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A Semi-Analytic method for Solving Nonlinear Partial Differential Equations

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Abstract
In this article, Adomian's decomposition method is used to give an analytical solution to homogeneous partial differential equations modeling problems in sciences and engineering. The solution algorithm yields a rapidly convergent sequence of analytic approximants, which is readily computable, without recourse to linearization, perturbation and discretization as practiced by the traditional methods. The method provides direct scheme for solving the problems and is capable of greatly reducing the size of computational work while still maintaining high accuracy when compared with the theoretical solution. The method can also help to overcome the problems caused by the shortage of analytical methods for the computation of solutions to nonlinear differential equations.
Keywords: Adomian's decomposition method, Nonlinear differential equations, Adomian's polynomials, Polynomial approximations,

1. Introduction

The usefulness of nonlinear partial differential equations in addressing scientific problems is becoming ever more generally recognized in the scientific community as a result of the success of its application in fields as wide ranging as finance, biology, engineering, psychology, economics, physics etc. Applications of these equations depend strongly on the existence of theoretical or exact solutions which can be computed by many standard known methods. However, some nonlinear problems can only be suitably tackled by resorting to numerical methods, which requires continuous development of innovative numerical approaches that tailored for these complicated deferential equations [6,7,8,9,10]. Adomian's decomposition method has received much attention in the last few decades. Unlike the existing numerical methods, ADM does not require massive computations inherent in discretization methods. The representation of nonlinear terms using Adomian's polynomial avoids the truncation error inherent in computational grids methods, which employ a linear approximation between the grid points.
In this work, we are concerned with the solution of nonlinear differential equations of the form;

$$Fu = Lu + Ru + Nu = g \qquad 1.1$$

Here g is a specified analytic function and F represents a general nonlinear differential operator, L is the linear operator, R is the random or stochastic operator and N is the nonlinear operator [12]. For the purpose of this work, we shall assume that equation (1.1) corresponds to an initial value problem with a nonlinear deterministic partial differential equation.

1.1 Review of Adomian's method

We shall consider natural and chemical systems as modeled by equation (1.1) where $L = \frac{\partial^\alpha}{\partial x^\alpha}$, the highest order linear operator, $R = \sum_{v=0}^{\alpha-1} \gamma_v(x) \frac{\partial^v}{\partial x^v}$, the remainder operator of the linear terms and assume N to be a simple nonlinearity such as $N = R_2 f(u)$, where $f(u)$ is an analytic function of the solution u. R_2 may have a similar form as the remainder operator R on may not include derivatives [11] . For simplicity, we write $R(u) = -R_1(u)$ and $N(u) = -R_2 f(u)$, thus equation (1.1) becomes

$$Lu = g + R_1 u + R_2 f(u) \qquad 1.2$$

We noted from [11] that this is a usual form of Adomian's and co-workers except for the replacement of $R = -R_1$ and $Nu = -R_2 f(u)$. This does not affect the derivation except to eliminate the power of negative one and is adopted for the sake of simplicity. We next solve for solution u as modeled by equation (1.2)
Upon inverting the linear operator L on both sides of (1.2) we have

$$L^{-1}L = L^{-1}g + L^{-1}R_1u + L^{-1}R_2f(u) \tag{1.3}$$

Where L^{-1} is p-fold indefinite integral operator $\int\ldots\int(.)dx\ldots dx$

Equation (1.3) yields integral equation

$$u = \phi + L^{-1}g + L^{-1}R_1u + L^{-1}R_2f(u) \tag{1.4}$$

and $\phi = \sum_{v=0}^{\alpha-1}\frac{x^v\partial^v u}{v!\partial x^v}$ comprises the terms that arise from the initial or boundary conditions.

The notion of analytic parametrization is utilized to craft out the Adomian decomposition series [1]

$$u = \sum_{m=0}^{\infty}u_m\lambda^m \text{ and } f(u) = \sum_{m=0}^{\infty}A_m\lambda^m \tag{1.5}$$

By substituting (1.5) into (1.4) we have

$$\sum_{m=0}^{\infty}u_m\lambda^m = \phi + L^{-1}g + L^{-1}R_1\sum_{m=0}^{\infty}u_m\lambda^m + L^{-1}R_2\sum_{m=0}^{\infty}A_m\lambda^m \tag{1.6}$$

We observed that various functions for $\lambda(x)$ would subsume well-know series as special case of the Adomian decomposition series [2], thus requiring the components u_m and A_m to become constants to be determined. Adomian choice is a special case corresponding $\lambda = 1$, where the analytic parameter this becomes the grouping parameter [12]

$$\sum_{m=0}^{\infty}u_m = \phi + L^{-1}g + L^{-1}R_1\sum_{m=0}^{\infty}u_m + L^{-1}R_2\sum_{m=0}^{\infty}A_m \tag{1.7}$$

Next we compute the initial solution components u_0 such that

$$u_0 = \phi + L^{-1}g \tag{1.8}$$

$$u_{m+1} = L^{-1}R_1u_m + L^{-1}R_2A_m$$

The series $u = \sum_{m=0}^{\infty}u_m$ converges very rapidly and the m-term approximation $\phi = \sum_{n=0}^{m-1}u_n$ serves as a practical solution for purposes of synthesis and design.

