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# An Improved L-Stable Scheme for Initial Value Problems Under Variable Step-Size Approach

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**Abstract**

A non-linear explicit scheme has been studied for autonomous and non-autonomous initial value problems in ordinary differential equations (ODEs). This research proposed fifth order of convergence. The stability region of the scheme is also shown, as is the evolution of the scheme's associated local truncation error. A few numerical experiments showed that the scheme is fit for initial value problems with singular solutions, blowup the ODEs, singularly perturbed and stiff problems. MATLAB R2019a was used for the numerical computations and plotting of results produced by all methods.

**Keywords**. Nonlinear method, local truncation error, L-Stability, Variable step-size, autonomous and non-autonomous.

## 1. Introduction

The usage of ordinary differential equations (ODEs) is present everywhere in science, engineering, economics, social sciences, and health care. The field of ODEs is an extensive mathematical discipline and is widely used to study the mathematical modeling of many physical problems like radioactive decay, population dynamics, mechanical systems, fluid flows, electrical networks, rate of chemical reactions, and many others [8]. Initial value problems (IVPs) of ODEs are complex by nature and their pure analytical solutions are not possible, such ODEs can be solved numerically, just in a few cases, IVPs can be solved analytically [7]. In terms of convergence, the numerical schemes performed better and were easily result-oriented than analytical schemes. Many researchers formulated numerous schemes for the solutions of such ODEs [10, 12]. Numerical methods for solutions of ODEs are common. Mathematicians have developed several effective single and multi-step methods to get an approximate solution. Solving ODEs is a challenge in pure and applied mathematics so the numerical methods have significance for estimating the solution of the ODEs [6, 8].

Therefore, in this research, the method known as fifth order has been developed to resolve IVPs with integration interval considering constant and variable step size. This method has rational character and good stability characteristics i.e. L-stability. The proposed numerical technique discretizes the interval [xn-x0] as follows.

$$x_n = x_0 + nh, \quad (n=1,2,3\ldots,M)$$

where,

$$h = \frac{x_n - x_0}{M}$$

h is the step size along with the integration interval of $[x_n\text{-}x_0]$.

In these cases, this paper aims to implement the developed method in variable step size and to adjust the step sizes to retain the assessed local errors lesser than tolerance efficiently.

The paper is structured as: the derivation of fifth-order is performed in Section 2, while the error and linear stability analysis are conducted in Section 3 and 4 respectively. Implementation in variable step-size mode is formulated in Section 5. Then, the performance of this method has been assessed through different numerical experiments.

## 2. Derivation of the improved scheme

Here the general first-order ODE has been considered to demonstrate different numerical methods for the solution of ODEs with an initial condition as defined below.

$$\frac{dy}{dt} = f(t,y),\ y(t_0) = y_0 \quad y, f(t,y) \in \mathbb{R},\ t \in [a,b] \subset \mathbb{R}\,. \tag{1}$$

Existence of unique solution of (1) is assumed for the integration interval of $t \in [a,b]$ .

Here,

$$y_n \approx y(t_n),$$

Where, $y_n$ is the approximation to the theoretical solution $y(t)$ at the nodal points

$$t_n = a + nh\,;$$

taking the step size

$$h = \frac{x_{n-x_0}}{M}$$

where $n = 1,2,3 \dots M$.

After getting the motivation from [7], the method has been improved for order of convergence and L-stability.

The approximate solution at $t = t_{n+1}$;

$$y_{n+1} = \frac{A + B.h}{1 + C.h + d.h^2 + E.h^3 + F.h^4} \tag{2}$$

where A, B, C, d, E, and F are unknown co-efficient which depend on the known variables at $t_n$.

