Decomposition of Continuity and Separation Axioms Via Lower and Upper Approximation

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Abstract
In this paper we study the rough set theory by defined the concepts of rough regularity and rough normality in the topological spaces which we can consider them as results from the general relations on the approximation spaces.
Keywords: Topologized approximation space, rough regularity, rough normality, rough continuity, rough homeomorphism.

1. Introduction

In [3] pawlak introduced approximation spaces during the early 1980s as part of his research on classifying objects by means their feature. In [1] rough set theory introduced by Pawlak in 1982, as an extension of set theory, mainly in the domain of intelligent systems. In [4,5] m. Jamal and N. Duc rough set theory as a mathematical tool to deal with vagueness and incomplete information data or imprecise by dividing these data into equivalence classes using equivalence relations which result from the same data. This paper study the rough set theory by defined the concepts of rough regularity and rough normality in the topological spaces which are results from the general relations on the approximation spaces.

2. Preliminaries

In [4] pawlak noted that the approximation space $\kappa = (X, R)$ with equivalence relation R defined a uniquely topological space (X, τ_x) where τ_x is the family of all clopen sets in (X, τ_x) and X/R is a base of τ_x. Moreover the lower (resp. upper) approximation of any subset $A \subseteq X$ is exactly the interior (resp. closure) of the subset A . In this section we shall generalize Pawlak's concepts to the case of general relations. Hence the approximation space $\kappa = (X, R)$ with general relation R defines a uniquely topological space (X, τ_x) where τ_x is the topology associated to κ (i.e. τ_x is the family of all open sets in (X, τ_x) and $X/R = \{xR : x \in X\}$ is a subbase of τ_k , where $xR = \{y \in X : xRy\}$). We give this hypothesis in the following definition.

Definition 2.1 [4]. Let $\kappa = (X, R)$ be an approximation space with general relation R and τ_x is the topology associated to κ. Then the triple $K = (X, R, \tau_k)$ is called *a topologized approximation space*.

The following definition introduces the lower and the upper approximations in a topologized approximation space $K = (X, R, \tau_k)$.

Definition 2.2 [4]. Let $K = (X, R, \tau_k)$ be a topologized approximation space and A ⊆ X. The *lower approximation* (resp. *upper approximation*) of A is defined by

$$\underline{R}A = A^{\circ} \text{ where } A^{\circ} = \cup\{G \subseteq X : G \subseteq A \text{ and } G \in \tau\}$$

$$(\text{resp. } \overline{R}A = A^{-} \text{ where } A^{-} = \cap\{F \subseteq X : A \subseteq F \text{ and } F \in \tau^{*}\}).$$

In the following proposition from [4] we introduce some properties of the lower and upper approximations of a set A.

Proposition 2.3 [4]. Let $K = (X, R, \tau_k)$ be a topologized approximation space. If A and B are two subsets of X, then

1) $\underline{R}A \subseteq A \subseteq \overline{R}A$.
2) $\underline{R}\emptyset = \overline{R}\emptyset = \emptyset$ and $\underline{R}X = \overline{R}X = X$.
3) $\overline{R}(A \cup B) = \overline{R}A \cup \overline{R}B$.
4) $\underline{R}(A \cap B) = \underline{R}A \cap \underline{R}B$.
5) If $A \subseteq B$, then $\overline{R}A \subseteq \overline{R}B$.
6) If $A \subseteq B$, then $\underline{R}A \subseteq \underline{R}B$.
7) $\underline{R}(A \cup B) \supseteq \underline{R}A \cup \underline{R}B$.

8) $\overline{R}(A \cap B) \subseteq \overline{R}A \cap \overline{R}B$.
9) $\underline{R}(A)^c = [\overline{R}A]^c$.
10) $\overline{R}(A)^c = [\underline{R}A]^c$.
11) $\underline{R}\,\underline{R}A = \underline{R}A$.
12) $\overline{R}\,\overline{R}A = \overline{R}A$.

Definition 2.4. Let $K = (X, R, \tau_k)$ be a topologized approximation space and $x \in X$. A subset N of X is said to be *rough closed neighborhood* of x iff, there exists a subset G of X such that $x \in G \subseteq \overline{R}G \subseteq N$.

Theorem 2.5. Let $K = (X, R, \tau_k)$ be a topologized approximation space and $G \subseteq X$, then $G = \overline{R}G$ iff, G a rough closed neighborhood of each of its points.
Proof. Let $G = \overline{R}G$ and $x \in G$, then $x \in \overline{R}G \subseteq G$ and $G \subseteq G$, therefor G a rough closed neighborhood of each of its points.
Conversely. Let G be a rough closed neighborhood of each of its points. Let $x \in G$. Then there exists $N_x \subseteq G$ such that $x \in \overline{R}N_x \subseteq G$, therefore $G = \cup_{x \in G} \overline{R}N_x = \overline{R}[\cup_{x \in G} N_x] = \overline{R}G$, because $\forall\, x \in G\ \exists\ N_x \subseteq G$, so $\cup_{x \in G} N_x = G$. Hence $G = \overline{R}G$.

