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Quadruple Fixed Point Theorems in Partially Ordered Metric Spaces Depended on Another Function

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Abstract
In this article, we introduced concept of ICS mapping for quadruple fixed point in partially ordered metric space. The present results generalized the result of Karapinar E. [23] also we state some examples showing that our results are effective.
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Introduction and Preliminaries

The Banach contraction principle, which is the most famous metrical fixed point theorem, play a very important role in nonlinear analysis and its applications are well known. Many authors have extended this theorem, including more general contractive conditions, which imply the existence of a fixed point. Existence of fixed points in ordered metric spaces was investigated in 2004 by Ran and Reurings [17] and then by Nieto and Lopez [16]. After this various results in have been obtained in this direction, see e.g. [1,15,18].

Bhaskar and Lakshmikantham [8] introduced the concept of a coupled fixed point of mapping $F: X\times X \rightarrow X$ and investigated some coupled fixed point theorems in partially ordered metric spaces. Later, various results in coupled fixed point have been obtained, see e.g. [2, 3, 4,5,6,10,11,12,13,14,15].

On the other hand, Berinde and Borcut [9] introduced the concept of triple fixed point and proof some related fixed point theorem. After this various results on tripled fixed point have been obtaind .

Further studied by Nieto and Rodriguez - Lopez [16], Samet and Vetro [19] introduced the notion of fixed point of N order in case of single-valued mappings. In particular for N = 4 (Quadruple case), i.e.,Let $(X,\leq)$ be partially ordered set and (X,d) be a complete metric space. We consider the following partial order on the product space $X^4 = X\times X\times X\times X$:

$$(u,v,r,t) \leq (x,y,z,w) \text{ iff } x\geq u,\ y\leq v,\ z\geq r,\ t\leq w,$$

where $(u,v,r,t),(x,y,z,w) \in X^4$.

Regarding this partial order, Karapinar E. [23] introduced the concept of Quadruple fixed point and prove some new fixed point theorems. In [23] Karapinar E. defined the following concept of quadruple fixed point.

Definition1.1 :- let $(X,\leq)$ be a partially ordered set, $F: X^4 \rightarrow X$ mapping. The mapping F is said to have the mixed monotone property if for any $x,y,z,w \in X$.

i. $x_1, x_2 \in X,\ x_1 \leq x_2 \rightarrow F(x_1,y,z,w) \leq F(x_2,y,z,w)$,
ii. $y_1, y_2 \in X,\ y_1 \geq y_2 \rightarrow F(x,y_1,z,w) \geq F(x,y_2,z,w)$,
iii. $z_1, z_2 \in X,\ z_1 \leq z_2 \rightarrow F(x,y,z_1,w) \leq F(x,y,z_2,w)$
iv. $w_1, w_2 \in X,\ w_1 \geq w_2 \rightarrow F(x,y,z,w_1) \geq F(x,y,z,w_2)$.

Definition 1.2:- An element $(x,y,z,w) \in X^4$ is called a quadruple fixed point of $F: X^4 \rightarrow X$ if

$$F(x,y,z,w) = x,\ F(y,z,w,x) = y,\ F(z,w,x,y) = z \text{ and } F(w,x,y,z) = w.$$

In this paper, we give some quadruple fixed point theorems for mapping having the mixed monotone property in partially ordered metric spaces depended on another function.

Main Results

Definition 2.1:- Let (X,d) be a metric space. A mapping $T: X \rightarrow X$ is said to be ICS if T is injective, continuous and has the property: for every sequence $\{x_n\}$ in X, if $\{Tx_n\}$ is convergent then $\{x_n\}$ is also convergent.

