

Hereditary Modules over Path Algebras of a Quiver Containing Cycles

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Abstract

Let E be a quiver, K any field, and KE denotes the path algebra of E with coefficients in K. A module is called hereditary if all its submodules are projective. In this paper, we characterize hereditary modules over the path algebra KE of a quiver E which contains cycles.

Keywords: path algebras, representations of quivers, hereditary modules

1. Introduction

The concept of a hereditary module is a generalization of the concept of a hereditary ring, and this generalization was first introduced by Shrikhande (1973). The hereditary modules over a ring have been widely studied. Hill (1977) has examined the endomorphism structure of hereditary modules. Irawati (2011) developed the concept of hereditary Noetherian prime (HNP) modules which is a generalization of HNP ring. Dinh et al. (2006) has studied the Goldie dimension of hereditary modules.

In this paper, we study the hereditary modules over path algebras. The path algebra of a finite, connected, and acyclic quiver is hereditary (see Assem et al. (2006)). All projective modules over the path algebra of a finite, connected, and acyclic quiver are hereditary. Our goal is to characterize the hereditary modules over the path algebra of a quiver which contains cycles.

In Section 2, we will review some basic facts for path algebras, representations of quivers, and hereditary modules. In Section 3, we characterize the hereditary modules over self-injective Nakayama algebras and the hereditary modules over the path algebra of a quiver which contains cycles.

2. Path Algebras, Representations of Quivers, and Hereditary Modules

In this section, we introduce some notations and definitions and also review some previous results which are used in this paper (see in Assem et al. (2006), Passman (1991), Schiffler (2014)).

2.1 Path Algebras

A quiver (directed graph) $E = (E^0, E^1, r, s)$ consists of two sets E^0, E^1 and maps $r, s: E^1 \longrightarrow E^0$. The elements of E^0 are called points or vertices, and the elements of E^1 are called edges or arrows. For each arrow e, $s(e)$ is the source of e, and $r(e)$ is the range of e.

A path μ in the quiver E is a sequence of arrows $\mu = e_1 \cdots e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1 \cdots n-1$. In this case, $s(\mu) = s(e_1)$ is the source of μ, $r(\mu) = r(e_n)$ is the range of μ, and n is the length of μ. For any $a \in E^0$, we associate a path of length $\ell = 0$, called constant or trivial path and denoted by $\mathcal{E}_a$.

A path $\mu = e_1 e_2 \cdots e_n$ in quiver E is closed if $r(e_n) = s(e_1)$. The closed path μ is called a cycle if $s(e_i) \neq s(e_j)$ for every $i \neq j$. A cycle has an exit if there is a vertex a such that a is in the cycle and a is a source of an arrow outside the cycle.

Given an arbitrary quiver E and a field K, the path algebra KE is defined to be K-algebra generated by paths in E which satisfy the following relations:

(a) $v_i v_j = \delta_{ij} v_i$ for every $v_i, v_j \in E^0$
(b) $e_i = e_i r(e_i) = s(e_i) e_i$ for every $e_i \in E^1$

2.2 Representations of Quivers

Using a quiver E associated to an algebra A, any A-module M as a K-linear representation of E can be visualized. The explicit computation of the simple, the indecomposable projective, and the indecomposable injective A-modules as representations of E can be seen in (Assem et al., 2006).

First, we need the concept of an admissible ideal and a bound quiver algebra. Let E be a finite and connected quiver. The two-sided ideal of path algebra KE generated (as an ideal) by arrows of E is called the arrows ideal of KE and is denoted by R_E. A two-sided ideal $\mathcal{J}$ of KE is said to be admissible if there exists $m \geq 2$ such that $R_E^m \subseteq \mathcal{J} \subseteq R_E^2$. If $\mathcal{J}$ is an admissible ideal of KE, the pair $(E, \mathcal{J})$ is said to be a bound quiver. We call a quotient algebra $KE/\mathcal{J}$: algebra of the bound quiver $(E, \mathcal{J})$ or, simply, a bound quiver algebra.

Using a bound quiver $(E, \mathcal{J})$ associated to an algebra A, we visualize any finite dimensional A-module M as a K-linear representation of $(E, \mathcal{J})$, that is, a family of finite dimensional K-vector spaces M_a, with $a \in E^0$, connected by K-linear maps $\varphi_\alpha: M_a \to M_b$ corresponding to arrows $\alpha: a \to b$ in E and satisfying some relations induced by $\mathcal{J}$.