Definition 2.6. Let $K = (X, R, \tau_k)$ be a topologized approximation space and $Y \subseteq X$. Then $Q = (Y, R, \sigma_Q)$ where $\sigma_Q = \{G \cap Y | G \in \tau_k\}$ is a topologized approximation space of Y, called the relative topologized approximation space for Y. The fact that a subset of X is being given this topologized approximation space is signified by referring to it as a *subspace of* X.

Theorem 2.7. Let $Q = (Y, R, \tau_Q)$ be a subspace of a topologized approximation space $K = (X, R, \tau_k)$, then:

i) If $H \subseteq Y$ then $\underline{R}_Y H = H$ iff $H = G \cap Y$ where $\underline{R}_X G = G$.
ii) If $F \subseteq Y$ then $\overline{R}_Y F = F$ iff $F = B \cap Y$ where $\overline{R}_X B = B$.
iii) if $A \subseteq Y$, then $\overline{R}_Y A = Y \cap \overline{R}_X A$.

Proof. By definitions of subspace of a topologized approximation space and upper approximation, the proof is obvious.

Theorem 2.8. Let $Q = (Y, R, \tau_Q)$ be a subspace of a topologized approximation space $K = (X, R, \tau_k)$. Then for $A, B \subseteq Y$ we have:

i) If $\overline{R}_Y A = A$ and $\overline{R}_X Y = Y$ then $\overline{R}_X A = A$.
ii) If $\underline{R}_Y B = B$ and $\underline{R}_Y Y = Y$ then $\underline{R}_X B = B$.

Proof. By Theorem 2.7 the proof is obvious.

Definition 2.9. Let $K = (X, R_1, \tau_k)$ and $Q = (Y, R_2, \sigma_Q)$be two topologized approximation spaces. Then a mapping $f: K \longrightarrow Q$ is said to be *rough continuous* at a point x of X iff, for each subset V contains $f(x)$ in Y, there exists a subset U contains x in X such that $f(\overline{R}_1 U) \subseteq \overline{R}_2 V$. The mapping f is said to a *rough continuous* iff it is rough continuous at every point of X.
Theorem 2.10. Let $K = (X, R_1, \tau_k)$and $Q = (Y, R_2, \sigma_Q)$ be two topologized approximation spaces and $f: K \longrightarrow Q$ be a mapping, then the following statements are equivalent:

i) f is rough continuous.
ii) For each subset F of Y, $\overline{R}_1 f^{-1}(\overline{R}_2 F) = f^{-1}(\overline{R}_2 F)$.
iii) For each subset E of X, $f(\overline{R}_1 E) \subseteq \overline{R}_2 f(E)$.
iv) For each subset B of Y, $\overline{R}_1 f^{-1}(B) \subseteq f^{-1}(\overline{R}_2 B)$.

Proof. (i)$\Rightarrow$(ii). Let F be a set in Y. We are going to prove that $\overline{R}_1 f^{-1}(\overline{R}_2 F) = f^{-1}(\overline{R}_2 F)$. For this purpose, let p be a point in $f^{-1}(\overline{R}_2 F)$. Then $f(p)$ is a point in $\overline{R}_2 F$. Since f is rough continuous at the point p, there exits a subset U of X such that $p \in U$ and $f(\overline{R}_1 U) \subseteq \overline{R}_2[\overline{R}_2 F]$, then by (12) of Proposition 2.3 we have $f(\overline{R}_1 U) \subseteq \overline{R}_2 F$. This implies that $\overline{R}_1 U \subseteq f^{-1}(\overline{R}_2 F)$. By Theorem 2.5, it follows that for each subset F of Y, $\overline{R}_1 f^{-1}(\overline{R}_2 F) = f^{-1}(\overline{R}_2 F)$.

(ii) $\Longrightarrow$ (iii). (1) of Proposition 2.3 implies $f(E) \subseteq \overline{R}_2 f(E)$. Then $E \subseteq f^{-1}\left(\overline{R}_2 f(E)\right)$, thus again by (5) of Proposition 2.3 we have $\overline{R}_1 E \subseteq \overline{R}_1 f^{-1}\left(\overline{R}_2 f(E)\right)$. Then from (ii), we have $\overline{R}_1 f^{-1}\left(\overline{R}_2 f(E)\right) = f^{-1}\left(\overline{R}_2 f(E)\right)$, therefore $\overline{R}_1 E \subseteq f^{-1}\left(\overline{R}_2 f(E)\right)$. Hence $f\left(\overline{R}_1 E\right) \subseteq \overline{R}_2 f(E)$.

