpha,\theta) = \lim_{x\to\infty}\left\{1-\left[1+\frac{\theta^3x^3+3\theta^2x^2+6\theta x}{\theta^3+6}\right]e^{-\theta x}\right\}^{\alpha} = 1 \qquad (7)$$

Since $\lim_{x\to\infty} G(x,\alpha,\theta)=1$, the function $g(x;\alpha\theta)$ is a valid density function.

3. Properties of the Exponentiated Rama distribution

3.1 Moments

Theorem 1: Let X be a random variable that have an exponentiated Rama distribution, then the rth moment about the origin, $E(X^r)$ is given by

$$E(X^r)=\varphi_{i,j,k,l}\frac{(r+3j-k-l)!}{(i+1)^{(r+3j-k-l+1)}}+\eta_{i,j,k,l}\frac{(r+3j-k-l+3)!}{(i+1)^{(r+3j-k-l+4)}}$$

Proof: The rth moment of a random variable X is given by

$$E(X^r)=\int_0^\infty x^r g(x,\alpha,\theta)dx \tag{8}$$

$$=\int_0^\infty x^r\frac{\alpha\theta^4}{\theta^3+6}(1+x^3)\left\{1-\left[1+\frac{\theta x(\theta^2x^2+3\theta x+6)}{\theta^3+6}\right]e^{-\theta x}\right\}^{\alpha-1}e^{-\theta x}dx$$

$$=\left\{\begin{array}{l}\int_0^\infty \frac{\alpha\theta^4x^r}{(\theta^3+6)}e^{-\theta x}\left\{1-\left[1+\frac{\theta x(\theta^2x^2+3\theta x+6)}{\theta^3+6}\right]e^{-\theta x}\right\}^{\alpha-1}dx\\ +\int_0^\infty \frac{\alpha\theta^4x^{r+3}}{(\theta^3+6)}e^{-\theta x}\left\{1-\left[1+\frac{\theta x(\theta^2x^2+3\theta x+6)}{\theta^3+6}\right]e^{-\theta x}\right\}^{\alpha-1}dx\end{array}\right\} \tag{9}$$

Using the series expansions $(1-x)^a=\sum_{i=0}^{\infty}\binom{a}{i}(-1)^i x^i$ and $(1+x)^a=\sum_{i=0}^{\infty}\binom{a}{i}x^i$, (9) can be written as

$$=\sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i\sum_{j=0}^{\infty}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\frac{\alpha 3^k 2^l\theta^{3j-k-l+4}}{(\theta^3+6)^{j+1}}\int_0^\infty x^{r+3j-k-l}e^{-\theta x(i+1)}dx$$
$$+\sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i\sum_{j=0}^{\infty}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\frac{\alpha 3^k 2^l\theta^{3j-k-l+4}}{(\theta^3+6)^{j+1}}\int_0^\infty x^{r+3j-k-l+3}e^{-\theta x(i+1)}dx$$

Recall that $\int_0^\infty x^k e^{-\alpha x}dx=\frac{\Gamma(k+1)}{\alpha^{k+1}}$ and $\Gamma\alpha=(\alpha-1)!$, consequently, we have

$$E(X^r)=\varphi_{i,j,k,l}\frac{(r+3j-k-l)!}{(i+1)^{(r+3j-k-l+1)}}+\eta_{i,j,k,l}\frac{(r+3j-k-l+3)!}{(i+1)^{(r+3j-k-l+4)}} \tag{10}$$

where

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$$\varphi_{i,j,k,l} = \sum_{i=0}^{\infty} \binom{\alpha-1}{i} (-1)^i \sum_{j=0}^{\infty} \binom{i}{j} \sum_{k=0}^{j} \binom{j}{k} \sum_{l=0}^{k} \binom{k}{l} \frac{\alpha 3^k 2^l \theta^{-r+3}}{\left(\theta^3+6\right)^{j+1}}$$

and

$$\eta_{i,j,k,l} = \sum_{i=0}^{\infty} \binom{\alpha-1}{i} (-1)^i \sum_{j=0}^{\infty} \binom{i}{j} \sum_{k=0}^{j} \binom{j}{k} \sum_{l=0}^{k} \binom{k}{l} \frac{\alpha 3^k 2^l \theta^{-r}}{\left(\theta^3+6\right)^{j+1}}$$

