Rough Probability In Topological Spaces

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Abstract

In this paper we study the rough probability in the topological spaces which we can consider them as results from the general relations on the approximation spaces.

Keywords: Stochastic approximation space, rough expectation, rough variance, rough probability generating function, rough characteristic function.

1. Introduction

In [2] pawlak introduced approximation spaces during the early 1980s as part of his research on classifying objects by means their feature. In [1] rough set theory introduced by Pawlak in 1982, as an extension of set theory, mainly in the domain of intelligent systems. In [4,5] m. Jamal and N. Duc rough set theory as a mathematical tool to deal with vagueness and incomplete information data or imprecise by dividing these data into equivalence classes using equivalence relations which result from the same data. This paper study the rough probability in the topological spaces which we can consider them as results from the general relations on the approximation spaces.

2. Rough Probability in Topological Spaces

In [4] m. Jamal study stochastic approximation spaces from topological view that generalize the stochastic approximation space in the case of general relation. We generalize the stochastic approximation space in the case of general relation. Since the approximation space $K = (U, R)$ with general relation R defines a uniquely topological space (U, τ_k), where U/R is a subbase of τ_k, then the order triple $S = (U, R, P)$ is called the stochastic approximation space, where R is a general relation and P is a probability measure. We give this hypothesis in the following definition.

Definition 2.1. [4]. Let $K = (U, R)$ be an approximation space with general relation R and τ_k is the topology associated to K. Then the order 4-tuples $S = (U, R, P, \tau_k)$ is called a topologized stochastic approximation space.

2.1 Rough Probability

There is no problem to find the probability of an observable set as it will be the same as the usual probability. The problem occurs when evaluating the probability of the unobservable sets. In order to investigate this problem we obtain some rules to find lower and upper probabilities in topologized stochastic approximation spaces with general relations [4].

Definition 2.2. [4]. Let A be an event in the topologized stochastic approximation space $S = (U, R, P, \tau_k)$. Then the lower (resp. upper) probability of A is given by:

$$\underline{P}(A) = P(A^{\circ}) \text{ (resp. } \overline{P}(A) = P(A^{-})\text{)}.$$

where $A^{\circ} = \cup\{G \subseteq X : G \subseteq A \text{ and } G \in \tau_k\}$ and $A^{-} = \cap\{F \subseteq X : A \subseteq F \text{ and } F \in \tau_k{}^*\}$. Clearly, $0 \le \underline{P}(A) \le 1$ and $0 \le \overline{P}(A) \le 1$.

Proposition 2.3. [4]. Let A, B be two events in the topologized stochastic approximation space $S = (U, R, P, \tau_k)$. Then

1) $\underline{P}(A) \le P(A) \le \overline{P}(A)$.
2) $\underline{P}(\emptyset) = \overline{P}(\emptyset) = 0$.
3) $\underline{P}(U) = \overline{P}(U) = 1$.
4) $\underline{P}(A^c) = 1 - \overline{P}(A)$.
5) $\overline{P}(A^c) = 1 - \underline{P}(A)$.
6) $\overline{P}(A \cup B) \le \overline{P}(A) + \overline{P}(B) - \overline{P}(A \cap B)$.

7) $\underline{P}(A \cup B) \geq \underline{P}(A) + \underline{P}(B) - \underline{P}(A \cap B)$.

Definition 2.4. [4]. Let A be an event in the topologized stochastic approximation space $S = (U, R, P, \tau_k)$. The rough probability of A, denoted by $P^*(A)$, is given by:

$$P^*(A) = \left(\underline{P}(A), \overline{P}(A)\right).$$

Example 2.5. Consider the experiment of choosing one card from four cards numbered from one to four. The collection of the four elements forms the outcome space. Hence

$$U = \{1,2,3,4\}.$$

Let R be a binary relation defined on U such that

$$R = \{(1,1),(2,2),(3,2),(3,4)\}.$$

Thus $U/R = \{\{1\},\{2\},\{2,4\}\}$. Let $K = (U, R)$ be an approximation space and τ_k is the topology associated to . Thus

$$\tau_k = \{U, \emptyset, \{1\}, \{2\}, \{1,2\}, \{2,4\}, \{1,2,4\}\}$$

$$\tau_k^* = \{U, \emptyset, \{3\}, \{1,3\}, \{3,4\}, \{1,3,4\}, \{2,3,4\}\}$$

Define the random variable X to be the number on the chosen card. We can construct Table 2.1 which contains the lower and the upper
probabilities of a random variable $X = x$ as following :

Table 2.1: Lower and upper probabilities of a random variable X

X	1	2	3	4
$\underline{P}(X = x)$	$\frac{1}{4}$	$\frac{1}{4}$	0	0
$\overline{P}(X = x)$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{2}{4}$

2.2 Rough Distribution Function

The distribution function of a random variable X gives the probability that X does not exceed x. We define the lower and upper distribution functions of a random variable X.

