Quadruple Fixed Point Theorems in Partially Ordered Metric Spaces Depended on Another Function

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Abstract
In this article, we introduced concept of ICS mapping for quadruple fixed point in partially ordered metric space. The present results generalized the result of Karapinar E. [23] also we state some examples showing that our results are effective.

Keywords: Quadruple Fixed Point, Mixed monotone, ICS Mapping, Partially ordered set 2000 Mathematics subject classification: 47H10,54H25

Introduction and Preliminaries

The Banach contraction principle, which is the most famous metrical fixed point theorem, play a very important role in nonlinear analysis and its applications are well known. Many authors have extended this theorem, including more general contractive conditions, which imply the existence of a fixed point. Existence of fixed points in ordered metric spaces was investigated in 2004 by Ran and Reurings [17] and then by Nieto and Lopez [16]. After this various results in have been obtained in this direction, see e.g. [1,15,18].

On the other hand, Berinde and Borcut [9] introduced the concept of triple fixed point and proved some related fixed point theorem. After this various results on tripled fixed point have been obtained, see e.g. [2,3,4,5,6,10,11,12,13,14,15].

Regarding this partial order, Karapinar E. [23] introduced the concept of Quadruple fixed point and investigated some coupled fixed point theorems in partially ordered metric spaces. Later, various results in coupled fixed point have been obtained, see e.g. [2,3,4,5,6,10,11,12,13,14,15].

In this paper, we give some quadruple fixed point theorems for mapping having the mixed monotone property in partially ordered metric spaces. Later, Berinde and Borcut [9] introduced the concept of tripled fixed point and proof some related fixed point theorem. After this various results on tripled fixed point have been obtained.

Further studied by Nieto and Rodriguez - Lopez [16], Samet and Vetro [19] introduced the notion of fixed point of N order in case of single-valued mappings. In particular for N = 4 (Quadruple case), i.e., Let (X, ≤) be partially ordered set and (X, d) be a complete metric space. We consider the following partial order on the product space

\[ X^4 = X \times X \times X \times X \]

\[ (u, v, r, t, x, y, z, w) \leq (x, y, z, w) \iff x \geq u, y \leq v, z \geq r, t \leq w, \]

where \((u, v, r, t, x, y, z, w) \in X^4\).

Regarding this partial order, Karapinar E. [23] introduced the concept of Quadruple fixed point and prove some new fixed point theorems. In [23] Karapinar E. defined the following concept of quadruple fixed point.

Definition 1.1: Let \((X, \leq)\) be a partially ordered set, \(F: X^4 \to X\) mapping. The mapping \(F\) is said to have the mixed monotone property if for any \(x, y, z, w \in X\):

i. \(x_1, x_2 \in X, x_1 \leq x_2 \to F(x_1, y, z, w) \leq F(x_2, y, z, w)\),
ii. \(y_1, y_2 \in X, y_1 \geq y_2 \to F(x, y_1, z, w) \geq F(x, y_2, z, w)\),
iii. \(z_1, z_2 \in X, z_1 \leq z_2 \to F(x, y, z_1, w) \leq F(x, y, z_2, w)\),
iv. \(w_1, w_2 \in X, w_1 \geq w_2 \to F(x, y, z, w_1) \geq F(x, y, z, w_2)\).

Definition 1.2: An element \((x, y, z, w) \in X^4\) is called a quadruple fixed point of \(F: X^4 \to X\) if

\(F(x, y, z, w) = x, F(y, z, w, x) = y, F(z, w, x, y) = z\) and \(F(w, x, y, z) = w\).

In this paper, we give some quadruple fixed point theorems for mapping having the mixed monotone property in partially ordered metric spaces depending on another function.