3.2 Moment Generating Function

Theorem 2: Given a random variable $X \sim ERD(\alpha,\theta)$, the moment generating function is given by

$$M_x(t) = \sum_{n=0}^{\infty} \left(\frac{t}{\theta}\right)^n \left[\tau_{i,j,k,l} \frac{(n+3j-k-l)!}{n!(i+1)^{(n+3j-k-l+1)}} + \phi_{i,j,k,l} \frac{(n+3j-k-l+3)!}{n!(i+1)^{(n+3j-k-l+4)}} \right] \tag{11}$$

Proof: The moment generating function of a continuous random variable X, is given by

$$M_x(t) = E\left(e^{tX}\right) = \int_0^{\infty} e^{tx} g(x)\,dx \tag{12}$$

$$= \frac{\alpha\theta^4}{\theta^3+6} \int_0^{\infty} e^{tx}\left(1+x^3\right) \left\{ 1 - \left[1 + \frac{\theta x\left(\theta^2 x^2 + 3\theta x + 6\right)}{\theta^3+6} \right] e^{-\theta x} \right\}^{\alpha-1} e^{-\theta x}\,dx \tag{13}$$

Applying binomial series expansion to (13), and bearing in mind that $\int_0^{\infty} x^k e^{-\alpha x}\,dx = \frac{\Gamma(k+1)}{\alpha^{k+1}}$, $\Gamma\alpha = (\alpha-1)!$

and $e^{tx} = \sum_{n=0}^{\infty} \frac{(tx)^n}{n!}$, we obtain

$$M_x(t) = \sum_{i=0}^{\infty} \binom{\alpha-1}{i} (-1)^i \sum_{j=0}^{\infty} \binom{i}{j} \sum_{k=0}^{j} \binom{j}{k} \sum_{l=0}^{k} \binom{k}{l} \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\alpha 3^k 2^l \theta^{3j-k-l+4}}{\left(\theta^3+6\right)^{j+1}} \int_0^{\infty} x^{n+3j-k-l} e^{-\theta x(i+1)}\,dx$$

$$+ \sum_{i=0}^{\infty} \binom{\alpha-1}{i} (-1)^i \sum_{j=0}^{\infty} \binom{i}{j} \sum_{k=0}^{j} \binom{j}{k} \sum_{l=0}^{k} \binom{k}{l} \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\alpha 3^k 2^l \theta^{3j-k-l+4}}{\left(\theta^3+6\right)^{j+1}} \int_0^{\infty} x^{n+3j-k-l+3} e^{-\theta x(i+1)}\,dx$$

$$= \sum_{i=0}^{\infty} \binom{\alpha-1}{i} (-1)^i \sum_{j=0}^{\infty} \binom{i}{j} \sum_{k=0}^{j} \binom{j}{k} \sum_{l=0}^{k} \binom{k}{l} \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\alpha 3^k 2^l \theta^3}{\left(\theta^3+6\right)^{j+1}} \frac{(n+3j-k-l)!}{(i+1)^{(n+3j-k-l+1)}}$$

$$+ \sum_{i=0}^{\infty} \binom{\alpha-1}{i} (-1)^i \sum_{j=0}^{\infty} \binom{i}{j} \sum_{k=0}^{j} \binom{j}{k} \sum_{l=0}^{k} \binom{k}{l} \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\alpha 3^k 2^l}{\left(\theta^3+6\right)^{j+1}} \frac{(n+3j-k-l+3)!}{(i+1)^{(n+3j-k-l+4)}}$$

Let

$$\varsigma_{i,j,k,l} = \sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i\sum_{j=0}^{\infty}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\frac{\alpha 3^k 2^l\theta^3}{n!\left(\theta^3+6\right)^{j+1}}$$

and

$$\phi_{i,j,k,l} = \sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i\sum_{j=0}^{\infty}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\frac{\alpha 3^k 2^l}{n!\left(\theta^3+6\right)^{j+1}}$$