Definition 2.6 [4]. Let X be a random variable in the topologized stochastic approximation space $S = (U, R, P, \tau_k)$. The lower distribution (resp. upper distribution) function of X is given by:

$$\underline{F}(x) = \underline{P}(X \leq x) \left(\text{resp.} \overline{F}(x) = \overline{P}(X \leq x)\right).$$

Definition 2.7. [4]. Let X be a random variable in the topologized stochastic approximation space $S = (U, R, P, \tau_k)$. The rough distribution function of X, denoted by $F^*(x)$, is given by:

$$F^*(x) = \left(\underline{F}(x), \overline{F}(x)\right).$$

Example 2.8. Consider the same experiment as in Example 2.5. The lower and upper distribution functions of X are

$$\underline{F}(x) = \begin{cases} 0 & -\infty < x < 1 \\ \frac{1}{4} & 1 \leq x < 2 \\ \frac{2}{4} & 2 \leq x < \infty \end{cases}$$

And

$$\overline{F}(x) = \begin{cases} 0 & -\infty < x < 1 \\ \frac{2}{4} & 1 \le x < 2 \\ \frac{5}{4} & 2 \le x < 3 \\ \frac{6}{4} & 3 \le x < 4 \\ \frac{9}{4} & 4 \le x < \infty \end{cases}$$

Therefore $F^*(2) = \left(\frac{2}{4}, \frac{5}{4}\right)$.

2.3 Rough Expectation

We define the lower and upper expectations of a random variable X in the topologized stochastic approximation space $S = (U, R, P, \tau_k)$.

Definition 2.9 [4]. Let X be a random variable in the topologized stochastic approximation space $S = (U, R, P, \tau_k)$. The lower (resp. upper) expectation of X is given by:

$$\underline{E}(X) = \sum_{k=1}^{n} x_k \, \underline{P}(X = x_k)$$

resp.

$$\overline{E}(X) = \sum_{k=1}^{n} x_k \, \overline{P}(X = x_k).$$

Definition 2.10 [4]. Let X be a random variable in the topologized stochastic approximation space $S = (U, R, P, \tau_k)$. The rough expectation of X is denoted by $E^*(X)$ and is given by:

$$E^*(X) = \left(\underline{E}(X), \overline{E}(X)\right).$$

Example 2.11. Consider the same experiment as in Example 2.5. Then the lower and upper expectations of X are

$$\underline{E}(X) = 0.75, \quad \overline{E}(X) = 4.75$$

Hence rough expectation of X is

$$E^*(X) = (0.75, 4.75).$$

Theorem 2.12. Let X be a random variable in the topologized stochastic approximation space $S = (U, R, P, \tau_k)$. For any constants a and b, we have

$$\underline{E}(aX + b) = a\underline{E}(X) + bc \text{ where } 0 \le c \le 1$$

Proof.

$$\underline{E}(aX + b) = \sum_{k=1}^{n} (ax_k + b)\underline{P}(x_k) = \sum_{k=1}^{n} \left(ax_k\underline{P}(x_k) + b\underline{P}(x_k)\right)$$

$$= a\sum_{k=1}^{n} x_k\underline{P}(x_k) + b\sum_{k=1}^{n} \underline{P}(x_k)$$

$$= a\underline{E}(X) + bc \text{ where } c = \textstyle\sum_{k=1}^{n} \underline{P}(x_k) \text{ (i.e. } 0 \le c \le 1).$$

Theorem 2.13. Let X be a random variable in the topologized stochastic approximation space $S = (U, R, P, \tau_k)$. For any constants a and b, we have

$$\overline{E}(aX + b) = a\overline{E}(X) + bd \text{ where } 1 \le d \le n, n \in N^+.$$

Proof.

The proof is similar to Theorem 2.12.

2.4 Rough Variance

We define the lower and upper variances of a random variable X in the topologized stochastic approximation space $S = (U, R, P, \tau_k)$.