(iii) $\Longrightarrow$ (iv). Let B be a subset of Y. Then by (iii), we have $f\left(\overline{R}_1 f^{-1}(B)\right) \subseteq \overline{R}_2 f\left(f^{-1}(B)\right) \subseteq \overline{R}_2 B$, therefore $f\left(\overline{R}_1 f^{-1}(B)\right) \subseteq \overline{R}_2 B$. Hence $\overline{R}_1 f^{-1}(B) \subseteq f^{-1}\left(\overline{R}_2 B\right)$.

(iv) $\Longrightarrow$ (i). Let p be a point of X and let V be a subset of Y such that $f(p) \in V$. Our hypothesis (iv) and (12) of Proposition 2.3 lead to $\overline{R}_1 f^{-1}\left(\overline{R}_2 V\right) \subseteq f^{-1}\left(\overline{R}_2 \overline{R}_2 V\right) = f^{-1}\left(\overline{R}_2 V\right)$. So

$$\overline{R}_1 f^{-1}\left(\overline{R}_2 V\right) \subseteq f^{-1}\left(\overline{R}_2 V\right). \quad \ldots (1)$$

On the other hand (1) of Proposition 2.3 implies

$$f^{-1}\left(\overline{R}_2 V\right) \subseteq \overline{R}_1 f^{-1}\left(\overline{R}_2 V\right). \quad \ldots (2)$$

From (1) and (2) we obtain

$\overline{R}_1 f^{-1}\left(\overline{R}_2 V\right) = f^{-1}\left(\overline{R}_2 V\right)$. Then $p \in \overline{R}_1 f^{-1}\left(\overline{R}_2 V\right)$ and $f\left(\overline{R}_1 f^{-1}\left(\overline{R}_2 V\right)\right) \subseteq \overline{R}_2 V$, therefore f is a rough continuous at p. Hence f is a rough continuous.

Definition 2.11. Let $K = (X, R_1, \tau_k)$ and $Q = (Y, R_2, \sigma_Q)$ be two topologized approximation spaces. Then a mapping $f: K \longrightarrow Q$ is said to be a *rough closed mapping* iff, $f\left(\overline{R}_1 F\right) = \overline{R}_2 f\left(\overline{R}_1 F\right)$ for each subset F of X.

Definition 2.12. Let $K = (X, R_1, \tau_k)$ and $Q = (Y, R_2, \sigma_Q)$ be two topologized approximation spaces. Then a mapping $f: K \longrightarrow Q$ is said to be a *rough open mapping* iff, $f\left(\underline{R}_1 G\right) = \underline{R}_2 f\left(\underline{R}_1 G\right)$ for each subset G of X.

Definition 2.13. Let $K = (X, R_1, \tau_k)$ and $Q = (Y, R_2, \sigma_Q)$ be two topologized approximation spaces. Then a mapping $f : K \longrightarrow Q$ is said to be a *rough homeomorphism* iff:

i) f is bijective.
ii) f is rough continuous.
iii) f^{-1} is rough continuous.

In this case, we say X and Y are *rough homeomorphic*.

Theorem 2.14. Let $K = (X, R_1, \tau_k)$ and $Q = (Y, R_2, \sigma_Q)$ be two topologized approximation spaces and $f: K \longrightarrow Q$ be an onto mapping, then f is rough closed iff f is rough open.

Proof. Assume that f is rough closed and F is a subset of X, with $f\left(X - \underline{R}_1 F\right) = \overline{R}_2 f\left(X - \underline{R}_1 F\right)$, by (9) and (10) of Proposition 2.3, we have

$$f\left(\underline{R}_1 F\right) = Y - \overline{R}_2 f\left(X - \underline{R}_1 F\right) = Y - \overline{R}_2\left(f(X) - f\left(\underline{R}_1 F\right)\right) = Y - \overline{R}_2\left(Y - f\left(\underline{R}_1 F\right)\right) = \left(\overline{R}_2\left(f\left(\underline{R}_1 F\right)\right)^c\right)^c = \underline{R}_2\left(\left(f\left(\underline{R}_1 F\right)\right)^c\right)^c = \underline{R}_2 f\left(\underline{R}_1 F\right)$$

Conversely. Similarly to the first part.

Theorem 2.15. Let $K = (X, R_1, \tau_k)$ and $Q = (Y, R_2, \sigma_Q)$ be two topologized approximation spaces and $f: K \longrightarrow Q$ be a bijective mapping, then f is rough homeomorphism iff f is rough continuous and rough closed.

Proof. Assume that f is a bijective and f is a rough homeomorphism, then by Definition 2.12, we have f is rough continuous. To prove f is a rough closed, let g be the inverse mapping of f, therefore $g = f^{-1}$ and $f = g^{-1}$, since f is bijective, then g is bijective. Let F be a subset of X, then by Definition 2.12, g is a rough continuous, therefore $g^{-1}\left(\overline{R}_1 F\right) = \overline{R}_2 g^{-1}\left(\overline{R}_1 F\right)$, since $f = g^{-1}$, then $f\left(\overline{R}_1 F\right) = \overline{R}_2 f\left(\overline{R}_1 F\right)$, Hence f is a rough closed.