Therefore, the moment generating function of an exponentiated Rama distribution becomes

$$M_x(t) = \sum_{n=0}^{\infty}\left(\frac{t}{\theta}\right)^n\left[\varsigma_{i,j,k,l}\frac{(n+3j-k-l)!}{(i+1)^{(n+3j-k-l+1)}}+\phi_{i,j,k,l}\frac{(n+3j-k-l+3)!}{(i+1)^{(n+3j-k-l+4)}}\right] \quad (14)$$

3.3 Order statistics

Theorem 3: Suppose $X_1, X_2,..., X_n$ constitutes a random sample of size n from an exponentiated Rama distribution. Let $X_{(1)}, X_{(2)},..., X_{(n)}$ denote the corresponding order statistics. Then, the PDF and the CDF of the ωth order statistics, say $Y = X_{(\omega)}$ are, respectively, given by

$$f_X(x) = \frac{\alpha n!\left(1+x^3\right)e^{-\theta x(j+1)}}{(\omega-1)!(n-\omega)!}\sum_{i=0}^{n-\omega}(-1)^i\sum_{j=0}^{\infty}\binom{\alpha(\omega+i)-1}{j}(-1)^j\sum_{k=0}^{\infty}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}$$
$$\sum_{m=0}^{l}\binom{l}{m}\frac{3^l 2^m\theta^{3k-l-m}}{\left(\theta^3+6\right)^{k+1}}x^{3k-l-m}$$

and

$$F_X(x) = \sum_{j=\omega}^{n}\binom{n}{j}\sum_{k=0}^{n-j}\binom{n-j}{k}(-1)^k\sum_{l=0}^{\infty}\binom{\alpha(j+k)}{l}(-1)^l\sum_{m=0}^{\infty}\binom{l}{m}$$
$$\sum_{\tau=0}^{m}\binom{m}{\tau}\sum_{u=0}^{\tau}\binom{\tau}{u}\frac{3^\tau 2^u\theta^{3m-\tau-u}}{\left(\theta^3+6\right)^m}x^{3m-\tau-u}e^{-l\theta x}$$

Proof: First, we shall derive the PDF of the ωth order statistics. If X is a random variable from a continuous distribution, the PDF of the order statistics is given as

$$f_X(x) = \frac{n!}{(\omega-1)!(n-\omega)!}\sum_{i=0}^{n-\omega}\binom{n-\omega}{i}(-1)^i G^{\omega+i-1}(x)g(x) \quad (15)$$

Substituting (15) and (16) for $G(x)$ and $g(x)$ in (15), we have

$$f_X(x) = \frac{\alpha\theta^4 n!\left(1+x^3\right)e^{-\theta x}}{\left(\theta^3+6\right)(\omega-1)!(n-\omega)!}\sum_{i=0}^{n-\omega}\binom{n-\omega}{i}(-1)^i\left\{1-\left[1+\frac{\theta x\left(\theta^2x^2+3\theta x+6\right)}{\theta^3+6}\right]e^{-\theta x}\right\}^{\alpha(\omega+i)-1} \quad (16)$$

The series expansion of $\left\{1-\left[1+\frac{\theta x\left(\theta^2x^2+3\theta x+6\right)}{\theta^3+6}\right]e^{-\theta x}\right\}^{\alpha(\omega+i)-1}$ gives

$$=\sum_{j=0}^{\infty}\binom{\alpha(\omega+i)-1}{j}(-1)^j\sum_{k=0}^{\infty}\binom{j}{k}\sum_{l=0}^{k}\binom{l}{k}\sum_{m=0}^{l}\binom{l}{m}\frac{\theta^k x^k}{\left(\theta^3+6\right)^k}3^l 2^m\theta^{2k-l-m}x^{2k-l-m}e^{-j\theta x} \tag{17}$$