Definition 2.14. [4]. Let X be a random variable in the topologized stochastic approximation space $S = (U, R, P, \tau_k)$. The lower (resp. upper) variance of X is given by:

$$\underline{V}(X) = \underline{E}\big(X - \underline{E}(X)\big)^2 \left(\text{resp. } \overline{V}(X) = \overline{E}\left(X - \overline{E}(X)\right)^2\right).$$

Definition 2.15. [4]. Let X be a random variable in the topologized stochastic approximation space $S = (U, R, P, \tau_k)$. The rough variance of X is denoted by $V^*(Z)$ and is given by:

$$V^*(X) = \left(\underline{V}(X), \overline{V}(X)\right).$$

Example 2.16. Consider the same experiment as in Example 2.5. Then the lower and upper variances of X are

$$\underline{V}(X) = 0.4\,,\ \overline{V}(X) = 13.75$$

The rough variance of X is $V^*(X) = (0.4\,, 13.75)$.

Theorem 2.17. Let X be a random variable in the topologized stochastic approximation space $S = (U, R, P, \tau_k)$. Then

$$\underline{V}(X) = \underline{E}(X)^2 - (2 - c)\left(\underline{E}(X)\right)^2 \text{ where } c = \textstyle\sum_{k=1}^{n} \underline{P}(x_k)$$

Proof. We have

$$\underline{E}\left(X - \underline{E}(X)\right)^2 = \underline{E}\left(X^2 - 2X\underline{E}(X) + \left(\underline{E}(X)\right)^2\right)$$

$$= \underline{E}(X)^2 - 2\underline{E}(X)\,\underline{E}(X) + c\left(\underline{E}(X)\right)^2 \text{ where } c = \sum_{k=1}^{n} \underline{P}(x_k)$$

$$= \underline{E}(X)^2 - 2\left(\underline{E}(X)\right)^2 + c\left(\underline{E}(X)\right)^2$$

$$= \underline{E}(X)^2 - (2 - c)\underline{E}(X)^2 .$$

Theorem 2.18. Let X be a random variable in the topologized stochastic approximation space $S = (U, R, P, \tau_k)$. Then

$$\overline{V}(X) = \overline{E}(X)^2 - (2 - d)\left(\overline{E}(X)\right)^2 \text{ where } d = \textstyle\sum_{k=1}^{n} \overline{P}(x_k).$$

Proof.

The proof is similar to Theorem 2.17.

Theorem 2.19. Let X be a random variable in the topologized stochastic approximation space $S = (U, R, P, \tau_k)$. Then for any constants a and b, we have

$$\underline{V}(aX + b) = a^2\underline{E}(X^2) - (2a - c)\left(\underline{E}(X)\right)^2 + 2b(a - c)\underline{E}(X) + b^2 c$$

Where $c = \sum_{k=1}^{n} \underline{P}(x_k)$.

Proof. We have

$$\underline{V}(aX + b) = \underline{E}\left((aX + b) - \underline{E}(X)\right)^2$$

$$= \underline{E}(aX + b)^2 - 2(aX + b)\underline{E}(X) + \left(\underline{E}(X)\right)^2$$

$$= \underline{E}\left(a^2X^2 + 2abX + b^2 - 2aX\underline{E}(X) - 2b\underline{E}(X) + \left(\underline{E}(X)\right)^2\right)$$

$$= a^2\underline{E}(X^2) + 2ab\underline{E}(X) + b^2c - 2a\left(\underline{E}(X)\right)^2 - 2bc\underline{E}(X) + c\left(\underline{E}(X)\right)^2$$

Where $c = \sum_{k=1}^{n} \underline{P}(x_k)$

$$= a^2\underline{E}(X^2) - (2a - c)\left(\underline{E}(X)\right)^2 + 2b(a - c)\underline{E}(X) + b^2c .$$

Theorem 2.20. Let X be a random variable in the topologized stochastic approximation space $S = (U, R, P, \tau_k)$. Then for any constants a and b, we have

$$\overline{V}(aX+b) = a^2\overline{E}(X^2) - (2a-d)\left(\overline{E}(X)\right)^2 + 2b(a-d)\overline{E}(X) + b^2 d$$

Where $d = \sum_{k=1}^{n} \overline{P}(x_k)$.

Proof.

The proof is similar to Theorem 2.19.