Main Results

Definition 2.1: Let \((X, d)\) be a metric space. A mapping \(T: X \to X\) is said to be ICS if \(T\) is injective, continuous and has the property: for every sequence \(\{x_n\}\) in \(X\), if \(\{Tx_n\}\) is convergent then \(\{x_n\}\) is also convergent. Let \(\phi\) be the set of all functions \(\phi: [0, \infty) \to [0, \infty)\) such that

i. \(\phi\) is non-decreasing,
ii. \(\phi(t) < t\) for all \(t > 0\),
iii. \(\lim_{t \to +\infty} \phi(r) < t\) for all \(t > 0\)

From now on, we denote \(X^4 = X \times X \times X \times X\). Our first result is given by the following:

Theorem 2.2: Let \((X, \leq)\) be a partially ordered set and suppose there is a metric \(d\) in \(X\) such that \((X, d)\) is a complete metric space. Suppose \(T: X \to X\) is a ICS mapping and \(F: X^4 \to X\) is such that \(F\) has the mixed monotone property. Assume that there exists \(\phi \in \Phi\) such that

\[d(TF(x, y, z, w), TF(u, v, p, q)) \leq \phi(\max\{d(Tx, Tu), d(Ty, Tv), d(Tz, Tp), d(Tw, Tq)\})\]
for any \(x, y, z, w \in X\) for which \(x \leq u, v \leq y, z \leq p, q \leq w\). Suppose either

1. \(F\) is continuous, or
2. \(X\) has the following property:
   1. if non decreasing sequence \(x_n \rightarrow x\) (respectively, \(z_n \rightarrow z\)), then \(x_n \leq x, (respectively, z_n \leq z)\) for all \(n\).
   2. if non increasing sequence \(y_n \rightarrow x\) (respectively, \(w_n \rightarrow z\)), then \(y_n \geq y, (respectively, w_n \geq w)\) for all \(n\).

If there exists \(x_0, y_0, z_0, w_0 \in X\) such that \(x_0 \geq F(x_0, y_0, z_0, w_0), y_0 \leq F(y_0, z_0, w_0, x_0), z_0 \leq F(z_0, w_0, x_0, y_0)\) and \(w_0 \geq F(w_0, x_0, y_0, z_0)\), then there exist \(x, y, z, w \in X\) such that

\[
\begin{align*}
  x &= F(x, y, z, w), \quad y = F(y, z, w, x), \\
  z &= F(z, w, x, y), \quad w = F(w, x, y, z)
\end{align*}
\]

that is, \(F\) has a quadrupled fixed point.

**Proof:** Let \(x_0, y_0, z_0, w_0 \in X\) such that \(x_0 \geq F(x_0, y_0, z_0, w_0), y_0 \leq F(y_0, z_0, w_0, x_0), z_0 \leq F(z_0, w_0, x_0, y_0)\) and \(w_0 \geq F(w_0, x_0, y_0, z_0)\) set

\[
x_1 = F(x_0, y_0, z_0, w_0), \quad y_1 = F(y_0, z_0, w_0, x_0)
\]

Continuing this process, we can construct sequences \(\{x_n\}, \{y_n\}, \{z_n\}\) and \(\{w_n\}\) in \(X\) such that

\[
x_{n+1} = F(x_n, y_n, z_n, w_n), \quad y_{n+1} = F(y_n, z_n, w_n, x_n)
\]

\[
z_{n+1} = F(z_n, w_n, x_n, y_n), \quad w_{n+1} = F(w_n, x_n, y_n, z_n)
\]

Since \(F\) has the mixed monotone property, then using the mathematical induction it is easy that

\[
x_n \leq x_{n+1}, \quad y_n \geq y_{n+1}, \quad z_n \leq z_{n+1}, \quad w_n \geq w_{n+1}
\]

for \(n = 0, 1, 2, 3, \ldots\).

Assume for some \(n \in \mathbb{N}\)

\[
x_n = x_{n+1}, \quad y_n = y_{n+1}, \quad z_n = z_{n+1}, \quad w_n = w_{n+1}
\]

then by (2.3), \((x_n, y_n, z_n, w_n)\) is the quadrupled fixed point of \(F\). From now on, assume for any \(n \in \mathbb{N}\) that

\[
x_n \neq x_{n+1}, \quad y_n \neq y_{n+1}, \quad z_n \neq z_{n+1}, \quad w_n \neq w_{n+1}
\]

Since \(T\) is injective, then by (2.6), for any \(n \in \mathbb{N}\)

\[
0 \leq \phi(\max\{d(Tx_n, Tx_n), d(Ty_n, Ty_n), d(Tz_n, Tz_n), d(Tw_n, Tw_n)\})
\]

in the account of (2.1) and (2.3), we have

\[
d(Tx_{n+1}) = \phi(\max\{d(Tx_n, Tx_n), d(Ty_n, Ty_n), d(Tz_n, Tz_n), d(Tw_n, Tw_n)\})
\]

and

\[
d(Ty_{n+1}) = \phi(\max\{d(Tx_n, Tx_n), d(Ty_n, Ty_n), d(Tz_n, Tz_n), d(Tw_n, Tw_n)\})
\]