Conversely. Assume that f is a bijective, rough continuous and rough closed. To prove f is a rough homeomorphism, we must show that f^{-1} is a rough continuous. Let g be the inverse mapping of f, therefore $g = f^{-1}$ and $f = g^{-1}$. Let F be a subset of X, since f is a rough closed then $f\left(\overline{R}_1 F\right) = \overline{R}_2 f\left(\overline{R}_1 F\right)$, thus $g^{-1}\left(\overline{R}_1 F\right) = \overline{R}_2 g^{-1}\left(\overline{R}_1 F\right)$, therefore g is a rough continuous. So f^{-1} is a rough continuous. Hence f is a rough homeomorphism.

Theorem 2.16. Let $K = (X, R_1, \tau_k)$ and $Q = (Y, R_2, \sigma_Q)$ be two topologized approximation spaces and $f: K \longrightarrow Q$ be a bijective mapping, then f is rough homeomorphism iff f is rough continuous and rough open.

Proof. From Theorem 2.14 and Theorem 2.15 the proof is obvious.

Definition 2.17. A rough property of a topologized approximation space $K = (X, R, \tau_k)$ is said to be a *rough hereditary* iff, every subspace of the topologized approximation space K has that rough property.

Definition 2.18. A rough property of a topologized approximation space $K = (X, R, \tau_k)$ is said to be a topologized approximation rough property iff, each *rough homeomorphic space* of K has that rough property whenever K has that rough property.

3. Rough Regular Spaces

We define rough regular space and introduce several theorems about rough regularity in topological spaces which are results from the general relations on the approximation spaces.

Definition 3.1. Let $K = (X, R, \tau_k)$ be a topologized approximation space. Then K is said to be a *rough regular space* if, for every subset F of X and $x \notin \overline{R}F$, there exist two subsets A and B of X such that $x \in \underline{R}A$, $\overline{R}F \subseteq \underline{R}B$ and $\underline{R}A \cap \underline{R}B = \emptyset$.

Definition 3.2 [4]. Let $K = (X, R, \tau_k)$ be a topologized approximation space. Then K is said to be a *rough* k_1 *space* (briefly $k_1 -$ space), if for every two distinct points $x, y \in X$, there exist two subsets A and B of X such that $x \in \underline{R}A$, $y \notin \underline{R}A$ and $y \in \underline{R}B$, $x \notin \underline{R}B$.

Definition 3.3 [4]. Let $K = (X, R, \tau_k)$ be a topologized approximation space. Then K is said to be a *rough* k_2 *space* (briefly k_2 −space), if for every two distinct points $x, y\ X$, there exist two subsets A and B of X such that $x \in \underline{R}A$, $y \in \underline{R}B$ and $\underline{R}A \cap \underline{R}B = \emptyset$.

Theorem 3.4 [4]. Let $K = (X, R, \tau_k)$ be a topologized approximation space. Then K is a $k_1 -$ space if and only if $\{x\} = \overline{R}\{x\}$ for every $x \in X$.

Definition 3.5. Let $K = (X, R, \tau_k)$ be a topologized approximation space. Then K is said to be a *rough* k_3*space* (briefly k_3–space) if, it is both rough regular space and k_1–space.

Theorem 3.6. Every k_3–space is k_2–space.

Proof. Let $K = (X, R, \tau_k)$ be a k_3–space (i.e. K is a rough regular k_1–space). Let $x, y \in X$ such that $x \neq y$, then by Theorem 3.4, we have $\{x\} = \overline{R}\{x\}$. Since $y \notin \{x\} = \overline{R}\{x\}$ and K is rough regular space, then there exist two subsets A and B of X such that $\overline{R}\{x\} \subseteq \underline{R}A$, $y \in \underline{R}B$ and $\underline{R}A \cap \underline{R}B = \emptyset$, thus $x \in \underline{R}A$, $y \in \underline{R}B$ and $\underline{R}A \cap \underline{R}B = \emptyset$. Hence K is k_2–space.

Theorem 3.7. Rough regularity is rough hereditary property.