Inserting (17) in (16) gives the PDF of the ωth order statistics as

$$f_X(x)=\frac{\alpha n!\left(1+x^3\right)e^{-\theta x(j+1)}}{(\omega-1)!(n-\omega)!}\sum_{i=0}^{n-\omega}(-1)^i\sum_{j=0}^{\infty}\binom{\alpha(\omega+i)-1}{j}(-1)^j\sum_{k=0}^{\infty}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}$$
$$\sum_{m=0}^{l}\binom{l}{m}\frac{3^l 2^m\theta^{3k-l-m}}{\left(\theta^3+6\right)^{k+1}}x^{3k-l-m} \tag{18}$$

Similarly, the PDF of the ωth order statistics is given by

$$F_X(x)=\sum_{j=0}^{n}\binom{n}{j}F^j(x)\{1-F(x)\}^{n-j}=\sum_{j=\omega}^{n}\binom{n}{j}\sum_{k=0}^{n-j}\binom{n-j}{k}(-1)^k F^{j+k}(x) \tag{19}$$

$$=\sum_{j=\omega}^{n}\binom{n}{j}\sum_{k=0}^{n-j}\binom{n-j}{k}(-1)^k\left\{1-\left[1+\frac{\theta x\left(\theta^2x^2+3\theta x+6\right)}{\theta^3+6}\right]e^{-\theta x}\right\}^{\alpha(j+k)} \tag{20}$$

Where the series expansion of $\left\{1-\left[1+\frac{\theta x\left(\theta^2x^2+3\theta x+6\right)}{\theta^3+6}\right]e^{-\theta x}\right\}^{\alpha(j+k)}$ yields

$$=\sum_{l=0}^{\infty}\binom{\alpha(j+k)}{l}(-1)^l\sum_{m=0}^{\infty}\binom{l}{m}\sum_{\tau=0}^{m}\binom{m}{\tau}\sum_{u=0}^{\tau}\binom{\tau}{u}\frac{\theta^{3m-\tau-u}x^{3m-\tau-u}}{\left(\theta^3+6\right)^m}3^\tau 2^u e^{-l\theta x}$$

A substitution of this result from the series expansion yields the cdf of the ωth order statistics of Rama distribution. Consequently,

$$F_X(x)=\sum_{j=\omega}^{n}\binom{n}{j}\sum_{k=0}^{n-j}\binom{n-j}{k}(-1)^k\sum_{l=0}^{\infty}\binom{\alpha(j+k)}{l}(-1)^l\sum_{m=0}^{\infty}\binom{l}{m}$$
$$\sum_{\tau=0}^{m}\binom{m}{\tau}\sum_{u=0}^{\tau}\binom{\tau}{u}\frac{3^\tau 2^u\theta^{3m-\tau-u}}{\left(\theta^3+6\right)^m}x^{3m-\tau-u}e^{-l\theta x} \tag{21}$$

3.4 Rényi entropy

Theorem 4: Suppose X is a random variable that follows Exponentiated Rama distribution, the entropy is given by

$$R_e(\gamma)=\frac{1}{1-\gamma}\log\left[\beta_{i,j,k,l}\frac{(3j-k-l)!}{(i\gamma+1)^{(3j-k-l+1)}}+\Lambda_{i,j,k,l}\frac{(3\gamma+3j-k-l)!}{(i\gamma+1)^{(3\gamma+3j-k-l+1)}}\right]$$

Proof: Given a random variable X which follows exponentiated Rama distribution, the Rényi entropy is given by

$$R_e(\gamma)=\frac{1}{1-\gamma}\log\left[\int f_{Ra}^{\gamma}(x)\,dx\right];\ \gamma>0,\gamma\neq 1 \tag{22}$$

$$R_e(\gamma)=\frac{1}{1-\gamma}\log\left\{\left(\int_0^\infty \frac{\alpha\theta^4}{\theta^3+6}\left(1+x^3\right)\left\{1-\left[1+\frac{\theta^3x^3+3\theta^2x^2+6\theta x}{\theta^3+6}\right]e^{-\theta x}\right\}^{\alpha-1}e^{-\theta x}\right)^{\gamma}dx\right\}$$

$$=\frac{1}{1-\gamma}\log\left\{\int_0^\infty \frac{\alpha^{\gamma}\theta^{4\gamma}}{\left(\theta^3+6\right)^{\gamma}}\left(1+x^3\right)^{\gamma}\left\{1-\left[1+\frac{\theta^3x^3+3\theta^2x^2+6\theta x}{\theta^3+6}\right]e^{-\theta x}\right\}^{\gamma(\alpha-1)}e^{-\gamma\theta x}dx\right\} \tag{23}$$

