Subclasses of Analytic and Multivalent Functions Defined by Extended Derivative Operator of Ruscheweyh's Type

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ABSTRACT
By means of certain extended derivative operator of Ruscheweyh's type, we introduce and investigate two new subclasses $S^p_{n,m}(\lambda, b, \delta)$ and $G^p_{n,m}(\lambda, b, \delta)$ of p-valently analytic functions of complex order. The various results obtained here for each of these subclasses included coefficient estimate, distortion theorem, radius of starlikeness, convexity and closure theorem.
Keywords & Phrases: - Multivalent function, coefficient estimate, distortion theorem, radius of starlikeness, differential operator.

1. Introduction

Let $A(n)$ denote the class of functions $f(z)$ normalized by

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k , \qquad \text{.........(1)}$$

$$a_k \geq 0 \quad and \quad n, p \in \mathbb{N} = \{1,2,3, \dots\}$$

which are analytic and p-valent in the open unit disc $U = \{z : z \in \mathbb{C} \; and \; |z| < 1\}$
We introduce here an extended linear derivative operator of Ruscheweyh's type :
$D^{\lambda,p} : A(1) \to A(1)$ which is defined by

$$D^{\lambda,p}(f(z)) = z^p - \sum_{k=n+p}^{\infty} \binom{\lambda+k-1}{k-p} a_k z^k \text{ where } \lambda > -p, f \in A(n) \quad \text{......(2)}$$

In particular when $\lambda = n, n \in \mathbb{N}$, it is easily observed from (2) that

$$D^{n,p}(f(z)) = \frac{z^p (z^{n-p} f(z))^{(n)}}{n!}$$

$n \in \mathbb{N} \cup \{0\} . p \in \mathbb{N}$
So that

$$D^{1,p}(f(z)) = (1-p)f(z) + zf'(z)$$

$$D^{2,p}(f(z)) = \frac{(1-p)(2-p)}{2!} f(z) + (2-p)zf'(z) + \frac{z^2}{2!} f''(z)$$

And so on .

$$(D^{\lambda,p}(f(z)))^{(m)} = \binom{p}{m} z^{p-m} - \sum_{k=n+p}^{\infty} \binom{k}{m} \binom{\lambda+k-1}{k-p} a_k z^{k-m}$$

Where $\binom{k}{m} = \frac{k(k-1)(k-2)\dots\dots(k-m+1)}{m!}$

By using the operator $D^{\lambda,p}(f(z))$, we introduce new subclass $S^p_{n,m}(\lambda, b, \delta)$ of p-valently analytic function $f(z)$ satisfying the following inequality

$$\left| \frac{1}{b} \left(\frac{\delta z \left(D^{\lambda,p}(f(z))\right)^{(m+1)} + \lambda z^2 \left(D^{\lambda,p}(f(z))\right)^{(m+2)}}{\lambda z \left(D^{\lambda,p}(f(z))\right)^{(m+1)} + (\delta - \lambda) \left(D^{\lambda,p}(f(z))\right)^{(m)}} - (p-m) \right) \right| < 1$$

$p \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$ $z \in U$, $p > \max(m, -\lambda)$, $b \in \mathbb{C} \cup \{0\}$, $\lambda \geq 0, 0 < \delta \leq 1$

Furthermore a function $f(z)$ is said to belong to the class $G^p_{n,m}(\lambda, b, \delta)$ if and only if $zf'(z) \in S^p_{n,m}(\lambda, b, \delta)$

The object of the present paper is to investigate the various properties and characteristics of analytic p-valent functions belonging to the subclasses $S^p_{n,m}(\lambda, b, \delta)$ and $G^p_{n,m}(\lambda, b, \delta)$ which we have defined here. Apart from deriving a set of coefficient bounds for each of these function classes, we establish distortion theorem, radius of starlikeness, convexity and closure theorem.

Our definitions of the function classes $S^p_{n,m}(\lambda, b, \delta)$ and $G^p_{n,m}(\lambda, b, \delta)$ are motivated by the investigation of H. M. Srivastava and others [2], we have relaxed the parametric constraint $0 \le \lambda \le 1$.

2. COEFFICIENT ESTIMATES

THEOREM 1:- A function $f(z) \in A(n)$ and defined by
$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k$, $a_k \ge 0$ *and* $p \in \mathbb{N}$, is in $S^p_{n,m}(\lambda, b, \delta)$ if and only if

$$\sum_{k=n+p}^{\infty} \binom{\lambda+k-1}{k-p}\binom{k}{m}[\lambda(k-m-1)+\delta)][k-p+|b|]a_k \le |b|\binom{p}{m}[\lambda(p-m-1)+\delta] \quad \ldots..(1.1)$$

PROOF: - Suppose that $f(z) \in S^p_{n,m}(\lambda, b, \delta)$
Therefore we have

$$\left|\frac{1}{b}\left(\frac{\delta z\left(D^{\lambda,p}(f(z))\right)^{(m+1)}+\lambda z^2\left(D^{\lambda,p}(f(z))\right)^{(m+2)}}{\lambda z\left(D^{\lambda,p}(f(z))\right)^{(m+1)}+(\delta-\lambda)\left(D^{\lambda,p}(f(z))\right)^{(m)}}-(p-m)\right)\right|<1 \quad \ldots\ldots\ldots (1.2)$$

$$\left(D^{\lambda,p}(f(z))\right)^{(m)} = \binom{p}{m} z^{p-m} - \sum_{k=n+p}^{\infty}\binom{k}{m}\binom{\lambda+k-1}{k-p} a_k z^{k-m}$$

$$\left(D^{\lambda,p}(f(z))\right)^{(m+1)} = \binom{p}{m}(p-m)z^{p-m-1} - \sum_{k=n+p}^{\infty}\binom{k}{m}\binom{\lambda+k-1}{k-p}(k-m)a_k z^{k-m-1}$$

$$\left(D^{\lambda,p}(f(z))\right)^{(m+2)} = \binom{p}{m}(p-m)(p-m-1)z^{p-m-2} - \sum_{k=n+p}^{\infty}\binom{k}{m}\binom{\lambda+k-1}{k-p}(k-m)(k-m-1)a_k z^{k-m-2}$$

$$\delta z\left(D^{\lambda,p}(f(z))\right)^{(m+1)} + \lambda z^2\left(D^{\lambda,p}(f(z))\right)^{(m+2)} = \binom{p}{m}(p-m)[\delta+\lambda(p-m-1)z^{p-m} - \sum_{k=n+p}^{\infty}\binom{k}{m}\binom{\lambda+k-1}{k-p}(k-m)[\delta+\lambda(k-m-1)]a_k z^{k-m}$$

$$\lambda z\left(D^{\lambda,p}(f(z))\right)^{(m+1)} + (\delta-\lambda)\left(D^{\lambda,p}(f(z))\right)^{(m)} = \binom{p}{m}[\lambda(p-m)+\delta-\lambda]z^{p-m} - \sum_{k=n+p}^{\infty}\binom{k}{m}\binom{\lambda+k-1}{k-p}[\lambda(k-m)+\delta-\lambda]a_k z^{k-m}$$