Proof. Let $K = (X, R, \tau_k)$ be a rough regular space and let Y be a subset of X. To proof $Q = (Y, R, \sigma_Q)$ is rough regular space. Let H be a subset of $Y$and $y \notin \overline{R}_Y H, \forall\, y \in Y$. But by Theorem 2.7, we have $\overline{R}_Y H = \overline{R}_X H \cap Y$, $y \notin \overline{R}_Y H$, so we get that $y \notin \overline{R}_X H$. Now, K rough regular then there exist two subsets A and B of X such that $y \in \underline{R}_X A$, $\overline{R}_X H \subseteq \underline{R}_X B$ and $\underline{R}_X A \cap \underline{R}_X B = \emptyset$. Therefore $y \in \underline{R}_X A \cap Y$ and $\overline{R}_X H \cap Y \subseteq \underline{R}_X B \cap Y$, thus $\overline{R}_Y H \subseteq \underline{R}_X B \cap Y$. Also by Theorem 2.7, we have $(\underline{R}_X A \cap Y)$ and $(\underline{R}_X B \cap Y)$ are subsets of Y such that $\underline{R}_Y(\underline{R}_X A \cap Y) = (\underline{R}_X A \cap Y)$ and $\underline{R}_Y(\underline{R}_X B \cap Y) = (\underline{R}_X B \cap Y)$. Then by (4) of Proposition 2.3, we have $\underline{R}_Y(\underline{R}_X A \cap Y) \cap \underline{R}_Y(\underline{R}_X B \cap Y) = \underline{R}_Y[(\underline{R}_X A \cap Y) \cap (\underline{R}_X B \cap Y)] = \underline{R}_Y[(\underline{R}_X A \cap \underline{R}_X B) \cap Y] = \underline{R}_Y[\emptyset \cap Y] = \underline{R}_Y \emptyset = \emptyset$. Therefore Q is rough regular space. Hence rough regularity is rough hereditary property.

Theorem 3.8. Rough regularity is a topologized approximation rough property.

Proof. Let $K = (X, R_1, \tau_k)$ be a rough regular space and let $Q = (Y, R_2, \sigma_Q)$ be a rough homeomorphic image of $K = (X, R_1, \tau_k)$ under a map f. Let F be a subset of Y and y be a point of Y which is not in $\overline{R}_2 F$. Since f is bijective function. There exits $x \in X$ such that $f(x) = y$. Now f being rough continuous, $f^{-1}(\overline{R}_2 F)$ is a subset of X such that $f^{-1}(\overline{R}_2 F) = \overline{R}_1\big(f^{-1}(\overline{R}_2 F)\big)$. Since $y \notin \overline{R}_2 F$ then $x = f^{-1}(y) \notin f^{-1}(\overline{R}_2 F) = \overline{R}_1\big(f^{-1}(\overline{R}_2 F)\big)$, thus $x \notin \overline{R}_1(f^{-1}(\overline{R}_2 F)$ and since K is rough regular, then there exist two subsets M, N of X such that $x \in \underline{R}_1 M$, $\overline{R}_1 f^{-1}(\overline{R}_2 F) \subseteq \underline{R}_1 N$ and $\underline{R}_1 M \cap \underline{R}_1 N = \emptyset$. Therefore $f(x) \in f\big(\underline{R}_1 M\big)$, $\overline{R}_2 F \subseteq f\big(f^{-1}(\overline{R}_2 F)\big) = f\left(\overline{R}_1\big(f^{-1}(\overline{R}_2 F)\big)\right) \subseteq f(\underline{R}_1 N)$, thus $\overline{R}_2 F \subseteq f(\underline{R}_1 N)$. Since f is rough homeomorphism, then f is rough open, therefore $f\big(\underline{R}_1 M\big) = \underline{R}_2 f\big(\underline{R}_1 M\big)$ and $f\big(\underline{R}_1 N\big) = \underline{R}_2 f\big(\underline{R}_1 N\big)$. Moreover by Proposition 2.3 $\underline{R}_2 f\big(\underline{R}_1 M\big) \cap \underline{R}_2 f\big(\underline{R}_1 N\big) = \underline{R}_2 f(\underline{R}_1 M \cap \underline{R}_1 N) = \underline{R}_2 f(\emptyset) = \underline{R}_2 \emptyset = \emptyset$. Therefore Q is rough regular space. Hence rough regularity is a topologized approximation rough property.

Theorem 3.9. Let $K = (X, R, \tau_k)$ be a topologized approximation space. Then the following statements are equivalent:

i) K is rough regular space.

ii) For every subset F of X and $x \notin \overline{R}F$, there exist two subsets A and B of X such that $x \in \underline{R}A$, $\overline{R}F \subseteq \underline{R}B$ and $\overline{R}\underline{R}A \cap \underline{R}B = \emptyset$.

iii) For every subset F of X and $x \notin \overline{R}F$, there is a subset A of X such that $x \in \underline{R}A$ and $\overline{R}\underline{R}A \cap \overline{R}F = \emptyset$.

iv) For every subset G of X and $x \in \underline{R}$G, there is a subset $B$of Xsuch that $x \in \underline{R}B \subseteq \overline{R}\underline{R}B \subseteq \underline{R}G$.

v) For every subset F of X, we have:

$\overline{R}F = \cap \{\overline{R}\underline{R}B : B$ subset of X and $\overline{R}F \subseteq \underline{R}B\}$.