By binomial expansion series, the expansion of $\left\{1-\left[1+\frac{\theta^3x^3+3\theta^2x^2+6\theta x}{\theta^3+6}\right]e^{-\theta x}\right\}^{\gamma(\alpha-1)}$ gives

$$=\sum_{i=0}^{\infty}\binom{\gamma(\alpha-1)-1}{i}(-1)^i\sum_{j=0}^{\infty}\binom{i}{j}\frac{\theta^jx^j}{\left(\theta^3+6\right)^j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}3^k2^l\theta^{2j-k-l}x^{2j-k-l}e^{-i\theta x}$$

Substituting in (23), we obtain the following

$$=\frac{1}{1-\gamma}\log\left\{\begin{array}{l}\sum_{i=0}^{\infty}\binom{\gamma(\alpha-1)-1}{i}(-1)^i\sum_{j=0}^{\infty}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\frac{3^k2^l\alpha^{\gamma}\theta^{4\gamma+3j-k-l}}{\left(\theta^3+6\right)^{\gamma+j}}\int_0^\infty x^{3j-k-l}e^{-\theta x(i\gamma+1)}dx\\+\sum_{i=0}^{\infty}\binom{\gamma(\alpha-1)-1}{i}(-1)^i\sum_{j=0}^{\infty}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\frac{3^k2^l\alpha^{\gamma}\theta^{4\gamma+3j-k-l}}{\left(\theta^3+6\right)^{\gamma+j}}\int_0^\infty x^{3\gamma+3j-k-l}e^{-\theta x(i\gamma+1)}dx\end{array}\right\}$$

$$=\frac{1}{1-\gamma}\log\left\{\begin{array}{l}\sum_{i=0}^{\infty}\binom{\gamma(\alpha-1)-1}{i}(-1)^i\sum_{j=0}^{\infty}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\frac{3^k2^l\alpha^{\gamma}\theta^{-1+4\gamma}}{\left(\theta^3+6\right)^{\gamma+j}}\frac{(3j-k-l)!}{(i\gamma+1)^{(3j-k-l+1)}}\\+\sum_{i=0}^{\infty}\binom{\gamma(\alpha-1)-1}{i}(-1)^i\sum_{j=0}^{\infty}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\frac{3^k2^l\alpha^{\gamma}\theta^{-1+\gamma}}{\left(\theta^3+6\right)^{\gamma+j}}\frac{(3\gamma+3j-k-l)!}{(i\gamma+1)^{(3\gamma+3j-k-l+1)}}\end{array}\right\}$$

Let

$$\beta_{i,j,k,l}=\sum_{i=0}^{\infty}\binom{\gamma(\alpha-1)-1}{i}(-1)^i\sum_{j=0}^{\infty}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\frac{3^k2^l\alpha^{\gamma}\theta^{-1+4\gamma}}{\left(\theta^3+6\right)^{\gamma+j}}$$

and

$$\Lambda_{i,j,k,l} = \sum_{i=0}^{\infty} \binom{\gamma(\alpha-1)-1}{i} (-1)^i \sum_{j=0}^{\infty} \binom{i}{j} \sum_{k=0}^{j} \binom{j}{k} \sum_{l=0}^{k} \binom{k}{l} \frac{3^k 2^l \alpha^\gamma \theta^{-1+\gamma}}{(\theta^3+6)^{\gamma+j}}$$

Hence, the Rényi entropy of exponentiated Rama distribution is

$$R_e(\gamma) = \frac{1}{1-\gamma} \log \left[\beta_{i,j,k,l} \frac{(3j-k-l)!}{(i\gamma+1)^{(3j-k-l+1)}} + \Lambda_{i,j,k,l} \frac{(3\gamma+3j-k-l)!}{(i\gamma+1)^{(3\gamma+3j-k-l+1)}} \right] \tag{24}$$