From (1.2) we have

$$\left|\frac{1}{b}\left(\frac{\sum_{k=n+p}^{\infty}\binom{k}{m}\binom{\lambda+k-1}{k-p}[\lambda(k-m-1)+\delta][-k+m+p-m]a_k z^{k-m}}{\binom{p}{m}[\lambda(p-m-1)+\delta]z^{p-m} - \sum_{k=n+p}^{\infty}\binom{k}{m}\binom{\lambda+k-1}{k-p}[\lambda(k-m-1)+\delta]a_k z^{k-m}}\right)\right| < 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1.3)$$

We know that $Re\,(z) < |\,z\,|$, also putting $z = r_1$, $0 \le r_1 \le 1$ in (1.3) , we observe that expression in the denominator on left hand side of (1.3) is positive for $r_1 = 0$ and by letting $r_1 \to 1_-$ through real values , (1.3) leads us that

$$\frac{1}{|b|}\left[\frac{\sum_{k=n+p}^{\infty}\binom{k}{m}\binom{\lambda+k-1}{k-p}[\lambda(k-m-1)+\delta][k-p]a_k}{\binom{p}{m}[\lambda(p-m-1)+\delta]-\sum_{k=n+p}^{\infty}\binom{k}{m}\binom{\lambda+k-1}{k-p}[\lambda(k-m-1)+\delta]a_k}\right] \le 1$$

$$\sum_{k=n+p}^{\infty}\binom{k}{m}\binom{\lambda+k-1}{k-p}[\lambda(k-m-1)+\delta][k-p+|b|]a_k \le |b|\binom{p}{m}[\lambda(p-m-1)+\delta]$$

Hence we get (1.1)

Conversely, by applying (2) and setting $|z| = 1$ we find that

$$\left|\left(\frac{\delta z\left(D^{\lambda,p}(f(z))\right)^{(m+1)}+\lambda z^2\left(D^{\lambda,p}(f(z))\right)^{(m+2)}}{\lambda z\left(D^{\lambda,p}(f(z))\right)^{(m+1)}+(\delta-\lambda)\left(D^{\lambda,p}(f(z))\right)^{(m)}}-(p-m)\right)\right|$$

$$= \left| \left(\frac{\sum_{k=n+p}^{\infty} \binom{k}{m} \binom{\lambda+k-1}{k-p} [\lambda(k-m-1)+\delta][k-p] a_k z^{k-m}}{\binom{p}{m} [\lambda(p-m-1)+\delta] z^{p-m} - \sum_{k=n+p}^{\infty} \binom{k}{m} \binom{\lambda+k-1}{k-p} [\lambda(k-m-1)+\delta] a_k z^{k-m}} \right) \right|$$

$$\le |b| \frac{\binom{p}{m} [\lambda(p-m-1)+\delta] z^{p-m} - \sum_{k=n+p}^{\infty} \binom{k}{m} \binom{\lambda+k-1}{k-p} [\lambda(k-m-1)+\delta] a_k z^{k-m}}{\binom{p}{m} [\lambda(p-m-1)+\delta] z^{p-m} - \sum_{k=n+p}^{\infty} \binom{k}{m} \binom{\lambda+k-1}{k-p} [\lambda(k-m-1)+\delta] a_k z^{k-m}} = |b|$$

Hence by the maximum modulus principle, we infer that $f(z) \in S_{n,m}^{p}(\lambda, b, \delta)$ which completes the proof of Theorem 1.

COROLLARY 1.1:- $f(z) \in S_{n,m}^{p}(\lambda, b, \delta)$ then

$$a_k \le \frac{|b| \binom{p}{m} [\lambda(p-m-1)+\delta]}{\binom{k}{m} \binom{\lambda+k-1}{k-p} [\lambda(k-m-1)+\delta][k-p+|b|]}$$

COROLLARY 1.2:- for $p = 1, m = 0$ we have

$$a_k \le \frac{|b|\delta}{\binom{\lambda+k-1}{k-1} [\lambda(k-1)+\delta][k-1+|b|]}, \quad k \ge n+p$$

COROLLARY 1.3:- for $p = 1, m = 1$ we have

$$a_k \le \frac{|b|[\delta-\lambda]}{k \binom{\lambda+k-1}{k-p} [\lambda k+\delta][k-1+|b|]}$$

COROLLARY 1.4:- for $p = 1, m = 1, \lambda = 1$ we have

$$a_k \le \frac{|b|[\delta-1]}{k^2 [k+\delta][k-1+|b|]}$$

THEOREM 2:- A function $f(z) \in A(n)$ and defined by
$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k, \; a_k \ge 0 \quad and \quad p \in \mathbb{N}$, is in $G_{n,m}^{p}(\lambda, b, \delta)$ if and only if

$$\sum_{k=n+p}^{\infty} \binom{\lambda+k-1}{k-p} \binom{k}{m} k[\lambda(k-m-1)+\delta)][k-p+|b|] \le |b| \binom{p}{m} p[\lambda(p-m-1)+\delta] \quad \text{.....(2.1)}$$

PROOF:- $f(z) \in G_{n,m}^{p}(\lambda, b, \delta)$ if and only if $zf'(z) \in S_{n,m}^{p}(\lambda, b, \delta)$