Let ϕ be the set of all functions $\phi : [0,\infty) \rightarrow [0,\infty)$ such that

i. ϕ is non- decreasing,
ii. $\phi(t) < t$ for all $t > 0$,
iii. $\lim_{r\rightarrow t^+} \phi(r) < t$ for all $t > 0$

From now on, we denote $X^4 = X\times X\times X\times X$. Our first result is given by the following:

Theorem 2.2:- Let $(X,\leq)$ be a partially ordered set and suppose there is a metric d in X such that (X,d) is a complete metric space. Suppose $T: X \rightarrow X$ is a ICS mapping and $F: X^4 \rightarrow X$ is such that F has the mixed monotone property. Assume that there exists $\phi \in \Phi$ such that

$$d\big(TF(x,y,z,w),TF(u,v,p,q)\big) \leq \phi\,(\max\{d(Tx,Tu),d(Ty,Tv),d(Tz,Tp),d(w,Tq)\}) \quad 2.1$$

for any $x, y, z, w \in X$ for which $x \leq u, v \leq y, z \leq p, q \leq w$. Suppose either

i. F is continuous, or

ii. X has the following property:

(a) if non decreasing sequence $x_n \to x$ (respectively, $z_n \to z$), then $x_n \leq x$, (respectively, $z_n \leq z$) for all n,

(b) if non increasing sequence $y_n \to x$ (respectively, $w_n \to z$), then $y_n \geq y$, (respectively, $w_n \geq w$) for all n.

If there exists $x_0, y_0, z_0, w_0 \in X$ such that $x_0 \geq F(x_0, y_0, z_0, w_0), y_0 \leq F(y_0, z_0, w_0, x_0), z_0 \leq F(z_0, w_0, x_0, y_0)$ and $w_0 \geq F(w_0, x_0, y_0, z_0)$, then there exist $x, y, z, w \in X$ such that

$$x = F(x, y, z, w), \quad y = F(y, z, w, x),$$
$$z = F(z, w, x, y), \quad w = F(w, x, y, z)$$

that is, F has a quadrupled fixed point.

Proof: Let $x_0, y_0, z_0, w_0 \in X$ such that $x_0 \geq F(x_0, y_0, z_0, w_0), y_0 \leq F(y_0, z_0, w_0, x_0), z_0 \leq F(z_0, w_0, x_0, y_0)$ and $w_0 \geq F(w_0, Sx_0, y_0, z_0)$ set

$$x_1 = F(x_0, y_0, z_0, w_0), \quad y_1 = F(y_0, z_0, w_0, x_0)$$
$$z_1 = F(z_0, w_0, x_0, y_0), \quad w_1 = F(w_0, Sx_0, y_0, z_0) \qquad 2.2$$

Continuing this process, we can construct sequences $\{x_n\}, \{y_n\}, \{z_n\}$ and $\{w_n\}$ in X such that

$$x_{n+1} = F(x_n, y_n, z_n, w_n) \quad y_{n+1} = F(y_n, z_n, w_n, x_n)$$
$$z_{n+1} = F(z_n, w_n, x_n, y_n) \quad w_{n+1} = F(w_n, Sx_n, y_n, z_n) \qquad 2.3$$

Since F has the mixed monotone property, then using the mathematical induction it is easy that

$$x_n \leq x_{n+1}, \quad y_n \geq y_{n+1}, z_n \leq z_{n+1}, \quad w_n \geq w_{n+1} \qquad 2.4$$

for $n = 0,1,2,3, \ldots$.

Assume for some $n \in N$

$$x_n = x_{n+1}, \quad y_n = y_{n+1}, \quad z_n = z_{n+1}, \quad w\_n = w_{n+1} \qquad 2.5$$

then by (2.3), (x_n, y_n, z_n, w_n) is the quadrupled fixed point of F. From now on, assume for any $n \in N$ that atleast,

$$x_n \neq x_{n+1}, \quad y_n \neq y_{n+1}, z_n \neq z_{n+1}, \quad w_n \neq w_{n+1} \qquad 2.6$$