3.5 Survival function of Exponentiated Rama distribution

The survival function of an exponentiated Rama distribution is defined as

$$S(x;\alpha,\theta) = 1 - F(x;\alpha,\theta) \tag{25}$$

$$S(x,\alpha,\theta) = 1 - \left\{ 1 - \left[1 + \frac{\theta^3 x^3 + 3\theta^2 x^2 + 6\theta x}{\theta^3 + 6} \right] e^{-\theta x} \right\}^{\alpha} \tag{26}$$

3.6 Hazard rate function of the for Exponentiated Rama distribution

The hazard rate function for the exponentiated Rama distribution is defined as

$$h(x;\alpha,\theta) = \frac{f(x;\alpha,\theta)}{S(x;\alpha,\theta)} = \frac{\frac{\alpha\theta^4}{\theta^3+6}(1+x^3)\left\{1-\left[1+\frac{\theta^3x^3+3\theta^2x^2+6\theta x}{\theta^3+6}\right]e^{-\theta x}\right\}^{\alpha-1} e^{-\theta x}}{1-\left\{1-\left[1+\frac{\theta^3x^3+3\theta^2x^2+6\theta x}{\theta^3+6}\right]e^{-\theta x}\right\}^{\alpha}} \tag{27}$$

4. Maximum Likelihood Estimators of exponentiated Rama distribution

Here, we shall estimate the parameters of the exponentiated Rama distribution. Let $X_1, X_2, \ldots, X_n$ constitute a random sample of size n. Then, the likelihood function is defined as

$$L = L(\alpha,\theta|x) = \prod_{i=1}^{n} \frac{\alpha\theta^4}{\theta^3+6}(1+x^3)\left\{1-\left[1+\frac{\theta^3x^3+3\theta^2x^2+6\theta x}{\theta^3+6}\right]e^{-\theta x}\right\}^{\alpha-1} e^{-\theta x} \tag{28}$$

$$= \left(\frac{\alpha\theta^4}{\theta^3+6}\right)^n \prod_{i=1}^{n}(1+x^3) \prod_{i=1}^{n}\left\{1-\left[1+\frac{\theta^3x^3+3\theta^2x^2+6\theta x}{\theta^3+6}\right]e^{-\theta x}\right\}^{\alpha-1} e^{-\theta\sum_{i=1}^{n}x_i} \tag{29}$$

Taking the natural logarithm of (29), we obtain the log likelihood function as

$$\ln L = \left\{ \begin{array}{l} n\ln\alpha + 4n\ln\theta - n\ln(\theta^3+6) + \sum_{i=1}^{n} \ln(1+x^3) \\ +(\alpha-1)\sum_{i=1}^{n} \ln\left[1-\left(1+\frac{\theta^3x^3+3\theta^2x^2+6\theta x}{\theta^3+6}\right)e^{-\theta x}\right]^{\alpha-1} - \theta\sum_{i=1}^{n} x_i \end{array} \right\}$$

Differentiating with respect to α and θ, we have the following equations

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \ln\left\{1-\left[1+\frac{\theta^3x^3+3\theta^2x^2+6\theta x}{\theta^3+6}\right]e^{-\theta x}\right\} = 0 \quad (30)$$

$$\frac{\partial \ln L}{\partial \theta} = \frac{4n}{\theta} - \frac{3n\theta^2}{\left(\theta^3+6\right)} + (\alpha-1)\sum_{i=1}^{n}\left\{\frac{\theta x\left(\theta^5+24\theta^2+\theta^5x^3+3\theta^4x^2+9\theta^3x+6\theta^2x^3\right)e^{-\theta x}}{\left(\theta^3+6\right)^2\left[1-\left(1+\frac{\theta^3x^3+3\theta^2x^2+6\theta x}{\theta^3+6}\right)e^{-\theta x}\right]}\right\} - \sum_{i=1}^{n} x_i = 0 \quad (31)$$

The solution of $\frac{\partial \ln L}{\partial \alpha} = 0$ and $\frac{\partial \ln L}{\partial \theta} = 0$ gives the maximum likelihood estimates of the parameters. The R package is used to resolve the log-likelihood equations.