Equation (2) is expanded by Taylor's series, we get

$$y_{n+1} = A + (-AC + B)h + (-Ad + (AC - B)C)h^2 + (-AE + (AC - B)d + (-AC^2 + Ad + BC)C)h^3 + (-AF + (AC - B)E + (-AC^2 + Ad + BC)d + (AC^3 - 2ACd - BC^2 + AE + Bd)C)h^4 + ((AC - B)F + (-AC^2 + Ad + BC)E + (AC^3 - 2ACd - BC^2 + AE + Bd)d + (-AC^4 + 3AC^2d + BC^3 - 2ACE - Ad^2 - 2BCd + AF + BE)C)h^5 + ((-AC^2 + Ad + BC)F + (AC^3 - 2ACd - BC^2 + AE + Bd)E + (-AC^4 + 3AC^2d + BC^3 - 2ACE - Ad^2 - 2BCD + AF + BE)d + (AC^5 - 4AC^3d - BC^4 + 3AC^2E + 3ACd^2 + 3BC^2d - 2ACF - 2AEd - 2BCE - Bd^2 + BF)C)\, h^6 + O(h^7)\,. \tag{3}$$

Solving for A, B, C, d, E, and F, by equating the coefficients up to $h^5$ and comparing it with Taylor's series t= $t_n$, we get

$$A = y_n \quad (4)$$

$$B = -AC + B \quad (5)$$

$$C = 2AC^2 - 2Ad - 2BC \quad (6)$$

$$D = -6AC^3 + 12ACd + 6BC^2 - 6AE - 6Bd \quad (7)$$

$$E = 24AC^4 - 72AC^2d - 24BC^3 + 48ACE + 24Ad^2 + 48BCd - 24AF - 24BE \quad (8)$$

$$F = -120AC^5 + 480AC^3d + 120BC^4 - 360AC^2E - 360ACd^2 - 360BC^2d + 240ACF + 240AdE + 240BCE + 120Bd^2 - 120BF \quad (9)$$

After solving above system of nonlinear equations, we get

$$A = y \quad (10)$$

$$B = \frac{1/5\,(90(y'')^2y'y^2 - 240y''(y')^3y - 20y''y'''y^3 + 120(y')^5 + 60(y')^2y'''y^2 - 10y'y''''y^3 + y'''''y^4)}{(6y^2(y'')^2 - 36yy''(y')^2 + 24(y')^4 + 8y^2y'''y' - y^3y'''')} \quad (11)$$

$$C = \frac{-1/5\,(y'''''y^3 - 5y^2y''''y' - 20y^2y'''y'' + 20yy'''(y')^2 + 60(y'')^2y'y - 60y''(y')^3)}{(y^3(y'''' - 8y^2y'''y' - 6y^2(y'')^2 + 36yy''(y')^2 - 24(y')^4)} \quad (12)$$

$$d = \frac{1/10\,(-2y^2(y'''''y' - 5y^2y''''y'' - 10yy''''(y')^2 + 30y(y'')^3 + 40y'''(y')^3 - 60(y'')^2(y')^2)}{(y^3(y'''' - 8y^2y'''y' - 6y^2(y'')^2 + 36yy''(y')^2 - 24(y')^4)} \quad (13)$$

$$E = \frac{1/30\,(3y^2(y'''''y'' - 5y^2y''''y''' - 6yy'''''(y')^2 + 40y(y''')^2y' - 30yy'''(y'')^2 + 30y'''''(y')^3 - 120y'''y''(y')^2 + 90(y'')^3y')}{(y^3(y'''' - 8y^2y'''y' - 6y^2(y'')^2 + 36yy''(y')^2 - 24(y')^4)} \quad (14)$$

$$F = \frac{1/120\,(4y^2(y'''''y''' - 5y^2(y'''')^2 - 24yy'''''y''y' + 40yy''''y'''y' + 60yy''''(y'')^2 - 80y(y''')^2y'' + 24y'''''(y')^3 - 120y''''y''(y')^2 - 80(y''')^2(y')^2 + 360y'''(y'')^2y' - 180(y'')^4)}{(y^3(y'''' - 8y^2y'''y' - 6y^2(y'')^2 + 36yy''(y')^2 - 24(y')^4)} \quad (15)$$