Let $g(z) = zf'^{(z)} = p z^p - \sum_{k=n+p}^{\infty} k\, a_k z^k$

$$g(z) \in S_{n,m}^{p}(\lambda, b, \delta)$$

Therefore

$$\left| \frac{1}{b} \left(\frac{\delta z \left(D^{\lambda,p}(g(z))\right)^{(m+1)} + \lambda z^2 \left(D^{\lambda,p}(g(z))\right)^{(m+2)}}{\lambda z \left(D^{\lambda,p}(g(z))\right)^{(m+1)} + (\delta-\lambda)\left(D^{\lambda,p}(g(z))\right)^{(m)}} - (p-m) \right) \right| < 1 \quad \text{.......... (2.2)}$$

$$\left(D^{\lambda,p}(g(z))\right)^{(m)} = \binom{p}{m} p z^{p-m} - \sum_{k=n+p}^{\infty} \binom{k}{m} \binom{\lambda+k-1}{k-p} k\, a_k z^{k-m}$$

$$\left(D^{\lambda,p}(g(z))\right)^{(m+1)} = \binom{p}{m} p(p-m) z^{p-m-1} - \sum_{k=n+p}^{\infty} \binom{k}{m} \binom{\lambda+k-1}{k-p} (k-m) k a_k z^{k-m-1}$$

$$\left(D^{\lambda,p}(g(z))\right)^{(m+2)} = \binom{p}{m} p(p-m)(p-m-1) z^{p-m-2} - \sum_{k=n+p}^{\infty} \binom{k}{m} \binom{\lambda+k-1}{k-p} (k-m)(k-m-1) k a_k z^{k-m-2}$$

$$\delta z \left(D^{\lambda,p}(g(z))\right)^{(m+1)} + \lambda z^2 \left(D^{\lambda,p}(g(z))\right)^{(m+2)} = \binom{p}{m} p(p-m)[\delta + \lambda(p-m-1) z^{p-m} - \sum_{k=n+p}^{\infty} \binom{k}{m} \binom{\lambda+k-1}{k-p} k(k-m)[\delta + \lambda(k-m-1)] a_k z^{k-m}$$

$$\lambda z \left(D^{\lambda,p}(g(z))\right)^{(m+1)} + (\delta-\lambda) \left(D^{\lambda,p}(g(z))\right)^{(m)}$$

$$= p\binom{p}{m}[\lambda(p-m)+\delta-\lambda]z^{p-m} - \sum_{k=n+p}^{\infty}\binom{k}{m}\binom{\lambda+k-1}{k-p}k[\lambda(k-m)+\delta-\lambda]a_k z^{k-m}$$

From (2.2) we have

$$\left|\frac{1}{b}\left(\frac{\sum_{k=n+p}^{\infty}\binom{k}{m}\binom{\lambda+k-1}{k-p}[\lambda(k-m-1)+\delta]k[-k+p]a_k z^{k-m}}{\binom{p}{m}p[\lambda(p-m-1)+\delta]z^{p-m} - \sum_{k=n+p}^{\infty}\binom{k}{m}\binom{\lambda+k-1}{k-p}k[\lambda(k-m-1)+\delta]a_k z^{k-m}}\right)\right| < 1 \quad \ldots\ldots\ldots\ldots(2.3)$$

We know that $Re(z) < |z|$, also putting $z = r_1$, $0 \le r_1 \le 1$ in (2.3), we observe that expression in the denominator on left hand side of (2.3) is positive for $r_1 = 0$ and by letting $r_1 \to 1_-$ through real values, (2.3) leads us that

$$\frac{1}{|b|}\left[\frac{\sum_{k=n+p}^{\infty}\binom{k}{m}\binom{\lambda+k-1}{k-p}[\lambda(k-m-1)+\delta]k\,[k-p]a_k}{p\binom{p}{m}[\lambda(p-m-1)+\delta] - \sum_{k=n+p}^{\infty}\binom{k}{m}\binom{\lambda+k-1}{k-p}k\,[\lambda(k-m-1)+\delta]a_k}\right] \le 1$$

$$\sum_{k=n+p}^{\infty}\binom{k}{m}\binom{\lambda+k-1}{k-p}k[\lambda(k-m-1)+\delta][k-p+|b|]a_k \le |b|p\binom{p}{m}[\lambda(p-m-1)+\delta]$$

Hence we get (2.1)

Conversely, by applying (2) and setting $|z| = 1$ we find that

$$\left|\left(\frac{\delta z\left(D^{\lambda,p}(g(z))\right)^{(m+1)} + \lambda z^2\left(D^{\lambda,p}(g(z))\right)^{(m+2)}}{\lambda z\left(D^{\lambda,p}(g(z))\right)^{(m+1)} + (\delta-\lambda)\left(D^{\lambda,p}(g(z))\right)^{(m)}} - (p-m)\right)\right|$$

$$= \left|\left(\frac{\sum_{k=n+p}^{\infty}\binom{k}{m}\binom{\lambda+k-1}{k-p}k[\lambda(k-m-1)+\delta][k-p]a_k z^{k-m}}{p\binom{p}{m}[\lambda(p-m-1)+\delta]z^{p-m} - \sum_{k=n+p}^{\infty}\binom{k}{m}\binom{\lambda+k-1}{k-p}k[\lambda(k-m-1)+\delta]a_k z^{k-m}}\right)\right|$$

$$\le |b|\frac{p\binom{p}{m}[\lambda(p-m-1)+\delta]z^{p-m} - \sum_{k=n+p}^{\infty}\binom{k}{m}\binom{\lambda+k-1}{k-p}k[\lambda(k-m-1)+\delta]a_k z^{k-m}}{p\binom{p}{m}[\lambda(p-m-1)+\delta]z^{p-m} - \sum_{k=n+p}^{\infty}\binom{k}{m}\binom{\lambda+k-1}{k-p}k[\lambda(k-m-1)+\delta]a_k z^{k-m}} = |b|$$

Hence by the maximum modulus principle, we infer that $g(z) \in S_{n,m}^{p}(\lambda, b, \delta)$ which completes the proof of Theorem 2.