5. Applications

In this section, we use a real data to demonstrate the applicability of the Exponentiated Rama distribution. The data refers to the tensile strength of 100 carbon fibers as used by Nicholas et al. (2006) and reported by Flaih A. *et al*. (2012) as follows

3.7, 3.11, 4.42, 3.28, 3.75, 2.96, 3.39, 3.31, 3.15, 2.81, 1.41, 2.76, 3.19, 1.59, 2.17, 3.51, 1.84, 1.61, 1.57, 1.89, 2.74, 3.27, 2.41, 3.09, 2.43, 2.53, 2.81, 3.31, 2.35, 2.77, 2.68, 4.91, 1.57, 2.00, 1.17, 2.17, 0.39, 2.79, 1.08, 2.88, 2.73, 2.87, 3.19, 1.87, 2.95, 2.67, 4.20, 2.85, 2.55, 2.17, 2.97, 3.68, 0.81, 1.22, 5.08, 1.69, 3.68, 4.70, 2.03, 2.82, 2.50, 1.47, 3.22, 3.15, 2.97, 2.93, 3.33, 2.56, 2.59, 2.83, 1.36, 1.84, 5.56, 1.12, 2.48, 1.25, 2.48, 2.03, 1.61, 2.05, 3.60, 3.11, 1.69, 4.90, 3.39, 3.22, 2.55, 3.56, 2.38, 1.92, 0.98, 1.59, 1.73, 1.71, 1.18, 4.38, 0.85, 1.80, 2.12, 3.65

The values reported in Table 1 are the estimated parameters, their standard errors, the Log-likelihood values, K-S statistic and their p-values, AIC and BIC values for the data set. Also, the fitted distributions are shown in Figures 3 and 4.

Table 1: MLE’s, -LL, AIC, BIC, K-S Statistics of the fitted distributions of dataset

Model	Parameters	S.E	-LL	AIC	BIC	KS	p
ER	**α =5.454316** **θ = 1.987194**	**1.2344989** **0.1140793**	**-141.2543**	**286.5087**	**291.719**	**0.07766**	**0.5562**
RD	θ = 1.24975	0.05628291	-166.7183	335.4366	338.0418	0.96832	5.55×10^{-16}
EA	α= 5.588762 θ = 1.604269	1.1644025 0.1039922	-142.4991	288.9983	294.2086	0.091947	0.3452
EE	α =7.878956 θ= 1.021184	1.51700077 0.08801806	-145.4642	294.9284	300.1387	0.11084	0.1589

Table 1 shows the MLE’s, -LL, AIC, BIC, K-S and p values of the exponentiated Rama distribution and the competing distributions. The results reported in Table 1 reveals that, on the average, the values of AIC, BIC and K-S are smaller for the Exponentiated Rama (ER) distribution than the other distributions while the values of log-likelihood (LL) and p-values are higher for the ER distribution. Hence, ER distribution provides a better fit more than the Rama distribution (RD), Exponentiated Akash distribution (EA), and Exponentiated Exponential (EE) distribution. Hence, the ER distribution outperforms the other distributions.