Let E be a finite quiver. A K-linear representation or, more briefly, a representation M of E is defined by the following data:

(a) To each point $a \in E^0$ is associated a K-vector space M_a.
(b) To each arrow $\alpha: a \to b$ in E^1 is associated a K-linear map $\varphi_\alpha: M_a \to M_b$

The representation is denoted as $M = (M_a, \varphi_\alpha)_{\alpha \in E^0, \alpha \in E^1}$ or simply $M = (M_a, \varphi_\alpha)$. We call M finite dimensional if each vector space M_a finite dimensional.

Given a quiver E, the representation of E together with morphisms of representations form a category which we denote by $Rep(E)$. We denote by $rep(E)$ the full subcategory of $Rep(E)$ consisting of finite dimensional representations.

Let E be a finite, connected, and acyclic quiver. There exists an equivalence of categories $Mod\ KE \cong Rep_K(E)$ that restricts to an equivalence $mod\ KE \cong rep_K(E)$.

Now we will define the simple, the indecomposable projective, and the indecomposable injective A-modules as bound representation of $(E, \mathcal{J})$.

Let $a \in E^0$, we denote by $S(a)$ the representation $(S(a)_b, \varphi_\alpha)$ of E defined as follows

$$S(a)_b = \begin{cases} 0 \ if\ b \neq a \\ K \ if\ b = a \end{cases}$$

$$\varphi_\alpha = 0\ for\ all\ \alpha \in E^1$$

$S(a)$ is the simple A-module corresponding to the point $a \in E^0$.

Next, we describe the indecomposable projective A-module. Let $\{e_a: a \in E^0\}$ be a complete set of orthogonal primitive idempotent of A. Let $(E, \mathcal{J})$ be a bound quiver, $A = KE/\mathcal{J}$, and $P(a) = e_a A$, where $a \in E^0$. If $P(a) = \left(P(a)_b, \varphi_\beta\right)$, then $P(a)_b$ is K-vector space with basis the set of all the $\varpi = \omega + \mathcal{J}$, with ω a path from a to b and, for an arrow $\beta: b \to c$, the K-linear map $\varphi_\beta: P(a)_b \to P(a)_c$ is given by the right multiplication by $\bar{\beta} = \beta + \mathcal{J}$. $P(a)$ is the indecomposable projective A-module corresponding to the point $a \in E^0$.

If $I(a) = \left(I(a)_b, \varphi_\beta\right)$, then $I(a)_b$ is the dual of K-vector space with basis the set of all $\varpi = \omega + \mathcal{J}$, with ω a path from b ke a and, for an arrow $\beta: b \to c$, the K-linear map $\varphi_\beta: I(a)_b \to I(a)_c$ is given by the dual of the left multiplication by $\bar{\beta} = \beta + \mathcal{J}$. $I(a)$ is the indecomposable injective A-module corresponding to the point $a \in E^0$.

2.3 Hereditary Modules

An algebra A is said to be right hereditary if any right ideal of A is projective as an A-module. If E is a finite, connected, and acyclic quiver, then the algebra $A = KE$ is hereditary. An A-Module M is said to be hereditary if M and all its submodules are projective.

3. Hereditary Module over Path Algebras of a Quiver Containing Cycles

In this section, we obtain the characterization of hereditary modules over the path algebra of a quiver which contains cycles.

First, we see the projective modules over the path algebra of a cycle. Without loss of generality, we can consider the Nakayama path algebra $N_n^m = KE/\mathcal{I}$.

Theorem 3.1. Let $N_n^m = KE/\mathcal{I}$ be a self-injective Nakayama path algebra, where E is a cycle with n vertices, and $\mathcal{I}$ is the ideal of the path algebra KE generated by the paths of length $m+1$. Then all projective N_n^m-modules are not hereditary modules.

Proof. Let $q = m+1$ be the length of paths generating $\mathcal{I}$. We prove this theorem by dividing the proof into three cases:

Case 1: $n = q$

Case 2: $n > q$

Case 3: $n < q$

Case 1: $n = q$, that is, the number of vertices of the cycle E is equal to the length of paths generating the ideal of the path algebra KE. Let E be a cycle with n vertices (see Figure 1) bound by $\alpha_1\alpha_2 \cdots \alpha_n = 0, \alpha_2\alpha_3 \cdots \alpha_1 = 0, \cdots, \alpha_n\alpha_{1\ldots}\alpha_{n-1} = 0$.

The indecomposable projective and injective N_n^m-modules $P(1)$, $P(2), \ldots, P(n), I(1), \ldots, I(n)$ as in Figure 2. The projective module $P(1)$ has (2) as its submodule, $P(2)$ has (3) as its submodule, and so on until $P(n)$ has (1) as its submodule. All these simple submodules are not projective. So we can conclude that all projective N_n^m-modules are not hereditary.