COROLLARY 2.1:- $f(z) \in G_{n,m}^{p}(\lambda, b, \delta)$ then

$$a_k \le \frac{|b|p\binom{p}{m}[\lambda(p-m-1)+\delta]}{\binom{k}{m}\binom{\lambda+k-1}{k-p}k[\lambda(k-m-1)+\delta][k-p+|b|]}, \quad k \ge n+p$$

3. GROWTH AND DISTORTION THEOREM

THEOREM 3:- If $f(z) \in S_{n,m}^{p}(\lambda, b, \delta)$ then

$$|\boldsymbol{z}|^{\boldsymbol{p}} - |\boldsymbol{z}|^{\boldsymbol{n+p}}\frac{|b|\binom{p}{m}[\lambda(p-m-1)+\delta]}{\binom{n+p}{m}\binom{\lambda+n+p-1}{n}[\lambda(n+p-m-1)+\delta][n+|b|]} \le |f(z)|$$

$$\le |\boldsymbol{z}|^{\boldsymbol{p}} + |\boldsymbol{z}|^{\boldsymbol{n+p}}\frac{|b|\binom{p}{m}[\lambda(p-m-1)+\delta]}{\binom{n+p}{m}\binom{\lambda+n+p-1}{n}[\lambda(n+p-m-1)+\delta][n+|b|]}$$

With equality hold for

$$f(z) = z^p - z^{n+p}\frac{|b|\binom{p}{m}[\lambda(p-m-1)+\delta]}{\binom{n+p}{m}\binom{\lambda+n+p-1}{n}[\lambda(n+p-m-1)+\delta][n+|b|]}$$

PROOF :- $f(z) \in S_{n,m}^{p}(\lambda, b, \delta)$

Therefore from (1.1)

$$\sum_{k=n+p}^{\infty}\binom{\lambda+k-1}{k-p}\binom{k}{m}[\lambda(k-m-1)+\delta)][k-p+|b|] \le |b|\binom{p}{m}[\lambda(p-m-1)+\delta]$$

Therefore

$$\sum_{k=n+p}^{\infty} a_k \le \frac{|b|\binom{p}{m}[\lambda(p-m-1)+\delta]}{\binom{k}{m}\binom{\lambda+k-1}{k-p}[\lambda(k-m-1)+\delta][k-p+|b|]}$$

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k$$

$$|f(z)| \geq |\boldsymbol{z}|^{\boldsymbol{p}} - \sum_{\boldsymbol{k=n+p}}^{\infty} |\boldsymbol{a_k}|\ |\boldsymbol{z}|^{\boldsymbol{k}} \geq |\boldsymbol{z}|^{\boldsymbol{p}} - |\boldsymbol{z}|^{\boldsymbol{n+p}} \sum_{\boldsymbol{k=n+p}}^{\infty} |\boldsymbol{a_k}|$$

$$\geq |\boldsymbol{z}|^{\boldsymbol{p}} - |\boldsymbol{z}|^{\boldsymbol{n+p}} \frac{|b|\binom{p}{m}[\lambda(p-m-1)+\delta]}{\binom{n+p}{m}\binom{\lambda+n+p-1}{n}[\lambda(n+p-m-1)+\delta][n+|b|]}$$

Similarly

$$|f(z)| \leq |\boldsymbol{z}|^{\boldsymbol{p}} + \sum_{\boldsymbol{k=n+p}}^{\infty} |\boldsymbol{a_k}|\ |\boldsymbol{z}|^{\boldsymbol{k}} \leq |\boldsymbol{z}|^{\boldsymbol{p}} + |\boldsymbol{z}|^{\boldsymbol{n+p}} \sum_{\boldsymbol{k=n+p}}^{\infty} |\boldsymbol{a_k}|$$

$$\leq |\boldsymbol{z}|^{\boldsymbol{p}} + |\boldsymbol{z}|^{\boldsymbol{n+p}} \frac{|b|\binom{p}{m}[\lambda(p-m-1)+\delta]}{\binom{n+p}{m}\binom{\lambda+n+p-1}{n}[\lambda(n+p-m-1)+\delta][n+|b|]}$$

Therefore

$$|\boldsymbol{z}|^{\boldsymbol{p}} - |\boldsymbol{z}|^{\boldsymbol{n+p}} \frac{|b|\binom{p}{m}[\lambda(p-m-1)+\delta]}{\binom{n+p}{m}\binom{\lambda+n+p-1}{n}[\lambda(n+p-m-1)+\delta][n+|b|]} \leq |f(z)|$$

$$\leq |\boldsymbol{z}|^{\boldsymbol{p}} + |\boldsymbol{z}|^{\boldsymbol{n+p}} \frac{|b|\binom{p}{m}[\lambda(p-m-1)+\delta]}{\binom{n+p}{m}\binom{\lambda+n+p-1}{n}[\lambda(n+p-m-1)+\delta][n+|b|]}$$

THEOREM 4:- If $f(z) \in G^p_{n,m}(\lambda, b, \delta)$ then

$$|\boldsymbol{z}|^{\boldsymbol{p}} - |\boldsymbol{z}|^{\boldsymbol{n+p}} \frac{|b|p\binom{p}{m}[\lambda(p-m-1)+\delta]}{\binom{n+p}{m}\binom{\lambda+n+p-1}{n}(n+p)[\lambda(n+p-m-1)+\delta][n+|b|]} \leq |f(z)|$$

$$\leq |\boldsymbol{z}|^{\boldsymbol{p}} + |\boldsymbol{z}|^{\boldsymbol{n+p}} \frac{|b|p\binom{p}{m}[\lambda(p-m-1)+\delta]}{\binom{n+p}{m}\binom{\lambda+n+p-1}{n}(n+p)[\lambda(n+p-m-1)+\delta][n+|b|]}$$

With equality hold for

$$f(z) = z^p - z^{n+p} \frac{|b|p\binom{p}{m}[\lambda(p-m-1)+\delta]}{\binom{n+p}{m}\binom{\lambda+n+p-1}{n}(n+p)[\lambda(n+p-m-1)+\delta][n+|b|]}$$

PROOF :- $f(z) \in G^p_{n,m}(\lambda, b, \delta)$
Therefore from (2.1)

$$\sum_{k=n+p}^{\infty} \binom{\lambda+k-1}{k-p}\binom{k}{m} k[\lambda(k-m-1)+\delta)][k-p+|b|] \leq |b|p\binom{p}{m}[\lambda(p-m-1)+\delta]$$