After putting the equations, (10) - (15) into (2), we get

$y_{n+1} =$

$$\frac{(24(-y^4(y'''''h + 20y^3y'''y''h + 10y^3y''''y'h - 60y^2y'''(y')^2h - 90y^2(y'')^2y'h + 240yy''(y')^3h - 120y(y')^5h + 5y^4y'''' - 40y^3y'''y' - 30y^3(y'')^2 + 180y^2y''(y')^2 - 120y(y')^4)}{(-180(y'')^4h^4 - 24y^3y'''''h + 1440y''(y')^3h - 5y^2(y'''')^2h^4 - 80(y''')^2(y')^2h^4 + 24\ y'''''\ (y')^3\ h^4 + 360(y'')^3y'\ h^3 + 120\ y''''\ (y')^3\ h^3 + 360y\ (y'')^3h^2 + 480\ y'''(y')^3h^2 - 720\ (y'')^2(y')^2h^2 + 40yy'''y''''\ y'\ h^4 - 24y\ y''y'''''y'\ h^4 - 120yy''''(y')^2\ h^2 + 480y^2y'''y''h + 120\ y^2y''''y'h - 480yy'''(y')^2h - 1440y(y'')^2y'h + 4\ y^2y'''y'''''h^4 - 80y(y''')^2y''h^4 + 60y(y'')^2y''''h^4 + 360y'''(y'')^2y'h^4 - 120y''y''''\ (y')^2h^4 - 20y^2\ y'''y''''h^3 + 12\ y^2y''y''''h^3 + 160\ y(y''')^2y'h^3 - 120yy'''(y'')^2h^3 - 24yy'''''(y')^2h^3 - 480y'''y''(y')^2h^3 - 60y^2y''y''''h^2 + 24y^2y'''''y'h^2 + 120y''''y^3 - 720(y'')^2y^2 - -2880(y')^4 - 960y'''y'y^2 + 4320y''(y')^2y)} \tag{16}$$

## 3. Local truncation error

The local truncation error (LTE) associated to a numerical method is:

$$LTE = Ch^{p+1}y^{(p+1)}(x) + O(h^{p+2}), \tag{17}$$

where C is an error constant and p is order of accuracy for the numerical method.

Now, consider the following operator in (16);

$$\mathcal{L}(\mathcal{Z}(t), h) = \mathcal{Z}(t+h) - \frac{L}{M}, \tag{18}$$

where $L\ and\ M$ is numerator and denominator respectively of proposed scheme given in (16)

where z(t) is an arbitrary analytic function defined on interval [a, b]. Expanding the above expression by Taylor series about t and collecting terms in h, after substituting z(t) by the solution y(t) and t by $t_n$ , one can obtain the following LTE for the method (16) which confirms its 5th order accuracy as follows :

$$\frac{1}{3600}\left(\frac{1}{(24(y_n')^4-36y_n''(y_n')^2y+8y_n'''y_n'(y_n)^2+6(y_n'')^2(y_n)^2-y_n^{(4)}(y_n)^3}\right)[120(y_n')^4y_n^{(6)} - 720(y_n')^3y_n''y_n^{(5)} - 1200(y_n')^3y_n'''y_n^{(4)} + 2700(y_n')^2(y_n'')^2y_n^{(4)} + 3600(y_n')^2y_n''(y_n''')^2 - 180(y_n')^2y_n''y_n^{(6)}y_n + 240(y_n')^2y_n'''y_n^{(5)}y_n + 150(y_n')^2(y_n^{(4)})^2y_n - 7200y_n'(y_n'')^3y_n''' + 720y_n'(y_n'')^2y_n^{(5)}y_n - 600y_n'y_n''y_n'''y_n^{(4)}y_n - 800y'(y_n''')^3y_n + 40y_n'y_n'''y_n^{(6)}(y_n)^2 + 2700(y_n'')^5 - 1350(y_n'')^3y_n{}^4y_n + 1800(y_n'')^2(y_n''')^2y_n + 30(y_n'')^2y_n^{(6)}(y_n)^2 - 240y_n''y_n'''y_n^{(5)}(y_n)^2 + 150y_n''(y_n^{(4)})^2(y_n)^2 + 100(y_n''')^2y_n^{(4)}(y_n)^2 - 5y_n^{(4)}y^{(6)}(y_n)^3 + 6(y_n^{(5)})^2(y_n)^3] \tag{19}$$