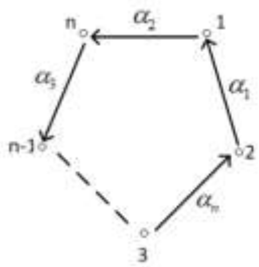

Figure 1. Cycle E with n Vertices

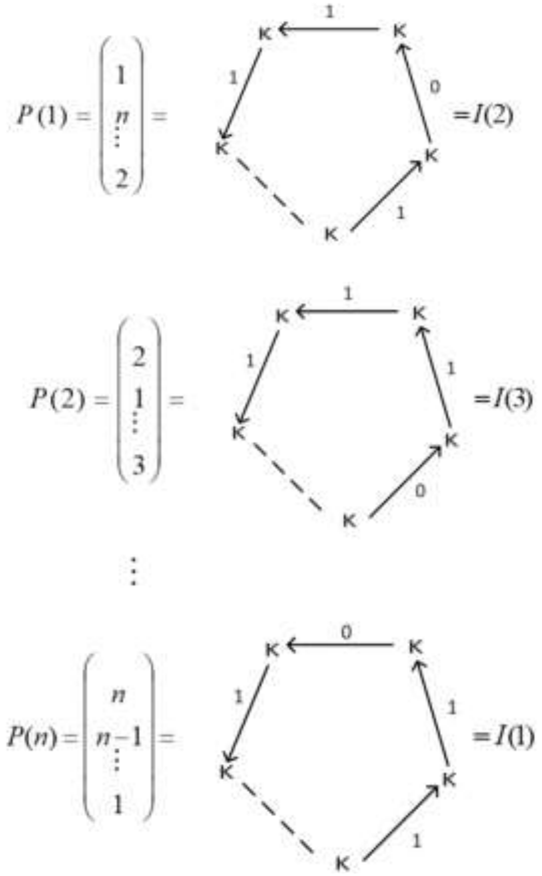

Figure 2. Indecomposable Projective and Injective N_n^m-Modules
$P(1), P(2), \ldots, P(n)$

Case 2: $n > q$, that is, the number of vertices of the cycle E is greater than the length of paths generating the ideal of the path algebra KE. Let E be a cycle with n vertices (see Figure 1). Let $\alpha_1\alpha_2\cdots\alpha_q = 0, \alpha_2\alpha_3\cdots\alpha_q\alpha_{q+1} = 0, \cdots, \alpha_{n-q+2}\alpha_{n-q+3}\cdots\alpha_n\alpha_1 = 0$. The projective module $P(1)$ has $(n-q+2)$ as its submodule, $P(2)$ has $(n-q+3)$ as its submodule, and so on until $P(n)$ has $(n-q+1)$ as its submodule. All these simple submodules are not projective. So we can conclude that all projective N_n^m-modules are not hereditary.

Case 3: $n < q$, that is, the number of vertices of the cycle E is less than the length of paths generating the ideal of the path algebra KE. Let E be a cycle with n vertices (see Figure 1), for any length q, we always obtain that the projective module $P(1)$ has (2) as its submodule, $P(2)$ has (3) as its submodule, and so on until $P(n)$ has (1) as its submodule. All these simple submodules are not projective. So we can conclude that all projective N_n^m-modules are not hereditary.■

Using a similar approach, we can conclude that all projective modules over the path algebra of a cycle are not hereditary.

Next, we characterize hereditary modules over the path algebra of a quiver which contains cycles.

Theorem 3.2. Let E be a quiver which is a union of a cycle and a line, K any field, and $A = KE/\mathcal{J}$ be a path algebra, where $\mathcal{J}$ is $\langle \alpha_i\alpha_{i+1}\cdots\alpha_{i+m+1},\ i = 1, \cdots, n \rangle$.

(a) If the cycle receives an oriented path with the same direction with the target on the cycle (see Figure 3), then all projective A-modules at vertices lie in the cycle, and the oriented path are not hereditary.
(b) If the cycle has an exit (see Figure 4), then all projective A-modules at vertices that lie in the cycle are not hereditary, and the projective A-module at the target of the exit is hereditary.