Therefore

$$\sum_{k=n+p}^{\infty} a_k \leq \frac{|b|p\binom{p}{m}[\lambda(p-m-1)+\delta]}{\binom{k}{m}\binom{\lambda+k-1}{k-p} k[\lambda(k-m-1)+\delta][k-p+|b|]}$$

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k$$

$$|f(z)| \geq |\boldsymbol{z}|^{\boldsymbol{p}} - \sum_{\boldsymbol{k=n+p}}^{\infty} |\boldsymbol{a_k}|\ |\boldsymbol{z}|^{\boldsymbol{k}} \geq |\boldsymbol{z}|^{\boldsymbol{p}} - |\boldsymbol{z}|^{\boldsymbol{n+p}} \sum_{\boldsymbol{k=n+p}}^{\infty} |\boldsymbol{a_k}|$$

$$\geq |\boldsymbol{z}|^{\boldsymbol{p}} - |\boldsymbol{z}|^{\boldsymbol{n+p}} \frac{|b|p\binom{p}{m}[\lambda(p-m-1)+\delta]}{\binom{n+p}{m}\binom{\lambda+n+p-1}{n}(n+p)[\lambda(n+p-m-1)+\delta][n+|b|]}$$

Similarly

$$|f(z)| \le |\boldsymbol{z}|^{\boldsymbol{p}} + \sum_{\boldsymbol{k=n+p}}^{\infty} |\boldsymbol{a_k}|\ |\boldsymbol{z}|^{\boldsymbol{k}} \le |\boldsymbol{z}|^{\boldsymbol{p}} + |\boldsymbol{z}|^{\boldsymbol{n+p}} \sum_{\boldsymbol{k=n+p}}^{\infty} |\boldsymbol{a_k}|$$

$$\le |\boldsymbol{z}|^{\boldsymbol{p}} + |\boldsymbol{z}|^{\boldsymbol{n+p}} \frac{|b|p\binom{p}{m}[\lambda(p-m-1)+\delta]}{\binom{n+p}{m}\binom{\lambda+n+p-1}{n}(n+p)[\lambda(n+p-m-1)+\delta][n+|b|]}$$

Therefore

$$|\boldsymbol{z}|^{\boldsymbol{p}} - |\boldsymbol{z}|^{\boldsymbol{n+p}} \frac{|b|p\binom{p}{m}[\lambda(p-m-1)+\delta]}{\binom{n+p}{m}\binom{\lambda+n+p-1}{n}(n+p)[\lambda(n+p-m-1)+\delta][n+|b|]} \le |f(z)|$$

$$\le |\boldsymbol{z}|^{\boldsymbol{p}} + |\boldsymbol{z}|^{\boldsymbol{n+p}} \frac{|b|p\binom{p}{m}[\lambda(p-m-1)+\delta]}{\binom{n+p}{m}\binom{\lambda+n+p-1}{n}(n+p)[\lambda(n+p-m-1)+\delta][n+|b|]}$$

THEOREM 5:- If $f(z) \in S^{p}_{n,m}(\lambda, b, \delta)$ then

$$p|z|^{p-1} - |z|^{n+p-1} \frac{|b|\binom{p}{m}(n+p)[\lambda(p-m-1)+\delta]}{\binom{n+p}{m}\binom{\lambda+n+p-1}{n}[\lambda(n+p-m-1)+\delta][n+|b|]} \le |f'(z)|$$

$$\le p|z|^{p-1} + |z|^{n+p-1} \frac{|b|\binom{p}{m}(n+p)[\lambda(p-m-1)+\delta]}{\binom{n+p}{m}\binom{\lambda+n+p-1}{n}[\lambda(n+p-m-1)+\delta][n+|b|]}$$

With equality hold for

$$f(z) = z^p - z^{n+p} \frac{|b|\binom{p}{m}[\lambda(p-m-1)+\delta]}{\binom{n+p}{m}\binom{\lambda+n+p-1}{n}[\lambda(n+p-m-1)+\delta][n+|b|]}$$

PROOF :- $f(z) \in S^{p}_{n,m}(\lambda, b, \delta)$

Therefore from (1.1)

$$\sum_{k=n+p}^{\infty} \binom{\lambda+k-1}{k-p}\binom{k}{m}[\lambda(k-m-1)+\delta)][k-p+|b|] \le |b|\binom{p}{m}[\lambda(p-m-1)+\delta]$$

Therefore

$$\sum_{k=n+p}^{\infty} a_k \le \frac{|b|\binom{p}{m}[\lambda(p-m-1)+\delta]}{\binom{k}{m}\binom{\lambda+k-1}{k-p}[\lambda(k-m-1)+\delta][k-p+|b|]}$$

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k$$

$$f'(z) = pz^{p-1} - \sum_{k=n+p}^{\infty} a_k\ kz^{k-1}$$

$$|f'(z)| \ge p\,|z|^{p-1} - \sum_{k=n+p}^{\infty} |a_k|\, k\,|z|^{k-1} \ge p|z|^{p-1} - |z|^{n+p-1}(n+p)\sum_{k=n+p}^{\infty} |a_k|$$

$$\ge p|z|^{p-1} - |z|^{n+p-1} \frac{|b|\binom{p}{m}(n+p)[\lambda(p-m-1)+\delta]}{\binom{n+p}{m}\binom{\lambda+n+p-1}{n}[\lambda(n+p-m-1)+\delta][n+|b|]}$$

Similarly

$$|f'(z)| \le p\,|z|^{p-1} + \sum_{k=n+p}^{\infty} |a_k|\, k\,|z|^{k-1} \le p|z|^{p-1} + |z|^{n+p-1}(n+p)\sum_{k=n+p}^{\infty} |a_k|$$

$$\le p|z|^{p-1} + |z|^{n+p-1} \frac{|b|\binom{p}{m}(n+p)[\lambda(p-m-1)+\delta]}{\binom{n+p}{m}\binom{\lambda+n+p-1}{n}[\lambda(n+p-m-1)+\delta][n+|b|]}$$