## 4. Linear stability analysis

Dahlquist's test problem has been performed for the linear stability analysis

$$\mathcal{Y}' = \lambda\mathcal{Y}, \quad R(\lambda) < 0, \tag{20}$$

The following difference equation is readily obtained from equation (20):

$$\frac{24(h\lambda+5)}{h^4\lambda^4-8h^3\lambda^3+36h^2\lambda^2-96h\lambda+120}. \tag{21}$$

If $\mathcal{Z} = h\lambda\ with\ Re(\lambda) < 0$ then, the stability function is in the form of:

$$\emptyset(\mathcal{Z}) = \frac{1+{}^{1}/_{5}\mathcal{Z}}{1-{}^{4}/_{5}\mathcal{Z}+{}^{3}/_{10}\,\mathcal{Z}^{2}-{}^{1}/_{15}\mathcal{Z}^{3}\ {}^{1}/_{120}\mathcal{Z}^{4}}. \tag{22}$$

 

Stability region of this method has been shown by the shaded portion in fig.1

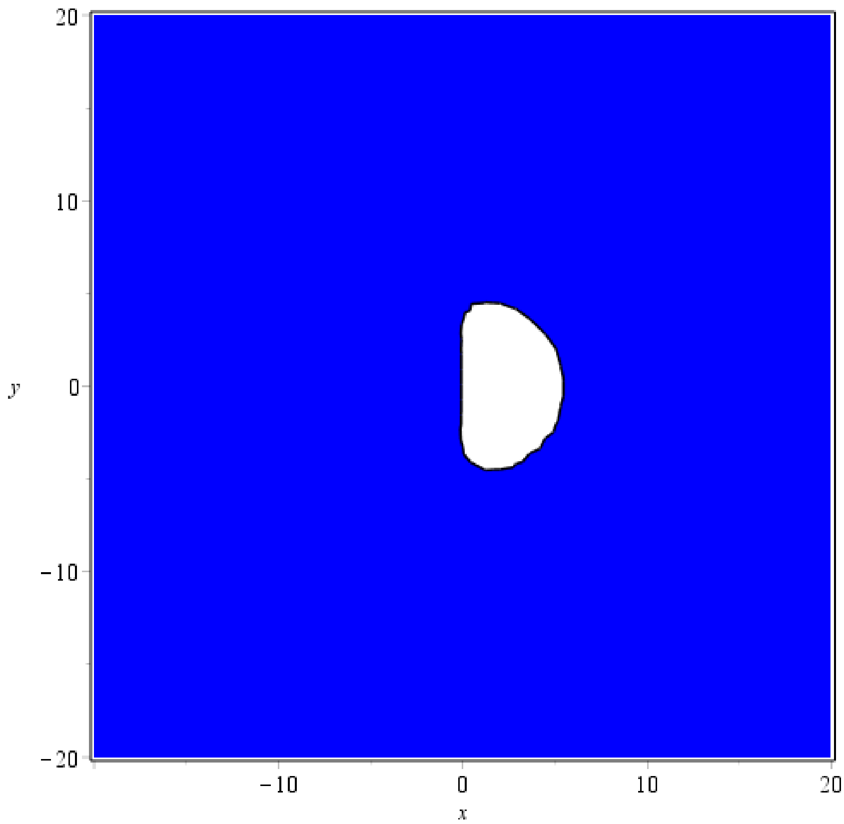


**Fig 1.** Absolute stability region of the fifth order method (16)

The method's absolute stability region (16), depicted in Fig. 1, contains the left half complex plane, satisfying the condition for the method to be A-stable [7]. Furthermore, the proposed method satisfies the following condition:

$$\lim_{z\to\infty} \phi(z) = 0$$

Since this method is accurate to the fifth order and L-stable.

## 5. Numerical Experiments

Some numerical experiments have been presented by implementing the proposed scheme(16). The numerical experiments have been performed on several IVPs having different behavior of solution . MATLAB version R2019a(9.6.0.1072779) on 64-bit operating-system has been used for the computational work.