Since we have \(\phi(t) < t\) for all \(t > 0\), so from (2.8)-(2.11) we obtain that

\[
0 < \max\{d(Tx_n, Tx_n), d(Ty_n, Ty_n), d(Tz_n, Tz_n), d(Tw_n, Tw_n)\}
\]

\[
\leq \phi(\max\{d(Tx_n, Tx_n), d(Ty_n, Ty_n), d(Tz_n, Tz_n), d(Tw_n, Tw_n)\})
\]

It follows that

\[
\max\{d(Tx_n, Tx_{n+1}), d(Ty_n, Ty_{n+1}), d(Tz_n, Tz_{n+1}), d(Tw_n, Tw_{n+1})\}
\]

Thus, \(\lim_{n \to +\infty} \max\{d(Tx_n, Tx_n), d(Ty_n, Ty_n), d(Tz_n, Tz_n), d(Tw_n, Tw_n)\} = r\)

Suppose that \(r > 0\). Letting \(n \to +\infty\) in (2.12), we obtain that

\[
0 < r < \lim_{n \to +\infty} \phi(\max\{d(Tx_n, Tx_n), d(Ty_n, Ty_n), d(Tz_n, Tz_n), d(Tw_n, Tw_n)\})
\]

it is a contradiction. We deduce that

\[
\lim_{n \to +\infty} \max\{d(Tx_n, Tx_n), d(Ty_n, Ty_n), d(Tz_n, Tz_n), d(Tw_n, Tw_n)\} = 0
\]