Since T is injuctive, then by (2.6), for any $n \in N$

$$0 \leq \phi(\max\{d(Tx_{n+1}, Tx_n), d(Ty_{n+1}, Ty_n), d(Tz_{n+1}, Tz_n), d(w_{n+1}, Tw_n)\}) \quad 2.7$$

in the account of (2.1) and (2.3), we have

$$d(Tx_n, Tx_{n+1}) = d\big(TF(x_{n-1}, y_{n-1}, z_{n-1}, w_{n-1}), TF(x_n, y_n, z_n, w_n)\big)$$
$$\leq \phi(\max\{d(Tx_{n-1}, Tx_n), d(Ty_{n-1}, Ty_n), d(Tz_{n-1}, Tz_n), d(w_{n-1}, Tw_n)\})$$

2.8

$$d(Ty_n, Ty_{n+1}) = d\big(TF(y_{n-1}, z_{n-1}, w_{n-1}, x_{n-1}), TF(y_n, z_n, w_n, x_n)\big)$$
$$\leq \phi(\max\{d(Tx_{n-1}, Tx_n), d(Ty_{n-1}, Ty_n), d(Tz_{n-1}, Tz_n), d(w_{n-1}, Tw_n)\}) \qquad 2.9$$

$$d(Tz_n, Tz_{n+1}) = d\big(TF(z_{n-1}, w_{n-1}, x_{n-1}, y_{n-1}), TF(z_n, w_n, x_n, y_n)\big)$$
$$\leq \phi(\max\{d(Tx_{n-1}, Tx_n), d(Ty_{n-1}, Ty_n), d(Tz_{n-1}, Tz_n), d(w_{n-1}, Tw_n)\}) \qquad 2.10$$

and

$$d(Tw_n, Tw_{n+1}) = d\big(TF(w_{n-1}, x_{n-1}, y_{n-1}, z_{n-1}), TF(w_n, x_n, y_n, z_n)\big)$$
$$\leq \phi(\max\{d(Tx_{n-1}, Tx_n), d(Ty_{n-1}, Ty_n), d(Tz_{n-1}, Tz_n), d(w_{n-1}, Tw_n)\}) \qquad 2.11$$

Since we have $\phi(t) < t$ for all $t > 0$, so from (2.8)-(2.11) we obtain that

$$0 < max\{d(Tx_n, Tx_{n+1}), d(Ty_n, Ty_{n+1}), d(Tz_n, Tz_{n+1}), d(Tw_n, Tw_{n+1})\}$$
$$\leq \phi(\max\{d(Tx_{n-1}, Tx_n), d(Ty_{n-1}, Ty_n), d(Tz_{n-1}, Tz_n), d(w_{n-1}, Tw_n)\}) \quad 2.12$$
$$\leq \max\{d(Tx_{n-1}, Tx_n), d(Ty_{n-1}, Ty_n), d(Tz_{n-1}, Tz_n), d(w_{n-1}, Tw_n)\}$$

It follows that

$$\max\{d(Tx_n, Tx_{n+1}), d(Ty_n, Ty_{n+1}), d(Tz_n, Tz_{n+1}), d(Tw_n, Tw_{n+1})\}$$
$$< max\{d(Tx_{n-1}, Tx_n), d(Ty_{n-1}, Ty_n), d(Tz_{n-1}, Tz_n), d(w_{n-1}, Tw_n)\}$$

Thus, $\{\max\{d(Tx_n, Tx_{n+1}), d(Ty_n, Ty_{n+1}), d(Tz_n, Tz_{n+1}), d(Tw_n, Tw_{n+1})\}\}$ is positive decreasing sequence. Hence, there exists $r \geq 0$ such that

$$\lim_{n \to +\infty} \max\{d(Tx_n, Tx_{n+1}), d(Ty_n, Ty_{n+1}), d(Tz_n, Tz_{n+1}), d(Tw_n, Tw_{n+1})\} = r$$

Suppose that $r > 0$. Letting $n \to +\infty$ in (2.12), we obtain that

$$0 < r < lim_{n \to +\infty} \phi(\max\{d(Tx_{n-1}, Tx_n), d(Ty_{n-1}, Ty_n), d(Tz_{n-1}, Tz_n), d(w_{n-1}, Tw_n)\}) \qquad 2.13$$

it is a contradiction. We deduce that

$$\lim_{n \to +\infty} \max\{d(Tx_n, Tx_{n+1}), d(Ty_n, Ty_{n+1}), d(Tz_n, Tz_{n+1}), d(Tw_n, Tw_{n+1})\} = 0 \qquad 2.14$$