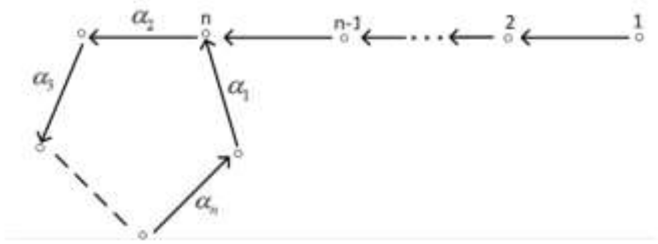

Figure 3. Quiver E Contains a Cycle Receiving an Oriented Path with the Same Direction with Target on the Cycle

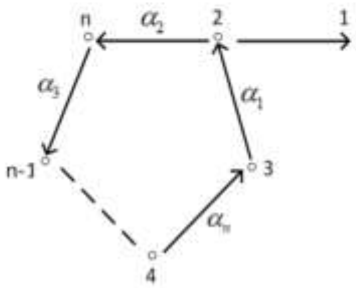

Figure 4. Quiver E Contains Cycle Having an Exit

Proof.

(a) Let E be a quiver which contains a cycle receiving an oriented path with the same direction with the target on the cycle (see Figure 3). All indecomposable projective A-modules at vertices that lie in the cycle are not hereditary. The indecomposable projective modules at vertices 1, 2, ..., $n-1$ are given by $P(1)$, $P(2),\ldots,P(n-1)$, as can be seen in Figure 5. The projective modules $P(1),P(2),\ldots,P(n-1)$ have the same submodule, which is Q, as in Figure 6. Since projective modules $P(1),P(2),\ldots,P(n-1)$ have the simple submodule Q which is not projective, then $P(1),P(2),\ldots,P(n-1)$ are not hereditary. Therefore, the indecomposable projective A-modules on E are not hereditary.

(b) Let E be a quiver which contains a cycle having an exit (see Figure 4). All projective A-modules at vertices that lie in the cycle are not hereditary. The indecomposable projective A-module on vertex 1 is $P(1)$ as in Figure 7, which is a projective simple module, so $P(1)$ is a hereditary module. ■