We shall show that \(\{Tx_n\}, \{Ty_n\}, \{Tz_n\}\) and \(\{Tw_n\}\) are Cauchy sequences. Assume the contrary, that is \(\{Tx_n\}, \{Ty_n\}, \{Tz_n\}\) and \(\{Tw_n\}\) are not a Cauchy sequence. that is
Now, using inequality (2.1) we obtain
\[
\lim_{n,m \to +\infty} d(Tx_n, Tx_m) \neq 0, \quad \lim_{n,m \to +\infty} d(Ty_n, Ty_m) \neq 0
\]
This means that there exists \( \epsilon > 0 \) for which we can find subsequences of integers \((m_k)\) and \((n_k)\) with \( n_k > m_k > k \) such that
\[
\max \{ d(Tx_{n_k}, Tx_{m_k}), d(Ty_{n_k}, Ty_{m_k}), d(Tz_{n_k}, Tz_{m_k}), d(Tw_{n_k}, Tw_{m_k}) \} \geq \epsilon
\]
Further corresponding to \( m_k \) we can choose \( n_k \) in such a way that it is the smallest integer with \( n_k > m_k \) and satisfying (2.15). Then
\[
\max \{ d(Tx_{n_k-1}, Tx_{m_k}), d(Ty_{n_k-1}, Ty_{m_k}), d(Tz_{n_k-1}, Tz_{m_k}), d(Tw_{n_k-1}, Tw_{m_k}) \} < \epsilon
\]
By triangular inequality (2.16), we have
\[
d(Tx_{n_k}, Tx_{n_k}) \leq d(Tx_{n_k}, Tx_{n_k-1}) + d(Tx_{n_k-1}, Tn_{n_k}) < \epsilon + d(Tx_{n_k-1}, Tn_{n_k})
\]
Thus, by (2.14) we obtain
\[
\lim_{k \to +\infty} d(Tx_{n_k}, Tx_{n_k}) \leq \lim_{k \to +\infty} d(Tx_{m_k}, Tx_{n_k-1}) \leq \epsilon
\]
Similarly, we have
\[
\lim_{k \to +\infty} d(Ty_{n_k}, Ty_{n_k}) \leq \lim_{k \to +\infty} d(Ty_{m_k}, Ty_{n_k-1}) \leq \epsilon
\]
\[
\lim_{k \to +\infty} d(Tz_{n_k}, Tz_{n_k}) \leq \lim_{k \to +\infty} d(Tz_{m_k}, Tz_{n_k-1}) \leq \epsilon
\]
\[
\lim_{k \to +\infty} d(Tw_{n_k}, Tw_{n_k}) \leq \lim_{k \to +\infty} d(Tw_{m_k}, Tw_{n_k-1}) \leq \epsilon
\]
Again by (2.16), we have
\[
d(Tx_{n_k}, Tx_{n_k}) \leq d(Tx_{n_k}, Tx_{m_k-1}) + d(Tx_{m_k-1}, Tn_{n_k}) + d(Tx_{n_k}, Tn_{n_k}) < d(Tx_{m_k}, Tx_{m_k-1}) + d(Tn_{n_k}, Tn_{n_k}) + \epsilon
\]
Letting \( k \to +\infty \) and using (2.14), we get
\[
\lim_{k \to +\infty} d(Tx_{m_k}, Tx_{n_k}) \leq \lim_{k \to +\infty} d(Tx_{m_k}, Tx_{n_k-1}) \leq \epsilon
\]
Similarly, we have
\[
\lim_{k \to +\infty} d(Ty_{m_k}, Ty_{n_k}) \leq \lim_{k \to +\infty} d(Ty_{m_k-1}, Ty_{n_k-1}) \leq \epsilon
\]
\[
\lim_{k \to +\infty} d(Tz_{m_k}, Tz_{n_k}) \leq \lim_{k \to +\infty} d(Tz_{m_k-1}, Tz_{n_k-1}) \leq \epsilon
\]
\[
\lim_{k \to +\infty} d(Tw_{m_k}, Tw_{n_k}) \leq \lim_{k \to +\infty} d(Tw_{m_k-1}, Tw_{n_k-1}) \leq \epsilon
\]
Using (2.15) and (2.22) - (2.25), we have
\[
\lim_{k \to +\infty} \max \{ d(Tx_{n_k}, Tx_{n_k}), d(Ty_{n_k}, Ty_{n_k}), d(Tz_{n_k}, Tz_{n_k}), d(Tw_{n_k}, Tw_{n_k}) \}
\]
\[
= \lim_{k \to +\infty} \max \{ d(Tx_{n_k}, Tx_{n_k}), d(Ty_{n_k-1}, Ty_{n_k-1}), d(Tz_{n_k-1}, Tz_{n_k-1}), d(Tw_{n_k-1}, Tw_{n_k-1}) \}
\]
\[
= \epsilon
\]
Now, using inequality (2.1) we obtain
\[
d(Tx_{m_k}, Tx_{n_k}) = d(T(F(x_{m_k-1}; Y_{m_k-1}; Z_{m_k-1}; W_{m_k-1}), T(F(x_{n_k-1}; Y_{n_k-1}; Z_{n_k-1}; W_{n_k-1})) \leq \phi \left( \max \{ d(Tx_{m_k-1}, Tx_{n_k-1}), d(Ty_{m_k-1}, Ty_{n_k-1}), d(Tz_{m_k-1}, Tz_{n_k-1}), d(Tw_{m_k-1}, Tw_{n_k-1}) \} \right)
\]
\[
d(Ty_{m_k}, Ty_{n_k}) = d(T(F(y_{m_k-1}; Z_{m_k-1}; W_{m_k-1}; X_{m_k-1}), T(F(y_{n_k-1}; Z_{n_k-1}; W_{n_k-1}; X_{n_k-1})) \leq \phi \left( \max \{ d(Tx_{m_k-1}, Tx_{n_k-1}), d(Ty_{m_k-1}, Ty_{n_k-1}), d(Tz_{m_k-1}, Tz_{n_k-1}), d(Tw_{m_k-1}, Tw_{n_k-1}) \} \right)
\]
\[
d(Tz_{m_k}, Tz_{n_k}) = d(T(F(z_{m_k-1}; W_{m_k-1}; X_{m_k-1}; Y_{m_k-1}), T(F(z_{n_k-1}; W_{n_k-1}; X_{n_k-1}; Y_{n_k-1}) \leq \phi \left( \max \{ d(Tx_{m_k-1}, Tx_{n_k-1}), d(Ty_{m_k-1}, Ty_{n_k-1}), d(Tz_{m_k-1}, Tz_{n_k-1}), d(Tw_{m_k-1}, Tw_{n_k-1}) \} \right)
\]
\[
d(Tw_{m_k}, Tw_{n_k}) = d(T(F(w_{m_k-1}; X_{m_k-1}; Y_{m_k-1}; Z_{m_k-1}), T(F(w_{n_k-1}; X_{n_k-1}; Y_{n_k-1}; Z_{n_k-1}) \leq \phi \left( \max \{ d(Tx_{m_k-1}, Tx_{n_k-1}), d(Ty_{m_k-1}, Ty_{n_k-1}), d(Tz_{m_k-1}, Tz_{n_k-1}), d(Tw_{m_k-1}, Tw_{n_k-1}) \} \right)
\]
Using (2.27) - (2.30) that
\[
\max \{ d(Tx_{n_k}, Ty_{n_k}), d(Ty_{n_k}, Ty_{n_k}), d(Tz_{n_k}, Tz_{n_k}), d(Tw_{n_k}, Tw_{n_k}) \} \leq \phi \left( \max \{ d(Tx_{m_k-1}, Tx_{n_k-1}), d(Ty_{m_k-1}, Ty_{n_k-1}), d(Tz_{m_k-1}, Tz_{n_k-1}), d(Tw_{m_k-1}, Tw_{n_k-1}) \} \right)
\]
Letting \( k \to +\infty \) in (2.31) and having in mind (2.16), we get that
\[ 0 < \epsilon \leq \lim_{n \to +\infty} \phi(t) < \epsilon \]