Proof. (i) $\Longrightarrow$(ii). Let $x \in X$ and F be a subset of X such that $x \notin \overline{R}F$, since K is rough regular, there exist two subsets A and B of Xsuch that $x \in \underline{R}A$, $\overline{R}F \subseteq \underline{R}B$ and $\underline{R}A \cap \underline{R}B = \emptyset$. Then by Proposition 2.3, we have $\underline{R}A \subseteq \overline{R}B^c$, so $\overline{R}\underline{R}A \subseteq \overline{R}\,\overline{R}B^c = \overline{R}B^c$. Hence $\overline{R}\underline{R}A \cap \underline{R}B = \emptyset$.

(ii) $\Longrightarrow$ (iii). Let $x \in X$ and F be a subset of X such that $x \notin \overline{R}F$. By (ii), there exist two subsets A and B of X such that $x \in \underline{R}$A, $\overline{R}F \subseteq \underline{R}B$ and $\overline{R}\underline{R}A \cap \underline{R}B = \emptyset$. Since $\overline{R}F \subseteq \underline{R}B$. Then $\overline{R}\underline{R}A \cap \underline{R}F = \emptyset$.

(iii) $\Longrightarrow$ (iv). Let $x \in X$ and G be a subset of X such that $x \in \underline{R}G$. Then by (9) of Proposition 2.3, we have $x \notin \overline{R}G^c$. By(iii), there is a subset B of X such that $x \in \underline{R}B$ and $\overline{R}\underline{R}B \cap \overline{R}G^c = \emptyset$. Therefore $x \in \underline{R}B \subseteq \overline{R}\underline{R}B \subseteq \underline{R}G$.

(iv) $\Longrightarrow$ (v). Let $y \in X$ and F be a subset of X such that $y \notin \overline{R}F$. Then by (10) of proposition 2.3, we have $y \in \underline{R}F^c$. By (iv), there is a subset B of X such that $y \in \underline{R}B \subseteq \overline{R}\underline{R}B \subseteq \underline{R}F^c$. Then $\overline{R}F \subseteq \underline{R}\overline{R}B^c \subseteq \overline{R}B^c$ and $y \notin \overline{R}B^c$. Let $\underline{R}W = \underline{R}\overline{R}B^c$, then $\overline{R}F \subseteq \underline{R}W$ since $\underline{R}W \subseteq \overline{R}B^c$ then by (5) of Proposition 2.3, we have $\overline{R}\underline{R}W \subseteq \overline{R}\overline{R}B^c = \overline{R}B^c$, thus $\overline{R}\underline{R}W \subseteq \overline{R}B^c$, therefore $y \notin \overline{R}\underline{R}W$. This implies that that

$y \notin \cap \{\overline{R}\underline{R}W : W$ subset of X and $\overline{R}F \subseteq \underline{R}W\}$. Then

$\cap \{\overline{R}\underline{R}W : W$ subset of X and $\overline{R}F \subseteq \underline{R}W\} \subseteq \overline{R}F$. But

$\overline{R}F \subseteq \cap \{\overline{R}\underline{R}W : W$ subset of X and $\overline{R}F \subseteq \underline{R}W\}$. Hence

$\overline{R}F = \cap \{\overline{R}\underline{R}W : W$ subset of X and $\overline{R}F \subseteq \underline{R}W\}$.

(v) $\Longrightarrow$ (i). Let $x \in X$ and F be a subset of X such that $x \notin \overline{R}F$. By (v), there is a subset B of X such that $\overline{R}F \subseteq \underline{R}B$ and $x \notin \overline{R}\underline{R}B$. Put $\underline{R}A = \underline{R}\overline{R}B^c$**.** Then $x \in \underline{R}$A. Moreover, $\underline{R}B \cap \underline{R}$A $= \emptyset$. Hence K is rough regular space.

4. Rough Normal Spaces

We define rough normal spaces and introduce several theorems about rough normality in topological spaces which are results from the general relations on the approximation spaces.

Definition 4.1. Let $K = (X, R, \tau_k)$ be a topologized approximation space. Then K is said to be a *rough normal space* if, for every two subsets G and H of X such that $\overline{R}G \cap \overline{R}H = \emptyset$, there exist two subsets A and B of Xsuch that $\overline{R}G \subseteq \underline{R}$A, $\overline{R}H \subseteq \underline{R}$B and $\underline{R}$A $\cap$ $\underline{R}$B $= \emptyset$.

Definition 4.2. Let $K = (X, R, \tau_k)$ be a topologized approximation space. Then K is said to be a *rough* k_4 space (briefly k_4–space) if, it is both rough normal and k_1–space

Theorem 4.3. Every k_4–space is k_3–space.