Therefore

$$p|z|^{p-1} - |z|^{n+p-1} \frac{|b|\binom{p}{m}(n+p)[\lambda(p-m-1)+\delta]}{\binom{n+p}{m}\binom{\lambda+n+p-1}{n}[\lambda(n+p-m-1)+\delta][n+|b|]} \le |f'(z)|$$
$$\le p|z|^{p-1} + |z|^{n+p-1} \frac{|b|\binom{p}{m}(n+p)[\lambda(p-m-1)+\delta]}{\binom{n+p}{m}\binom{\lambda+n+p-1}{n}[\lambda(n+p-m-1)+\delta][n+|b|]}$$

THEOREM 6:- If $f(z) \in G^p_{n,m}(\lambda, b, \delta)$ then

$$p|z|^{p-1} - |z|^{n+p-1} \frac{|b|\binom{p}{m}p[\lambda(p-m-1)+\delta]}{\binom{n+p}{m}\binom{\lambda+n+p-1}{n}[\lambda(n+p-m-1)+\delta][n+|b|]} \le |f'(z)|$$
$$\le p|z|^{p-1} + |z|^{n+p-1} \frac{|b|\binom{p}{m}p[\lambda(p-m-1)+\delta]}{\binom{n+p}{m}\binom{\lambda+n+p-1}{n}[\lambda(n+p-m-1)+\delta][n+|b|]}$$

With equality hold for

$$f(z) = z^p - z^{n+p} \frac{|b|p\binom{p}{m}[\lambda(p-m-1)+\delta]}{\binom{n+p}{m}\binom{\lambda+n+p-1}{n}(n+p)[\lambda(n+p-m-1)+\delta][n+|b|]}$$

PROOF :- $f(z) \in G^p_{n,m}(\lambda, b, \delta)$

Therefore from (1.1)

$$\sum_{k=n+p}^{\infty} \binom{\lambda+k-1}{k-p}\binom{k}{m} k[\lambda(k-m-1)+\delta)][k-p+|b|] \le |b|p\binom{p}{m}[\lambda(p-m-1)+\delta]$$

Therefore

$$\sum_{k=n+p}^{\infty} a_k \le \frac{|b|\binom{p}{m}p[\lambda(p-m-1)+\delta]}{\binom{k}{m}\binom{\lambda+k-1}{k-p} k[\lambda(k-m-1)+\delta][k-p+|b|]}$$

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k$$

$$f'(z) = pz^{p-1} - \sum_{k=n+p}^{\infty} a_k\, kz^{k-1}$$

$$|f'(z)| \ge p\,|z|^{p-1} - \sum_{k=n+p}^{\infty} |a_k|\,k\,|z|^{k-1} \ge p|z|^{p-1} - |z|^{n+p-1}(n+p)\sum_{k=n+p}^{\infty}|a_k|$$
$$\ge p|z|^{p-1} - |z|^{n+p-1} \frac{|b|\binom{p}{m}p[\lambda(p-m-1)+\delta]}{\binom{n+p}{m}\binom{\lambda+n+p-1}{n}[\lambda(n+p-m-1)+\delta][n+|b|]}$$

Similarly

$$|f'(z)| \le p\,|z|^{p-1} + \sum_{k=n+p}^{\infty} |a_k|\,k\,|z|^{k-1} \le p|z|^{p-1} + |z|^{n+p-1}(n+p)\sum_{k=n+p}^{\infty}|a_k|$$
$$\le p|z|^{p-1} + |z|^{n+p-1} \frac{|b|\binom{p}{m}p[\lambda(p-m-1)+\delta]}{\binom{n+p}{m}\binom{\lambda+n+p-1}{n}[\lambda(n+p-m-1)+\delta][n+|b|]}$$

Therefore

$$p|z|^{p-1} - |z|^{n+p-1} \frac{|b|\binom{p}{m}p[\lambda(p-m-1)+\delta]}{\binom{n+p}{m}\binom{\lambda+n+p-1}{n}[\lambda(n+p-m-1)+\delta][n+|b|]} \le |f'(z)|$$
$$\le p|z|^{p-1} + |z|^{n+p-1} \frac{|b|\binom{p}{m}p[\lambda(p-m-1)+\delta]}{\binom{n+p}{m}\binom{\lambda+n+p-1}{n}[\lambda(n+p-m-1)+\delta][n+|b|]}$$

4. RADII OF CLOSE-TO-CONVEXITY, STARLIKENESS AND CONVEXITY

A function $f(z) \in A(n)$ is said to be close to convex of order $\propto$ $(0 \le \propto < 1)$ if $Re\{f'(z)\} > \propto$ for all $z \in U$

A function $f(z) \in A(n)$ is said to be starlike of order $\propto$ $(0 \le \propto < 1)$ if

$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \propto$ for all $z \in U$

A function $f(z) \in A(n)$ is said to be convex of order $\propto$ $(0 \leq \propto < 1)$ if and only if $zf'(z)$ is starlike of order $\propto$, that is

$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \propto$ for all $z \in U$

THEOREM 7:- If $f(z) \in S^p_{n,m}(\lambda, b, \delta)$, then f is close to convex of order $\propto$ in $|z| < r_1(p,n,m,\lambda,b,\delta,\propto)$ where

$$r_1(p,n,m,\lambda,b,\delta,\propto) = \inf_k \left(\left(\binom{\lambda+k-1}{k-p}\binom{k}{m}\frac{(p-\propto)[\lambda(k-m-1)+\delta)][k-p+|b|]}{k|b|\binom{p}{m}[\lambda(p-m-1)+\delta]}\right)^{\frac{1}{k-p}}\right)$$

PROOF:- It is sufficient to show that $\left|\frac{f'(z)}{z^{p-1}} - p\right| < p-\propto$

$$f'^{(z)} = p\,z^{p-1} - \sum_{k=n+p}^{\infty} k\,a_k\,z^{k-1}$$

$$\frac{f'(z)}{z^{p-1}} = p - \sum_{k=n+p}^{\infty} k\,a_k\,z^{k-p}$$

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq \sum_{k=n+p}^{\infty} k|a_k||z|^{k-p} < p-\propto \qquad \ldots\ldots\ldots..(7.1)$$

From (1.1) we have

$$\sum_{k=n+p}^{\infty}\binom{\lambda+k-1}{k-p}\binom{k}{m}[\lambda(k-m-1)+\delta)][k-p+|b|]a_k \leq |b|\binom{p}{m}[\lambda(p-m-1)+\delta]$$