it is a contradiction. Thus \{Tx_n\}, \{Ty_n\}, \{Tz_n\} and \{Tw_n\} are Chauchy sequences in \((X, d)\). Since \(X\) is complete metric space, \{Tx_n\}, \{Ty_n\}, \{Tz_n\} and \{Tw_n\} are convergent sequences.

Since \(T\) is an ICS mapping, there exist \(x, y, z, w \in X\) such that
\[
\lim_{n \to +\infty} x_n = x, \lim_{n \to +\infty} y_n = y, \lim_{n \to +\infty} z_n = z, \lim_{n \to +\infty} w_n = w. \quad 2.32
\]

Suppose now the assumption (a) holds. Since \(T\) is an ICS mapping, there exist \(x, y, z, w \in X\) such that
\[
\lim_{n \to +\infty} x_n = x, \lim_{n \to +\infty} y_n = y, \lim_{n \to +\infty} z_n = z, \lim_{n \to +\infty} w_n = w. \quad 2.33
\]

Thus we proved that \(T\) has a quadruple fixed point. This complete proof of the Theorem 2.2.

**Corollary 2.3:**

Let
\[
\text{such that}
\]

**Corollary 2.4:**

If there exists a monotone property. Assume that there exists \(x, y, z, w \in X\) such that
\[
\lim_{n \to +\infty} x_n = x, \lim_{n \to +\infty} y_n = y, \lim_{n \to +\infty} z_n = z, \lim_{n \to +\infty} w_n = w. \quad 2.33
\]

Suppose now the assumption (b) holds. Since \(T\) is an ICS mapping, there exist \(x, y, z, w \in X\) such that
\[
\lim_{n \to +\infty} x_n = x, \lim_{n \to +\infty} y_n = y, \lim_{n \to +\infty} z_n = z, \lim_{n \to +\infty} w_n = w. \quad 2.33
\]

It suffices to remark that
\[
\lim_{n \to +\infty} y_n = y, \lim_{n \to +\infty} z_n = z, \lim_{n \to +\infty} w_n = w. \quad 2.33
\]

Thus we proved that \(T\) has a quadruple fixed point. This complete proof of the Theorem 2.2.

**Corollary 2.3:**

Let \((X, \leq)\) be a partially ordered set and suppose there is a metric \(d\) in \(X\) such that \((X, d)\) is a complete metric space. Suppose \(T: X \to X\) is a ICS mapping and \(F: X^4 \to X\) such that \(F\) has the mixed monotone property. Assume that there exists \(\phi \in \phi s\) such that
\[
d(T(F(x, y, z, w)), (F(y, z, w, x), (F(z, w, x, y), (F(w, x, y, z), ) \leq \phi (\max \{ d(Tx_n, Tx_n), d(Ty_n, Ty_n), d(Tz_n, Tz_n), d(Tw_n, Tw_n) \}) \quad 2.34
\]

Taking as \(n \to +\infty\) and using (2.33), the right hand side of (2.34) tends to \(0\), so we get that
\[
d(T(F(x, y, z, w), T(F(y, z, w, x), T(F(z, w, x, y), T(F(w, x, y, z, )) = 0. \quad 2.35
\]

Thus we proved that \(T\) has a quadruple fixed point. This complete proof of the Theorem 2.2.