Proof. Let $K = (X, R, \tau_k)$ be a k_4–space (i.e. K is a rough normal k_1–space). Let F be a subset of X and $x \in X$ such that $x \notin \overline{R}F$, then by Theorem 3.4, we have $\{x\} = \overline{R}\{x\}$, therefore $\overline{R}\{x\} \cap \overline{R}F = \emptyset$. Since K is rough normal space, then there exist two subsets A and $B$of Xsuch that $\overline{R}\{x\} \subseteq \underline{R}A$,$\overline{R}F \subseteq \underline{R}B$ and $\underline{R}A \cap \underline{R}B = \emptyset$. Thus $x \in \underline{R}$A, $\overline{R}F \subseteq \underline{R}B$ and $\underline{R}A \cap \underline{R}B = \emptyset$. Therefore K is rough regular space, since K is k_1–space. So K is k_3–space.

Theorem 4.4. If $K = (X, R, \tau_k)$ is a rough normal space and Y is a subset of X such that $Y = \overline{R}_X Y$. Then $Q = (Y, R, \sigma_Q)$is rough normal.

Proof. Assume that A and B are subsets of Y such that $\overline{R}_Y A \cap \overline{R}_Y B = \emptyset$**.** Since $Y = \overline{R}_X Y$, then by Theorem 2.8, we have $\overline{R}_Y A = \overline{R}_X A$ and $\overline{R}_Y B = \overline{R}_X B$. Therefore $\overline{R}_X A \cap \overline{R}_X B = \emptyset$. Since K rough normal space, then there exist two subsets G and H of X such that $\overline{R}_X A \subseteq \underline{R}_X G$, $\overline{R}_X B \subseteq \underline{R}_X H$ and $\underline{R}_X G \cap \underline{R}_X H = \emptyset$. Therefore $\overline{R}_Y A \subseteq \underline{R}_X G \cap Y$ and $\overline{R}_Y B \subseteq \underline{R}_X H \cap Y$. Also by Theorem 2.7, we have $\underline{R}_X G \cap Y$ and $\underline{R}_X H \cap Y$ are subsets of Y such that $\underline{R}_Y(\underline{R}_X G \cap Y) = (\underline{R}_X G \cap Y)$ and $\underline{R}_Y(\underline{R}_X H \cap Y) = (\underline{R}_X H \cap Y)$. So by (4) and (2) of Proposition 2.3, we have $[\underline{R}_Y(\underline{R}_X H \cap Y) \cap \underline{R}_Y(\underline{R}_X G \cap Y)] = \underline{R}_Y[(\underline{R}_X H \cap Y) \cap (\underline{R}_X G \cap Y)] = \underline{R}_Y[(\underline{R}_X H \cap \underline{R}_X G) \cap Y] = \underline{R}_Y[\emptyset \cap Y] = \underline{R}_Y \emptyset = \emptyset$. Hence Q is rough normal space.

Theorem 4.5. Rough normality is a topologized approximation rough property.

Proof. Let $K = (X, R_1, \tau_k)$ be a rough normal and let $Q = (Y, R_2, \tau_Q)$ be a rough homeomorphic image of $K = (X, R_1, \tau_k)$ under a map f. Let G ,H be two subsets of Y such that $\overline{R}_2 G \cap \overline{R}_2 H = \emptyset$. Since f is rough continuous, then $f^{-1}(\overline{R}_2 G) = \overline{R}_1(f^{-1}(\overline{R}_2 G))$ and $f^{-1}(\overline{R}_2 H) = \overline{R}_1(f^{-1}(\overline{R}_2 H))$. Then by (v) of Theorem 2.10 and (12) Proposition 2.3, we have $\overline{R}_1 f^{-1}(\overline{R}_2 G) \subseteq f^{-1}(\overline{R}_2 \overline{R}_2 G) = f^{-1}(\overline{R}_2 G)$ and $\overline{R}_1 f^{-1}(\overline{R}_2 H) \subseteq f^{-1}(\overline{R}_2 \overline{R}_2 H) =$

$f^{-1}(\overline{R}_2H)$, therefore $\overline{R}_1f^{-1}(\overline{R}_2G) \cap \overline{R}_1f^{-1}(\overline{R}_2H) \subseteq f^{-1}(\overline{R}_2G) \cap f^{-1}(\overline{R}_2H) = f^{-1}(\overline{R}_2G \cap \overline{R}_2H) = f^{-1}(\emptyset) = \emptyset$, hence $\overline{R}_1f^{-1}(\overline{R}_2G) \cap \overline{R}_1f^{-1}(\overline{R}_2H) = \emptyset$. Since K is rough normal, then there exist two subsets M, N of X such that $\overline{R}_1f^{-1}(\overline{R}_2G) \subseteq \underline{R}_1M$, $\overline{R}_1f^{-1}(\overline{R}_2H) \subseteq \underline{R}_1N$ and $\underline{R}_1M \cap \underline{R}_1N = \emptyset$. Therefore $f\left(f^{-1}(\overline{R}_2G)\right) = f\left(\overline{R}_1f^{-1}(\overline{R}_2G)\right) \subseteq f\left(\underline{R}_1M\right)$ and $f\left(f^{-1}(\overline{R}_2H)\right) = f\left(\overline{R}_1f^{-1}(\overline{R}_2H)\right) \subseteq f(\underline{R}_1N)$, thus $\overline{R}_2G \subseteq f\left(\underline{R}_1M\right)$ and $\overline{R}_2H \subseteq f\left(\underline{R}_1N\right)$. Since f is rough homeomorphism, then f rough open, therefore $f\left(\underline{R}_1M\right) = \underline{R}_2f\left(\underline{R}_1M\right)$ and $f\left(\underline{R}_1N\right) = \underline{R}_2f\left(\underline{R}_1N\right)$. Moreoverby (2) and (4) of Proposition 2.3, we have $\underline{R}_2f\left(\underline{R}_1M\right) \cap \underline{R}_2f\left(\underline{R}_1N\right) = \underline{R}_2f\left(\underline{R}_1M \cap \underline{R}_1N\right) = \underline{R}_2f(\emptyset) = \underline{R}_2\emptyset = \emptyset$. Therefore Q is rough normal space. Hence rough normality is a topologized approximation rough property.