That is

$$\sum_{k=n+p}^{\infty}\binom{\lambda+k-1}{k-p}\binom{k}{m}\frac{[\lambda(k-m-1)+\delta)][k-p+|b|]}{|b|\binom{p}{m}[\lambda(p-m-1)+\delta]}a_k \leq 1 \;\ldots\ldots\ldots\ldots\ldots\ldots.(7.2)$$

Observe that (7.1) is true if

$$\frac{k|z|^{k-p}}{p-\propto} \leq \binom{\lambda+k-1}{k-p}\binom{k}{m}\frac{[\lambda(k-m-1)+\delta)][k-p+|b|]}{|b|\binom{p}{m}[\lambda(p-m-1)+\delta]}$$

Therefore

$$|z| \leq \left(\binom{\lambda+k-1}{k-p}\binom{k}{m}\frac{(p-\propto)[\lambda(k-m-1)+\delta)][k-p+|b|]}{k|b|\binom{p}{m}[\lambda(p-m-1)+\delta]}\right)^{\frac{1}{k-p}},$$

$(p \neq k, p, k \in \mathbb{N})$, which complete the proof.

THEOREM 8:- If $f(z) \in S^p_{n,m}(\lambda, b, \delta)$, then f is starlike of order $\propto$ in $|z| < r_2(p,n,m,\lambda,b,\delta,\propto)$ where

$$r_2(p,n,m,\lambda,b,\delta,\propto) = \inf_k \left(\left(\binom{\lambda+k-1}{k-p}\binom{k}{m}\frac{(p-\propto)[\lambda(k-m-1)+\delta)][k-p+|b|]}{(k-\propto)|b|\binom{p}{m}[\lambda(p-m-1)+\delta]}\right)^{\frac{1}{k-p}}\right)$$

PROOF:- We must show that

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leq p-\propto$$

$$zf'(z) - p\,f(z) = pz^p - \sum_{k=n+p}^{\infty} ka_k\,z^k - pz^p + p\sum_{k=n+p}^{\infty} a_k\,z^k = -\sum_{k=n+p}^{\infty}(k-p)a_k\,z^k$$

We have

$$\left|\frac{zf'(z)}{f(z)} - p\right| = \left|\frac{-\sum_{k=n+p}^{\infty}(k-p)a_k z^k}{z^p - \sum_{k=n+p}^{\infty} a_k z^k}\right| \leq \frac{\sum_{k=n+p}^{\infty}(k-p)|a_k|\,|z|^{k-p}}{1-\sum_{k=n+p}^{\infty}|a_k|\,|z|^{k-p}} \leq p-\propto \qquad \ldots\ldots\ldots\ldots\ldots(8.1)$$

Hence (8.1) holds true if

$$\sum_{k=n+p}^{\infty}(k-p)|a_k|\,|z|^{k-p} \leq (p-\propto)\left(1-\sum_{k=n+p}^{\infty}|a_k|\,|z|^{k-p}\right)$$

Or equivalently

$$\sum_{k=p+1}^{\infty} \frac{(k-\propto)}{(p-\propto)} |a_k|\, |z|^{k-p} \le 1 \qquad \text{..........(8.2)}$$

From (1.1) we have

$$\sum_{k=n+p}^{\infty} \binom{\lambda+k-1}{k-p} \binom{k}{m} \frac{[\lambda(k-m-1)+\delta)][k-p+|b|]}{|b|\binom{p}{m}[\lambda(p-m-1)+\delta]} a_k \le 1 \quad \ldots\ldots\ldots\ldots\ldots.(8.3)$$

Hence by using (8.2) and (8.3) we get

$$\frac{(k-\propto)}{(p-\propto)} |z|^{k-p} \le \binom{\lambda+k-1}{k-p} \binom{k}{m} \frac{[\lambda(k-m-1)+\delta)][k-p+|b|]}{|b|\binom{p}{m}[\lambda(p-m-1)+\delta]}$$

$$|z|^{k-p} \le \binom{\lambda+k-1}{k-p} \binom{k}{m} \frac{(p-\propto)[\lambda(k-m-1)+\delta)][k-p+|b|]}{(k-\propto)|b|\binom{p}{m}[\lambda(p-m-1)+\delta]}$$

$$|\boldsymbol{z}| \le \left(\binom{\lambda+k-1}{k-p} \binom{k}{m} \frac{(p-\propto)[\lambda(k-m-1)+\delta)][k-p+|b|]}{(k-\propto)|b|\binom{p}{m}[\lambda(p-m-1)+\delta]} \right)^{\frac{1}{k-p}}$$

$(p \neq k, p, k \in \mathbb{N})$, which complete the proof.

THEOREM 9:- If $f(z) \in S_{n,m}^{p}(\lambda, b, \delta)$, then f is convex of order $\propto$ in $|z| < r_3(p,n,m,\lambda,b,\delta,\propto)$ where

$$r_3(p,n,m,\lambda,b,\delta,\propto) = \inf_{k} \left(\left(\binom{\lambda+k-1}{k-p} \binom{k}{m} \frac{p(p-\propto)[\lambda(k-m-1)+\delta)][k-p+|b|]}{k(k-\propto)|b|\binom{p}{m}[\lambda(p-m-1)+\delta]} \right)^{\frac{1}{k-p}} \right)$$

PROOF:- We know that f is convex if and only if zf' is starlike

We must show that

$$\left| \frac{zg'(z)}{g(z)} - p \right| \le p-\propto$$

Where $g(z) = zf'(z)$

$$g(z) = p\, z^p - \sum_{k=n+p}^{\infty} k\, a_k\, z^k$$

$$zg'(z) = p^2 z^p - \sum_{k=n+p}^{\infty} k^2 a_k z^k$$

$$zg'(z) - p\, g(z) = p^2 z^p - \sum_{k=n+p}^{\infty} k^2 a_k z^k - p^2 z^p + p \sum_{k=n+p}^{\infty} k\, a_k z^k = -\sum_{k=n+p}^{\infty} k(k-p)\, a_k z^k$$

$$\left| \frac{zg'(z)}{g(z)} - p \right| = \left| \frac{-\sum_{k=n+p}^{\infty} k(k-p)\, a_k z^k}{p\, z^p - \sum_{k=n+p}^{\infty} k\, a_k z^k} \right| \le \frac{\sum_{k=n+p}^{\infty} k(k-p)|a_k|\, |z|^{k-p}}{p - \sum_{k=n+p}^{\infty} k\, |a_k|\, |z|^{k-p}} \le p-\propto$$