2.5 Rough Moment Generating Function and Rough Characteristic Function

Definition 2.21. [4]. Let X be a random variable in the topologized stochastic approximation space $S = (U, R, P, \tau_k)$. Then the lower (resp. upper) moment generating function of X is defined by:

$$\underline{M_X}(t) = \underline{E}(e^{tX})$$

$$(\text{resp. } \overline{M_X}(t) = \overline{E}(e^{tX})).$$

Definition 2.22. [4]. Let X be a random variable in the topologized stochastic approximation space $S = (U, R, P, \tau_k)$. Then the rough moment generating function of X is defined by:

$$M_X{}^*(t) = \left(\underline{M_X}(t), \overline{M_X}(t)\right).$$

Example 2.23. Consider the same experiment as in Example 2.5. Then the lower and upper moment generating functions of X are

$$\underline{M_X}(X) = \frac{1}{4}(e^t + e^{2t}),\ \overline{M_X}(X) = \frac{1}{4}(2e^t + 3e^{2t} + e^{3t} + 2e^{4t}).$$

Theorem 2.24. Let X be a random variable in the topologized stochastic approximation space $S = (U, R, P, \tau_k)$. For any constants a and b, we have

$$\underline{M_Y}(t) = e^{\left(\frac{a}{b}\right)t} . \underline{M_X}\left(\frac{t}{b}\right)$$

Where $Y = \frac{X+a}{b}$.

Proof. We have

$$\underline{M_Y}(t) = \underline{E}(e^{tY}) = \underline{E}\left(e^{t\left(\frac{X+a}{b}\right)}\right)$$

$$= \underline{E}\left(e^{\left(\frac{a}{b}\right)t} . e^{\left(\frac{t}{b}\right)X}\right)$$

$$= e^{\left(\frac{a}{b}\right)t} . \underline{E}\left(e^{\left(\frac{t}{b}\right)X}\right)$$

$$= e^{\left(\frac{a}{b}\right)t} . \underline{M_X}\left(\frac{t}{b}\right).$$

Theorem 2.25. Let X be a random variable in the topologized stochastic approximation space $S = (U, R, P, \tau_k)$. For any constants a and b, we have

$$\overline{M_Y}(t) = e^{\left(\frac{a}{b}\right)t} . \overline{M_X}\left(\frac{t}{b}\right)$$

Where $Y = \frac{X+a}{b}$.

Proof.

The proof is similar to Theorem 2.24.

Definition 2.26. Let X be a random variable in the topologized stochastic approximation space $S = (U, R, P, \tau_k)$. The lower (resp. the upper) characteristic function of X is denoted by $\underline{\emptyset_X}(t)$ (resp. $\overline{\emptyset_X}(t)$) and is given by:

$$\underline{\emptyset_X}(t) = \underline{E}(e^{itX}) = \sum_X e^{itX}\underline{P}(X)$$
$$\left(\text{resp.}\, \overline{\emptyset_X}(t) = \overline{E}(e^{itX}) = \textstyle\sum_X e^{itX}\overline{P}(X)\right).$$

Definition 2.27. Let X be a random variable in the topologized stochastic approximation space $S = (U, R, P, \tau_k)$. Then The rough characteristic function of X is denoted by $\emptyset_X{}^*(t)$ and is given by:

$$\emptyset_X{}^*(t) = \left(\underline{\emptyset_X}(t), \overline{\emptyset_X}(t)\right).$$

Example 2.28. Consider the same experiment as in Example 2.5. Then the lower and upper characteristic functions of X are

$$\underline{\emptyset_X}(X) = \tfrac{1}{4}(e^{it} + e^{2it}),\ \overline{\emptyset_X}(X) = \tfrac{1}{4}(2e^{it} + 3e^{2it} + e^{3it} + 2e^{4it}).$$

Theorem 2.29. Let X be a random variable in the topologized stochastic approximation space $S = (U, R, P, \tau_k)$. For any constants a and b, we have

$$\underline{\emptyset_Y}(t) = e^{itb}\underline{\emptyset_X}(at)$$

Where $Y = aX + b$.

Proof. We have

$$\underline{\emptyset_Y}(t) = \underline{\emptyset_{aX+b}}(t) = \underline{E}\left(e^{it(aX+b)}\right)$$
$$= \underline{E}\left(e^{itb}\,.\,e^{i(ta)X}\right) = e^{itb}\underline{E}\left(e^{i(ta)X}\right)$$
$$= e^{itb}\underline{\emptyset_X}(at).$$

Theorem 2.30. Let X be a random variable in the topologized stochastic approximation space $S = (U, R, P, \tau_k)$. For any constants a and b, we have

$$\overline{\emptyset_Y}(t) = e^{itb}\overline{\emptyset_X}(at)$$

Where $Y = aX + b$.

Proof.

The proof is similar to Theorem 2.29.

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