Theorem 4.6. Let $K = (X, R, \tau_k)$ be a topologized approximation space. Then the following statements are equivalent :

i) K is rough normal space.

ii) For every two subsets G and H of X such that $\overline{R}G \cap \overline{R}H = \emptyset$, there are two subsets A and B of X such that $\overline{R}G \subseteq \underline{R}A$, $\overline{R}H \subseteq \underline{R}B$ and $\overline{R}\underline{R}A \cap \underline{R}B = \emptyset$.

iii) For every two subsets G and H of X such that $\overline{R}G \cap \overline{R}H = \emptyset$, there is a subset A of X such that $\overline{R}G \subseteq \underline{R}$A and $\overline{R}\underline{R}$A $\cap \overline{R}H = \emptyset$.

iv) For every two subsets G and H of X such that $\overline{R}G \subseteq \underline{R}H$, there is a subset B of X such that $\overline{R}G \subseteq \underline{R}B \subseteq \overline{R}\underline{R}B \subseteq \underline{R}H$.

Proof : (i)$\Longrightarrow$ (ii). Assume that G and H are subsets of X such that $\overline{R}G \cap \overline{R}H = \emptyset$, since K is rough normal, there exist two subsets A and B of X such that $\overline{R}G \subseteq \underline{R}A$, $\overline{R}H \subseteq \underline{R}B$ and $\underline{R}A \cap \underline{R}B = \emptyset$. Then by (9), (5), (12) of Proposition 2.3, we have $\underline{R}$A $\subseteq \overline{R}B^c$ and $\overline{R}\underline{R}A \subseteq \overline{R}\,\overline{R}B^c = \overline{R}B^c$. Hence $\overline{R}\underline{R}A \cap \underline{R}B = \emptyset$.

(ii) $\Longrightarrow$ (iii). Assume that G and H are subsets of X such that $\overline{R}G \cap \overline{R}H = \emptyset$. By (ii), there exist two subsets A and B of X such that $\overline{R}G \subseteq \underline{R}A$, $\overline{R}H \subseteq \underline{R}B$ and $\overline{R}\underline{R}A \cap \underline{R}B = \emptyset$. Since $\overline{R}H \subseteq \underline{R}B$, then $\overline{R}\underline{R}A \cap \overline{R}H = \emptyset$.

(iii) $\Longrightarrow$ (iv). Assume that G and H are subsets of X such that $\overline{R}G \subseteq \underline{R}H$, Then $\overline{R}G \cap \overline{R}H^c = \emptyset$. By (iii), there is a subset A of X such that $\overline{R}H^c \subseteq \underline{R}A$ and $\overline{R}\underline{R}A \cap \overline{R}G = \emptyset$. Then by (10) of Proposition 2.3, we have $\overline{R}G \subseteq \underline{R}\overline{R}A^c \subseteq \overline{R}A^c \subseteq \underline{R}H$. Let $\underline{R}B = \underline{R}\overline{R}A^c$ thus , $\overline{R}G \subseteq \underline{R}B \subseteq \overline{R}\underline{R}\,B \subseteq \overline{R}A^c \subseteq \underline{R}H$.

(iv) $\Longrightarrow$ (i). Let G and H be subsets of X such that $\overline{R}G \cap \overline{R}H = \emptyset$ and satisfies (iv). Then $\overline{R}G \subseteq \underline{R}H^c$. Now by hypothesis (iv) there exist a subset M of X such that $\overline{R}G \subseteq \underline{R}M$ and $\overline{R}\underline{R}M \subseteq \underline{R}H^c$, then $\overline{R}H \subseteq \underline{R}\overline{R}M^c$, also $\underline{R}M \cap \overline{R}M^c = \emptyset$. But by (1) of Proposition 2.3 $\underline{R}\overline{R}M^c \subseteq \overline{R}M^c$. Thus $\underline{R}M \cap \underline{R}\overline{R}M^c = \emptyset$. Hence K rough normal space.

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