The two well-known fifth order methods have been used for the comparison. One of the methods is Taylor's method given as:

Specify $x_0, y_0, z_0, h$

($(x_0, y_0)$ initial points,

$x_n$ point where the solution is required

h the step length to be used in the marching process)

Compute

$$f'(x_i,y_i),\ f''(x_i,y_i),\qquad f'''(x_i,y_i)\ .\ .\ .$$

Compute

$$y(x_i+h) = y(x_i) + h\,f(x_i,y_i) + {h^2}/{2}\,f'(x_i,y_i) + {h^3}/{6}\,f''(x_i,y_i) + {h^4}/{24}\,f'''(x_i,y_i) + {h^4}/{120}\,f'''(x_i,y_i) + .\ .\ . x_i = x_i + h$$

Until $xi$until $x_i = x_n$

And second on of the methods is the Fifth Order Runge-Kutta method given as:

$$y_{i+1} = y_i + \frac{1}{90}(7k_1 + 32k_3 + 12k_4 + 32k_5 + 7k_6)h$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{1}{4}h, y_i + \frac{1}{4}k_1h\right)$$

$$k_3 = f\left(x_i + \frac{1}{4}h, y_i + \frac{1}{8}k_1h + \frac{1}{8}k_2h\right)$$

$$k_4 = f\left(x_i + \frac{1}{2}h, y_i - \frac{1}{2}k_2h + k_3h\right)$$

$$k_5 = f\left(x_i + \frac{3}{4}h, y_i + \frac{3}{16}k_1h + \frac{9}{16}k_4h\right)$$

$$k_6 = f\left(x_i + h, y_i - \frac{3}{7}k_1h + \frac{2}{7}k_2h + \frac{12}{7}k_3h - \frac{12}{7}k_4h + \frac{8}{7}k_5h\right)$$

**Table 1. Nonlinear equation with Initial value problem**

| $\boldsymbol{y' = 1 + y^2}$, $\boldsymbol{y(0) = 1}$, $\boldsymbol{Exact = \tan(x + \frac{\pi}{4})}$ | | | | |
|---|---|---|---|---|
| **Methods** | Step size | | | |
| | xfinal= 0.5 | | xfinal= 1 | |
| | At h=0.001 | At h=0.01 | At h=0.05 | At h=0.01 |
| **Taylor Method** | 4.3565e-11 | 4.0338e-08 | Inf | Inf |
| | 4.3565e-11 | 4.0338e-08 | Inf | Inf |
| | 4.6014e-10 | 4.7609e-09 | Inf | Inf |
| | 0.0029 | 0.0094 | 0.0049 | 0.0102 |
| **RK Method** | 0.3228 | 0.3246 | 8.9598e+171 | 1.9754e+86 |
| | 0.3228 | 0.3246 | NAN | Nan |
| | 0.0779 | 0.0799 | NAN | Nan |
| | 0.0123 | 0.0080 | 0.0824 | 0.0019 |
| **Proposed Method** | 2.1432e-12 | 2.6731e-11 | 1.2892e-04 | 5.2e-3 |
| | 1.3332e-12 | 2.6731e-11 | 5.1565e-06 | 6.5957e-06 |
| | 1.6698e-13 | 5.6497e-12 | 1.0970e-05 | 7.8909e-05 |
| | 0.0255 | 0.0222 | 0.0227 | 0.0227 |