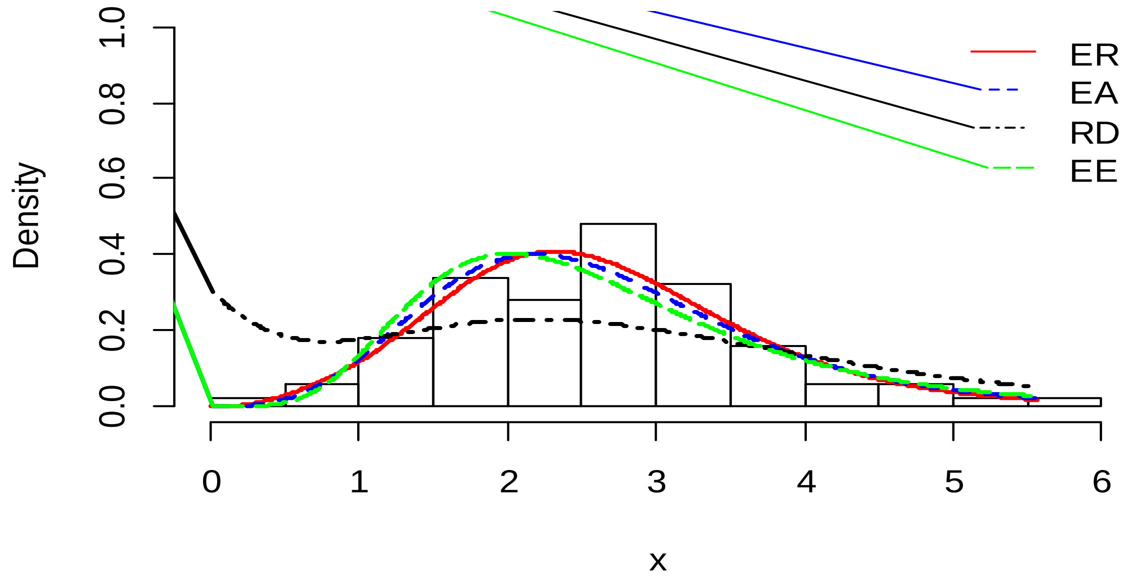

Figure. 3: The histogram and PDFs of fitted models for data set two

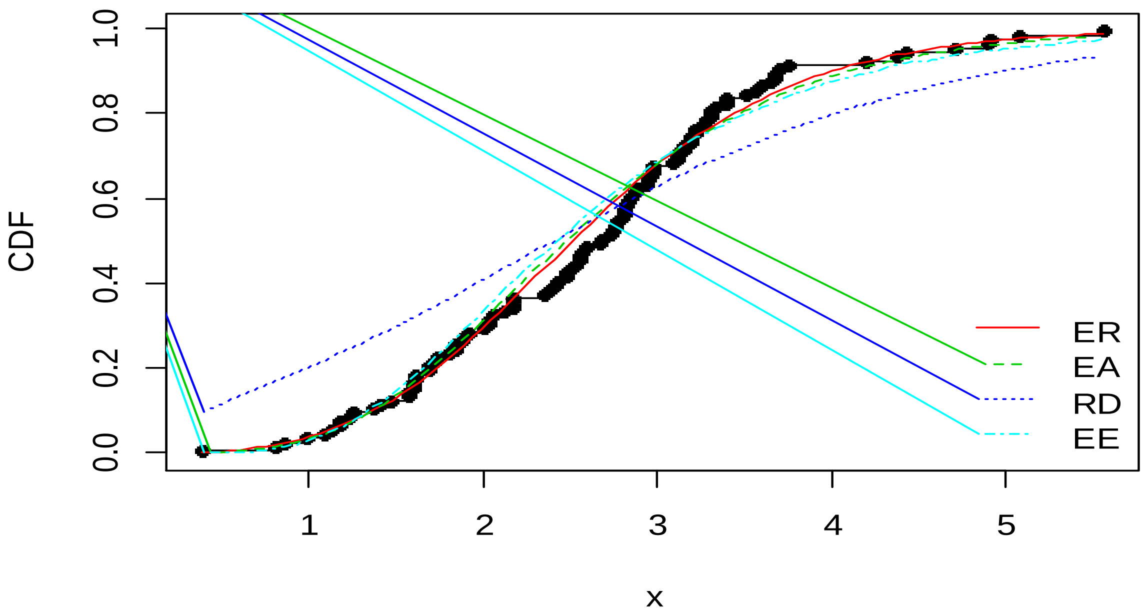

Figure. 4: CDFs of fitted models

From Figures 3 and 4, it is obvious that the ER distribution provided better fits than the other competing distributions.

6. Conclusion

In conclusion, this article has introduced an exponentiated Rama (ER) distribution, which generalizes the Rama distribution. Properties of the ER distribution have been explicitly derived. The maximum likelihood estimation of the ER distribution was presented with its application to a real data set. The ER distribution can be seen as an improvement over the RD, EA, and EE distributions since these distributions do not provide a better fits than the ER distribution.

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