**Proof:**

It suffices to remark that
\[
d(T(F(x, y, z, w)), T(F(y, z, w, x), T(F(z, w, x, y), T(F(w, x, y, z, )) \leq \max \{ d(Tx_n, Tx_n), d(Ty_n, Ty_n), d(Tz_n, Tz_n), d(Tw_n, Tw_n) \} \quad 2.36
\]

Then, we apply Theorem 2.2 because that \(\phi\) us non decreasing.

**Corollary 2.4:**

Let \((X, \leq)\) be a partially ordered set and suppose there is a metric \(d\) in \(X\) such that \((X, d)\) is a complete metric space. Suppose \(T: X \to X\) is a ICS mapping and \(F: X^4 \to X\) such that \(F\) has the mixed monotone property. Assume that there exists \(k \in [0, 1]\) such that
\[
d(T(F(x, y, z, w), T(F(y, z, w, x), T(F(z, w, x, y), T(F(w, x, y, z, )) \leq k \max \{ d(Tx_n, Tx_n), d(Ty_n, Ty_n), d(Tz_n, Tz_n), d(Tw_n, Tw_n) \} \quad 2.37
\]

for any \(x, y, z, w \in X\) for which \(x \leq u, v \leq y, z \leq p, q \leq w\). Suppose either
i. \(F\) is continuous,
ii. \(X\) has the following property:
Now, we shall prove the existence and uniqueness of a quadruple fixed point, for a product complete metric space. Suppose a monotone property. Assume that there exists Let Corollary 2.5:-

Proof:- It suffices if we take $\phi(t) = kt$ in Theorem 2.2.

Corollary 2.5:- Let $(X, \leq)$ be a partially ordered set and suppose there is a metric $d$ in $X$ such that $(X, d)$ is a complete metric space. Suppose $T: X \rightarrow X$ is a partially ordered set and suppose there is a metric $d$ in $X$ such that $(X, d)$ is a complete metric space. Suppose $T: X \rightarrow X$ is a partially ordered set and suppose there is a metric $d$ in $X$ such that $(X, d)$ is a complete metric space. Suppose $T: X \rightarrow X$ is a partially ordered set and suppose there is a metric $d$ in $X$ such that $(X, d)$ is a complete metric space. Suppose $T: X \rightarrow X$ is a mixed monotone property. Assume that there exists $k \in (0, 1)$ such that

$$d(TF(x, y, z, w), TF(u, v, p, q)) \leq \frac{k}{4} \left( d(Tx, Tu) + d(Ty, Tv) + d(Tz,Tp) + d(Tw, Tq) \right)$$

for any $x, y, z, w \in X$ for which $x \leq u, v \leq y, z \leq p, q \leq w$. Suppose either

i. $T$ is continuous, or
ii. $X$ has the following property:

(a) if non decreasing sequence $x_n \rightarrow x$ (respectively, $z_n \rightarrow z$), then $x_n \leq x_i, (respectively, $z_n \leq z$)

(b) if non increasing sequence $y_n \rightarrow y$ (respectively, $w_n \rightarrow w$), then $y_n \leq y_i, (respectively, $w_n \geq w$)

If there exists $x_0, y_0, z_0, w_0 \in X$ such that $x_0 \geq F(x_0, y_0, z_0, w_0), y_0 \leq F(y_0, z_0, w_0, x_0), z_0 \leq F(z_0, w_0, x_0, y_0)$ and $w_0 \geq F(w_0, x_0, y_0, z_0)$, then there exist $x, y, z, w \in X$ such that

$$x = F(x, y, z, w), y = F(y, z, w, x), z = F(z, w, x, y) \text{ and } w = F(w, x, y, z)$$

that is, $F$ has a quadrupled fixed point.

Proof:- It follows by taking $\phi(t) = kt$ in Corollary 2.3.