To continue our computation of the solution components further than the initial component u_0, we shall requires for Adomian polynomials A_m. Adomian and co-worker determined the polynomials by implicit differentiations

$$A_m = \left.\frac{\partial^m f(u(x;\lambda))}{m!\partial\lambda^m}\right|_{\lambda=0} \tag{1.10}$$

Where A_m are polynomials in $(u_0, u_1, u_2, \ldots\ u_m)$ The first few Adomian polynomials from A_0 to A_{10} may be found in Adomian and co-worker [1,2]

$$\left.\begin{aligned} A_0 &= f(u_0) \\ A_1 &= u_1f^{(1)}(u_0) \\ A_2 &= u_2f^{(1)}(u_0) + \frac{u_1^2f^{(2)}(u_0)}{2!} \\ A_3 &= u_3f^{(1)}(u_0) + u_1u_2f^{(2)}(u_0) + \frac{u_1^3f^{(3)}(u_0)}{3!} \end{aligned}\right\} \tag{1.11}$$

$$A_m = \sum_{v=1}^{m}c(v,m)f^v(u_0) \tag{1.12}$$

Where $f^{v}(u_0)=\dfrac{\partial^{v} f(u_0)}{\partial u_0^{v}}$, as discussed by V Choi, H.-W. and Shin, J.-G [3]

1.1.2 Applications

Problem 1
Consider the nonlinear equation

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}+u^2-u\frac{\partial u}{\partial x}=0,\ \ 0\le x\le 1, t>0 \qquad 1.13$$

with the initial conditions $u(x,0)=\dfrac{\partial u(x,0)}{\partial t}=e^{x}$

The exact solution of the above problem is $u(x,t)=e^{x+t}$

Using the Adomian approach as discussed above, we rewrite (1.13) in an operator;

$$L_t u - L_x u + u^2 - uu_x = 0 \qquad 1.14$$

$L_t=\dfrac{\partial^2}{\partial t^2}$, $L_x=\dfrac{\partial^2}{\partial x^2}$ and $u_x=\dfrac{\partial}{\partial x}$

Operating L^{-1} on both sides of (1.14) and imposing the initial conditions, we have

$$L^{-1}L_t u - L^{-1}L_x u + L^{-1}u^2 - L^{-1}uu_x = 0 \qquad 1.15$$

$$u(x,t)=u(x,0)+tu_t(x,0)+L^{-1}L_x u - L^{-1}u^2 + L^{-1}uu_x \qquad 1.16$$

Where $L^{-1}=\int_0^t\int_0^t (.)dtdt$

u^2 and uu_x are two nonlinear part in (1.16)

Let $Nu=u^2=\sum_{m=0}^{\infty}A_m$ and $Mu=uu_x=\sum_{m=0}^{\infty}B_m$

A_m and B_m are the Adomian polynomials generated for the nonlinearities in (1.16)

$$u(x,t)=u(x,0)+tu_t(x,0)+L^{-1}L_x\left[\sum_{n=0}^{\infty}u_n(x,t)\right]-L^{-1}\left[\sum_{n=0}^{\infty}A_n(x,t)\right]+L^{-1}\left[\sum_{n=0}^{\infty}B_n(x,t)\right] \qquad 1.17$$

By using scheme (1.11), we generate the followings;