Therefore we have

$$\sum_{k=n+p}^{\infty} k(k-p)|a_k|\, |z|^{k-p} \le (p-\propto)[p - \sum_{k=n+p}^{\infty} k\, |a_k|\, |z|^{k-p}]$$

$$\sum_{k=n+p}^{\infty} \frac{k(k-\propto)}{p(p-\propto)} |a_k| |z|^{k-p} \le 1 \qquad \ldots\ldots\ldots(9.1)$$

From (1.1) we have

$$\sum_{k=n+p}^{\infty} \binom{\lambda+k-1}{k-p} \binom{k}{m} \frac{[\lambda(k-m-1)+\delta)][k-p+|b|]}{|b|\binom{p}{m}[\lambda(p-m-1)+\delta]} a_k \le 1 \quad \ldots\ldots\ldots\ldots\ldots.(9.2)$$

Hence by using (9.1) and (9.2) we get

$$\frac{k(k-\propto)}{p(p-\propto)} |z|^{k-p} \le \binom{\lambda+k-1}{k-p} \binom{k}{m} \frac{[\lambda(k-m-1)+\delta)][k-p+|b|]}{|b|\binom{p}{m}[\lambda(p-m-1)+\delta]}$$

$$|\boldsymbol{z}| \leq \left(\binom{\lambda+k-1}{k-p}\binom{k}{m}\frac{p(p-\propto)[\lambda(k-m-1)+\delta)][k-p+|b|]}{k(k-\propto)|b|\binom{p}{m}[\lambda(p-m-1)+\delta]}\right)^{\frac{1}{k-p}}$$

$(p \neq k , p, k \in \mathbb{N})$, which complete the proof.

5. CLOSURE THEOREM

THEOREM 10 :

Let $f_1(z) = z^p$ and $f_k(z) = z^p - \binom{\lambda+k-1}{k-p}\binom{k}{m}\frac{[\lambda(k-m-1)+\delta)][k-p+|b|]}{|b|\binom{p}{m}[\lambda(p-m-1)+\delta]}z^k$ for $k \geq n+p$

Then $f(z) \in S_{n,m}^p(\lambda, b, \delta)$ if and only if $f(z)$ can be expressed in the form

$f(z) = \lambda_1 f_1(z) + \sum_{k=n+p}^{\infty} \lambda_k f_k(z)$ where $\lambda_k \geq 0$ and $\lambda_1 + \sum_{k=n+p}^{\infty} \lambda_k = 1$

PROOF: - Suppose $f(z)$ can be expressed in the form

$$f(z) = \lambda_1 f_1(z) + \sum_{k=n+p}^{\infty} \lambda_k f_k(z)$$

$$= \lambda_1 z^p + \sum_{k=n+p}^{\infty} \lambda_k \left[z^p - \binom{\lambda+k-1}{k-p}\binom{k}{m}\frac{[\lambda(k-m-1)+\delta)][k-p+|b|]}{|b|\binom{p}{m}[\lambda(p-m-1)+\delta]}z^k\right]$$

$$= [\lambda_1 + \sum_{k=n+p}^{\infty} \lambda_k] z^p - \sum_{k=n+p}^{\infty} \lambda_k \binom{\lambda+k-1}{k-p}\binom{k}{m}\frac{[\lambda(k-m-1)+\delta)][k-p+|b|]}{|b|\binom{p}{m}[\lambda(p-m-1)+\delta]}z^k$$

$$= z^p - \sum_{k=n+p}^{\infty} \lambda_k \binom{\lambda+k-1}{k-p}\binom{k}{m}\frac{[\lambda(k-m-1)+\delta)][k-p+|b|]}{|b|\binom{p}{m}[\lambda(p-m-1)+\delta]}z^k$$

Then

$$\sum_{k=n+p}^{\infty} \lambda_k \binom{\lambda+k-1}{k-p}\binom{k}{m}\frac{[\lambda(k-m-1)+\delta)][k-p+|b|]}{|b|\binom{p}{m}[\lambda(p-m-1)+\delta]} \cdot \frac{|b|\binom{p}{m}[\lambda(p-m-1)+\delta]}{\binom{\lambda+k-1}{k-p}\binom{k}{m}[\lambda(k-m-1)+\delta)][k-p+|b|]} z^k$$

$$= \sum_{k=n+p}^{\infty} \lambda_k = 1 - \lambda_1 \leq 1$$

Therefore $f(z) \in S_{n,m}^p(\lambda, b, \delta)$

Conversely, suppose that $f(z) \in S_{n,m}^p(\lambda, b, \delta)$

We have

$$a_k \leq \binom{\lambda+k-1}{k-p}\binom{k}{m}\frac{[\lambda(k-m-1)+\delta)][k-p+|b|]}{|b|\binom{p}{m}[\lambda(p-m-1)+\delta]}$$

We take

$$\lambda_k = \frac{|b|\binom{p}{m}[\lambda(p-m-1)+\delta]}{\binom{\lambda+k-1}{k-p}\binom{k}{m}[\lambda(k-m-1)+\delta)][k-p+|b|]}a_k$$

$$k \geq n+p \text{ and } \sum_{k=n+p}^{\infty} \lambda_k = 1 - \lambda_1$$

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k$$

$$= z^p - \sum_{k=n+p}^{\infty} \binom{\lambda+k-1}{k-p}\binom{k}{m}\frac{[\lambda(k-m-1)+\delta)][k-p+|b|]}{|b|\binom{p}{m}[\lambda(p-m-1)+\delta]}z^k$$

$$= z^p - \sum_{k=n+p}^{\infty} \lambda_k [z^p - f_k(z)] = z^p\left[1 - \sum_{k=n+p}^{\infty} \lambda_k\right] - \sum_{k=n+p}^{\infty} \lambda_k f_k(z) = \lambda_1 f_1(z) + \sum_{k=n+p}^{\infty} \lambda_k f_k(z)$$

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