Now, we shall prove the existence and uniqueness of a quadruple fixed point, for a product $X^4$ of a partially ordered set $(X, \leq)$, we define a partial ordering in the following way: for all $x, y, z, w \in X$ and $(u, v, p, q) \in X^4$

$$x \leq y \Leftrightarrow (x, y, z, w) \leq (u, v, p, q) \text{ or } (x, y, z, w) \geq (u, v, p, q)$$

We say that $(x, y, z, w), (u, v, p, q) \in X^4$ are comparable if

$$\begin{cases} (x, y, z, w) \leq (u, v, p, q) \text{ or } (x, y, z, w) \geq (u, v, p, q) \\ \text{Also we say that } (x, y, z, w) \text{ is equal to } (u, v, p, q) \text{ if and only if } x = u, y = v, z = p, w = q. \end{cases}$$

Theorem 2.6:- In addition to hypothesis of Theorem 2.2., suppose that for all $(x, y, z, w), (u, v, p, q) \in X^4$, there exists $(a, b, c, d) \in X^4$ such that

$$F(a, b, c, d), F(b, c, e, a), F(c, e, a, b), F(e, a, b, c)$$

is comparable to

$$F(x, y, z, w), F(y, z, w, x), F(z, w, x, y), F(w, x, y, z)$$

and

$$F(u, v, p, q), F(v, p, q, u), F(p, q, u, v), F(q, u, v, p).$$

Then, $F$ has a unique quadruple fixed point $(x, y, z, w)$.

Proof:- The set of quadruple fixed points of $F$ is non empty due to Theorem 2.2. Assume, now, $(x, y, z, w), (u, v, p, q) \in X^4$ are two quadruple fixed points of $F$, that is,

$$F(x, y, z, w) = x, F(u, v, p, q) = u, F(y, z, w, x) = y, F(v, p, q, u) = v$$

$$F(z, w, x, y) = z, F(p, q, u, v) = p, F(w, x, y, z) = w, F(q, u, v, p) = q$$

We shall show that $(x, y, z, w)$ and $(u, v, p, q)$ are equal. By assumption, there exists $(a, b, c, d) \in X^4$ such that

$$F(a, b, c, e), F(b, c, e, a), F(c, e, a, b), F(e, a, b, c)$$

is comparable to

$$F(x, y, z, w), F(y, z, w, x), F(z, w, x, y), F(w, x, y, z)$$

and

$$F(u, v, p, q), F(v, p, q, u), F(p, q, u, v), F(q, u, v, p).$$

Define sequences $\{ a_n \}, \{ b_n \}, \{ c_n \}$ and $\{ e_n \}$ such that

$$a_0 = a, \quad b_0 = b, \quad c_0 = c \text{ and } e_0 = e$$

and for any $n \geq 1$

$$a_n = F(a_{n-1}, b_{n-1}, c_{n-1}, e_{n-1}), \quad b_n = F(b_{n-1}, c_{n-1}, e_{n-1}, a_{n-1}), \quad c_n = F(c_{n-1}, e_{n-1}, a_{n-1}, b_{n-1}), \quad e_n = F(e_{n-1}, a_{n-1}, b_{n-1}, c_{n-1})$$

$$2.41$$
for all n. Further, set \( x_0 = x, y_0 = y, z_0 = z, w_0 = w \) and \( u_0 = u, v_0 = v, p_0 = p, q_0 = q \) and on the same way define the sequences \( \{ x_n \}, \{ y_n \}, \{ z_n \}, \{ w_n \} \) and \( \{ u_n \}, \{ v_n \}, \{ p_n \}, \{ q_n \} \). Then it is easy that

\[
\begin{align*}
x_n &= F(x, y, z, w), \\
y_n &= F(u, v, p, q), \\
z_n &= F(z, w, x, y), \\
w_n &= F(w, x, y, z).
\end{align*}
\]

Thus, from (2.49) it is known that

\[
\lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = \lim_{n \to \infty} z_n = \lim_{n \to \infty} w_n = x, y, z, w.
\]

Therefore, for each \( n \geq 1 \),

\[
\max \{ \| x_n - a_n \|, \| y_n - c_n \|, \| z_n - b_n \|, \| w_n - d_n \| \} 
\leq \phi \left( \max \{ \| x, y, z, w - x, y, z, w \| \} \right) = \phi (0) = 0.
\]