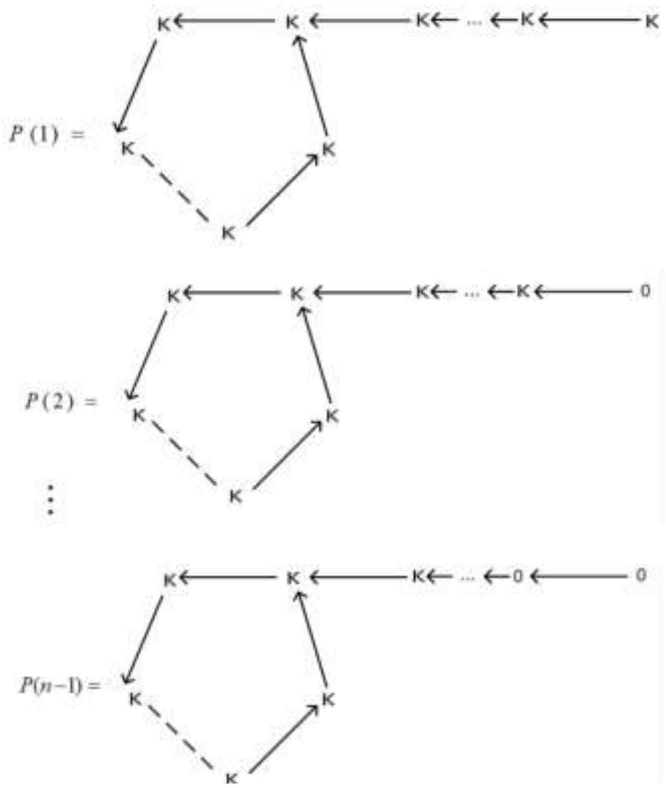

Figure 5. Indecomposable Projective A-Modules
$P(1), P(2), \dots, P(n-1)$

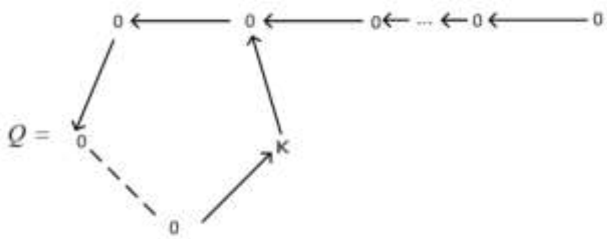

Figure 6. Submodule Q of Projective A-Modules
$P(1), P(2), \dots, P(n-1)$

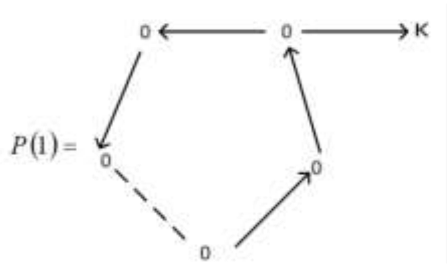

Figure 7. Projective A-Module $P(1)$

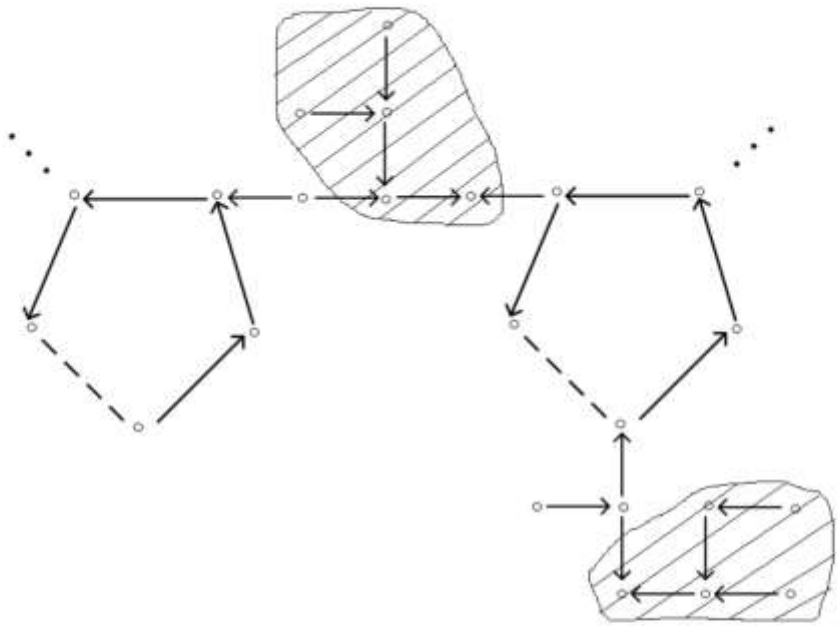

Figure 8. Quiver E Contains Some Cycles and Trees

The next theorem is the characterization of hereditary modules over the path algebra of a quiver which is a union of some cycles and trees.

Theorem 3.3. Let E be a quiver which is a union of some cycles and trees (see Figure 8). Let $S_1, \cdots, S_k$ be the cycles in E. Suppose K be a field, $A = KE/\mathcal{I}$ be a path algebra where $\mathcal{I}$ is the ideal generated by the paths in cycles $S_1, \cdots, S_k$ with length $m_1 + 1, \cdots, m_k + 1$, respectively. For T a tree in E, let $V(T) =$ the set of vertices in the trees, $V_0(T) =$ the subset of $V(T)$ which has oriented paths with the target on cycle, and $V_1(T) = V(T) - V_0(T)$. Then

(a) All projective A-modules at vertices lie in cycles, and $V_0(T)$ for all $T \subseteq E$ are not hereditary.
(b) The projective modules in $V_1(T)$ for all $T \subseteq E$ are hereditary.
(c) The projective modules at the target of an exit are hereditary.

Proof. Let E be a quiver which is a union of some cycles and trees (see Figure 8)

(a) By Theorem 3.2(a), we have that all projective A-modules at vertices lie in cycles and $V_0(T)$ for all $T \subseteq E$ are not hereditary.

(b) The path algebra of a quiver with the set of vertices $V_1(T)$ is hereditary, then all projective A-modules at vertices lie in $V_1(T)$ for all $T \subseteq E$ are hereditary.
(c) By Theorem 3.2, the projective A-modules at the target of an exit are hereditary.

Example 3.4. In Figure 8, the hereditary modules are the projective modules that correspond to the vertices in the shaded region.

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References

Assem, I., Skowronski, A., & Simson, D. (2006). *Elements of the Representation Theory of Associative Algebras*. London Math. Soc. Student Texts 65.

Dinh, H. Q., Guil Asensio, P. A., & López-Permouth, S. R. (2006). On the Goldie dimension of rings and modules. *Journal of Algebra*, *305*(2), 937–948.

Hill, D. A. (1977). Endomorphism Rings of Hereditary Modules. *Arch Mathematics*, *28*(1), 45–50.

Irawati. (2011). The Generalization of HNP Ring and Finitely Generated Module over HNP Ring. *International Journal of Algebra*, *5*(13), 611–626.

Passman, D. S. (1991). *a Course In Ring Theory*. Brooks/Cole Publishing Company.

Schiffler, R. (2014). *Quiver Representations*. Springer International Publishing Switzerland.

Shrikhande, M. S. (1973). On Hereditary and Cohereditary Modules. *Canadian Journal of Mathematics*, *25*(4), 892–896.