**Table 2. Nonlinear equation with Initial value problem.**

| $y' = \frac{1}{y^2}$, $y(0) = 1$, $Exact = (3x+1)^{1/3}$ | | | | |
|---|---|---|---|---|
| **Methods** | Step size | | | |
| | xfinal= 0.5 | | xfinal= 1 | |
| | At h=0.001 | At h=0.01 | At h=0.05, | At h=0.01 |
| **Taylor Method** | 1.0214e-14 | 9.3808e-10 | 3.6860e-07 | 9.3808e-9 |
| | 8.4377e-14 | 7.9860e-11 | 2.3144e-07 | 5.9635e-11 |
| | 3.2739e-14 | 8.0731e-11 | 2.8368e-07 | 7.4669e-11 |
| | 0.0087 | 0.0088 | 0.008 | 0.0089 |
| **RK Method** | 1.7282 | 1.7264 | 8.9598e+171 | 1.9754e+86 |
| | 1.7282 | 1.7264 | NAN | NAN |
| | 0.5681 | 0.5730 | NAN | NAN |
| | 0.0126 | 0.0046 | 0.0506 | 2.0500e-04 |
| **Proposed Method** | 5.5511e-15 | 1.0830e-11 | 3.8696e-08 | 1.0830e-11 |
| | 4.4409e-15 | 9.2293e-12 | 2.4459e-08 | 6.8914e-12 |
| | 1.5015e-15 | 9.3200e-12 | 2.9873e-08 | 8.6250e-12 |
| | 0.0072 | 0.0248 | 3.6000e-04 | 0.0026 |

**Table 3. Nonlinear equation with Initial value problem.**

| $y' = \frac{1}{y}$, $y(0) = 1$, $Exact = (\sqrt{2x+1})$ | | | | |
|---|---|---|---|---|
| **Methods** | Step size | | | |
| | xfinal= 0.5 | | xfinal= 1 | |
| | At h=0.001 | At h=0.01 | At h=0.05 | At h=0.01 |
| **Taylor Method** | 8.8818e-13 | 1.1501e-11 | 4.1515e-08 | 1.1501e-11 |
| | 8.6597e-14 | 1.1281e-11 | 3.4791e-08 | 9.6954e-12 |
| | 3.3688e-13 | 9.6005e-12 | 3.5376e-08 | 1.0031e-11 |
| | 0.0022 | 0.0145 | 0.001 | 0.0658 |
| **RK Method** | 1.8509e+16 | 43.5263 | 3.9412 | 1.8998e+03 |
| | 1.8509e+16 | 43.5263 | 3.9412 | 1.8998e+03 |
| | 5.1193e+14 | 10.7084 | 1.4619 | 259.3529 |
| | 0.0015 | 0.008 | 0.0104 | 0.0077 |
| **Proposed Method** | 4.4409e-15 | 2.2722e-12 | 7.8310e-09 | 2.2722e-12 |
| | 4.0432e-15 | 2.2291e-12 | 6.5793e-09 | 1.9194e-12 |
| | 2.0237e-15 | 1.8964e-12 | 6.6756e-09 | 1.9826e-12 |
| | 3.600e-3 | 7.3900e-04 | 4.3100e-04 | 0.0188 |

5.1 Results and discussion

Anew developed fifth-order improved scheme is able to solve nonlinear IVPs in computational and applied mathematics. The maximum-error, last errors and average errors have been tabularized. The proposed improved scheme has smaller errors than the other methods having same order of accuracy. In this article, it is proved that the proposed scheme has more accuracy and effectiveness in the comparison to some existing standard methods. The Tables 1-3 showed the errors and CPU times for all the numerical schemes considering the IVPs. Different step-sizes $h = 0.01, 0.05, 0.001$ at the $x_{final} = 0.5, 1$ have been used for obtaining numerical results. The first nonlinear IVP possesses a blow-up singular solution given as $y(x) = \tan\left(x + \frac{\pi}{4}\right)$with singularity at $x = \frac{\pi}{4}$. The second and third nonlinear IVPs have a singular solution, which is given as $y(x) = (3x+1)^{1/3}$ with singularity at $x = -\frac{1}{3}$ and $y(x) = \frac{1}{y}$ at $x = -\frac{1}{2}$ respectively.

## 6. Conclusion

It is concluded that the fifth order explicit non-linear scheme has good capability in dealing with IVPs in ODEs. Many standards and non-standard methods found in literature cannot handle these types of equations efficiently, but due to L-stable of this scheme, it is an appropriate option for solving singular and stiff ODEs at a low cost as it has been evidenced by the CPU values.

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