$A_0=u_0^2$

$A_1=2u_0u_1$

$A_2=u_1^2+2u_0u_2$

$A_3=2u_1u_2+2u_0u_3$

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$B_0=u_0u_0^{'}$

$B_1=u_0u_1^{'}+u_1u_0^{'}$

$B_2=u_0u_2^{'}+u_1u_1^{'}+u_2u_0^{'}$

$B_3=u_0u_1^{'}+u_2u_1^{'}+u_1u_2^{'}+u_3u_0^{'}$

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From equation (1.17)

$$u_0=u(x,0)+tu_t(x,0)=e^{x}+te^{x}=(t+1)e^{x}$$

$$u_1=L^{-1}L_x u_0 - L^{-1}A_0 + L^{-1}B_0 = e^{x}\left(\frac{t^2}{2!}+\frac{t^3}{3!}\right)$$

$$u_2 = L^{-1}L_x u_1 - L^{-1}A_1 + L^{-1}B_1 = e^x\left(\frac{t^4}{4!}+\frac{t^5}{5!}\right)$$

$$u_3 = L^{-1}L_x u_2 - L^{-1}A_2 + L^{-1}B_2 = e^x\left(\frac{t^6}{6!}+\frac{t^7}{7!}\right)$$

$$u_4 = L^{-1}L_x u_3 - L^{-1}A_3 + L^{-1}B_3 = e^x\left(\frac{t^8}{8!}+\frac{t^9}{9!}\right)$$

$$u_n = L^{-1}L_x u_{n-1} - L^{-1}A_{n-1} + L^{-1}B_{n-1} = e^x\left(\frac{t^{2n}}{(2n)!}+\frac{t^{2n+1}}{(2n+1)!}\right)$$

The solution $u(x,t)=\sum_{n=0}^{\infty}u_n = u_1+u_2+u_3+u_4+\ldots$ .is thus computed. This is equivalent to power series expansion of the exact solution e^{x+t}

Problem 2

Let us consider a non linear wave equation

$$\frac{\partial^2 u}{\partial x^2}-\frac{u\partial^2 u}{\partial t^2} = -2(x^2+t^2) \qquad 1.18$$

with the initial conditions $u(x,0)=x^2,\ u(0,t)=t^2,\ \frac{\partial u(0,t)}{\partial x}=0$

Using the Adomian's approach as discussed above, we rewrite (1.13) in an operator;

$$L_x u - uL_t u = -2(t^2+x^2) \qquad 1.19$$

$$L_x = \frac{\partial^2}{\partial x^2}, L_t = \frac{\partial^2}{\partial t^2}$$

Operating L^{-1} on both sides of (1.19) and imposing the initial conditions, we have

$$L_x^{-1}L_x u - L_x^{-1}uL_t u = L_x^{-1}[-2(t^2+x^2)]$$

1.20

$$u(x,t) = x^2+t^2+L_x^{-1}uL_t u - L_x^{-1}[-2(t^2+x^2)]$$

1.21

$$L_x^{-1} = \int_0^x\int_0^x (.)dxdx$$

$uL_t u$ is the nonlinear term in (1.20)

Let $L_t u = u' = \frac{\partial^2 u}{\partial t^2}$ and $Nu = uL_t u = uu' = \sum_{m=0}^{\infty}A_m$

$$u(x,t) = x^2+t^2+L_x^{-1}[\sum_{m=0}^{\infty}A_m]-2L_x^{-1}t^2-2L_x^{-1}x^2 \qquad 1.22$$

By using scheme (1.11), we generate the followings;

$A_0 = u_0 u_0'$

$A_1 = u_0 u_1' + u_1 u_0'$

$A_2 = u_0 u_2' + u_1 u_1' + u_2 u_0'$

$A_3 = u_0 u_1' + u_2 u_1' + u_1 u_2' + u_3 u_0'$

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From the scheme (1.22),

$$u_0 = x^2 + t^2 - 2L_x^{-1}t^2 - 2L_x^{-1}x^2 = x^2 + t^2 - x^2t^2 - \frac{x^4}{6}$$

$$u_1 = L_x^{-1}A_0 = x^2t^2 + \frac{x^4}{6} - \frac{x^4t^2}{3} - \frac{7x^6}{90} + \frac{2x^6t^2}{15} + \frac{x^4}{16}$$

$$u_2 = L_x^{-1}A_1 = \frac{x^4t^2}{3} + \frac{7x^6}{90} - \frac{2x^6t^2}{15} - \frac{x^4}{16}$$

The solution $u(x,t) = \sum_{n=0}^{\infty} u_n = u_1 + u_2 + u_3 + u_4 + . . .$ is computed. We observed that the self canceling "noise" terms appear between the various components. The non-canceled terms in u_0 yields the exact solution $u(x,t) = x^2 + t^2$.

1.1.4 Conclusion

In this work, we have revisited Adomian's methodology and implement his approach to obtain analytical solution to nonlinear partial differential equations. We observed that the method convergences rapidly to the exact solutions. This approach is elegant, powerful and accurate. The method provides a new approach to obtain an analytical solution to scientific and financial problems without modeling compromises merely for the sake of achieving linearization as commonly practiced by conventional numerical methods

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