This yield that

\[
\lim_{n \to \infty} d(Tx, Ta_n) = 0, \quad \lim_{n \to \infty} d(Ty, Tb_n) = 0, \quad \lim_{n \to \infty} d(Tz, Tc_n) = 0, \quad \lim_{n \to \infty} d(Tw, Te_n) = 0.
\]

Analogously, we show that

\[
\lim_{n \to \infty} d(Tu, Ta_n) = 0, \quad \lim_{n \to \infty} d(Tv, Tb_n) = 0, \quad \lim_{n \to \infty} d(Tp, Tc_n) = 0, \quad \lim_{n \to \infty} d(Tq, Te_n) = 0.
\]

Combining (2.50) and (2.51) yields that \((T, x, y, z, w)\) and \((Tu, Tv, Tp, Tq)\) are equal. The fact that \( T \) is injective gives us \( x = u, y = v, z = p \) and \( w = q \).

This complete prove of the Theorem 2.7.

### Examples

Now we state some examples showing that our results are effective.

**Example 3.1:** Let \( X = [\frac{1}{2}, 64] \) with the metric \( d(x, y) = |x - y| \) for all \( x, y \in X \) and the usual ordering \( \leq \).

Clearly, \((X, d)\) is complete metric space.

Let \( T : X \to X \) and \( F : X^4 \to X \) be defined by

\[
T(x) = \ln x + 1 \quad \text{and} \quad F(x, y, z, w) = 8 \left( \sqrt[4]{\frac{x^2}{yw}} \right)^{\frac{1}{3}}, \quad \forall x, y, z, w \in X.
\]

It is clear that \( T \) is an ICS mapping. \( F \) has mixed monotone property and continuous.

Set \( k = \frac{1}{2} \). Taking \( x, y, z, w, u, v, p, q \in X \) for which \( x \leq u, y \geq v, z \leq p \) and \( w \geq q \), we have

\[
d(TF(x, y, z, w), TF(u, v, p, q)) = \frac{1}{12} | \ln x + \ln z - 2 \ln y - 2 \ln w | - (\ln u + \ln p - 2 \ln v - 2 \ln q) |
\leq \frac{1}{12} | \ln x + \ln u | + \frac{1}{6} | \ln y + \ln v | + \frac{1}{6} | \ln z + \ln p | + \frac{1}{6} | \ln w + \ln q |
\leq \frac{1}{6} \left( | \ln x - \ln u | + | \ln y - \ln v | + | \ln z - \ln p | + | \ln w - \ln q | \right).
\]

\[
k = \frac{k}{3} (d(Tx, Tu) + d(Ty, Tv) + d(Tz, Tp) + d(Tw, Tq))
\]

which is the contractive condition (2.1). Moreover, taking \( x_0 = z_0 = 1 \) and \( y_0 = w_0 = 64 \), we have

\[
x_0 \leq F(x_0, y_0, z_0, w_0), \quad F(y_0, z_0, w_0, x_0), \quad z_0 \leq F(z_0, w_0, x_0, y_0), \quad w_0 \geq F(w_0, Sx_0, y_0, z_0)
\]
Therefore all the conditions of Corollary (2.5) hold and \((8,8,8,8)\) is the unique quadruple fixed point of \(F\), since also the hypotheses of Theorem 2.7 hold.

Finally following example shows that if \(T\) is not an ICS mapping then the conclusion of the Theorem 2.2 fails.

**Example 3.2:**- Let \(X = \mathbb{R}\) with the usual metric and the usual ordering. Let \(F: X^4 \rightarrow X\) be defined by
\[
F(x, y, z, w) = 2x - y + 2z - w + 1, \quad \text{for all } x, y, z, w \in X
\]
then \(F\) has the mixed monotone property and \(F\) is continuous. Also, there exists \(x_0 = 1, y_0 = 0, z_0 = 1\) and \(w_0 = 0\) such that
\[
x_0 \leq F(x_0, y_0, z_0, w_0), \quad y_0 \geq F(y_0, z_0, w_0, x_0), \quad z_0 \leq F(z_0, w_0, x_0, y_0), \quad w_0 \geq F(w_0, Sx_0, y_0, z_0)
\]
Let \(T: X \rightarrow X\) be defined by \(T(x) = 1\) for all \(x \in X\), then \(T\) is not an ICS mapping. It is obvious that the condition (2.1) holds for \(\phi \in \phi\). However, \(F\) has no quadruple fixed point.

**References**

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