We shall show that $\{Tx_n\}, \{Ty_n\}, \{Tz_n\}$ and $\{Tw_n\}$ are Chauchy sequences. Assume the contrary, that is $\{Tx_n\}, \{Ty_n\}, \{Tz_n\}$ and $\{Tw_n\}$ are not a Chauchy sequence. that is

$$\lim_{n,m\to+\infty} d(Tx_m, Tx_n) \neq 0, \ \lim_{n,m\to+\infty} d(Ty_m, Ty_n) \neq 0$$
$$\lim_{n,m\to+\infty} d(Tz_m, Tz_n) \neq 0, \ \lim_{n,m\to+\infty} d(Tw_m, Tw_n) \neq 0$$

This means that there exists $\epsilon > 0$ for which we can find subsequences of integers (m_k) and (n_k) with $n_k > m_k > k$ such that

$$\max\{ d(Tx_{n_k}, Tx_{m_k}), d(Ty_{n_k}, Ty_{m_k}), d(Tz_{n_k}, Tz_{m_k}), d(Tw_{n_k}, Tw_{m_k}) \} \geq \epsilon \quad 2.15$$

Further corresponding to m_k we can choose n_k in such a way that it is the smallest integer with $n_k > m_k$ and satisfying (2.15). Then

$$\max\{ d(Tx_{n_k-1}, Tx_{m_k}), d(Ty_{n_k-1}, Ty_{m_k}), d(Tz_{n_k-1}, Tz_{m_k}), d(Tw_{n_k-1}, Tw_{m_k}) \} < \epsilon \quad 2.16$$

By triangular inequality and (2.16), we have

$$d(Tx_{m_k}, Tx_{n_k}) \leq d(Tx_{m_k}, Tx_{n_k-1}) + d(Tx_{n_k-1}, Tn_{n_k})$$
$$< \epsilon + d(Tx_{n_k-1}, Tn_{n_k}) \quad 2.17$$

Thus, by (2.14) we obtain

$$\lim_{k\to+\infty} d(Tx_{m_k}, Tx_{n_k}) \leq \lim_{k\to+\infty} d(Tx_{m_k}, Tx_{n_k-1}) \leq \epsilon \quad 2.18$$

Similarly, we have

$$\lim_{k\to+\infty} d(Ty_{m_k}, Ty_{n_k}) \leq \lim_{k\to+\infty} d(Ty_{m_k}, Ty_{n_k-1}) \leq \epsilon \quad 2.19$$
$$\lim_{k\to+\infty} d(Tz_{m_k}, Tz_{n_k}) \leq \lim_{k\to+\infty} d(Tz_{m_k}, Tz_{n_k-1}) \leq \epsilon \quad 2.20$$
$$\lim_{k\to+\infty} d(Tw_{m_k}, Tw_{n_k}) \leq \lim_{k\to+\infty} d(Tw_{m_k}, Tw_{n_k-1}) \leq \epsilon \quad 2.21$$

Again by (2.16), we have

$$d(Tx_{m_k}, Tx_{n_k}) \leq d(Tx_{m_k}, Tx_{m_k-1}) + d(Tx_{m_k-1}, Tx_{n_k-1}) + d(Tx_{n_k-1}, Tn_{n_k})$$
$$< d(Tx_{m_k}, Tx_{m_k-1}) + d(Tx_{m_k-1}, Tx_{m_k})$$
$$+ d(Tx_{m_k}, Tx_{n_k-1}) + d(Tx\_(n\_k-1), Tn\_(n\_k))$$
$$< d(Tx_{m_k}, Tx_{m_k-1}) + d(Tx\_(m\_k-1), Tx\_(m\_k))$$
$$+ \epsilon + d(Tx_{n_k-1}, Tn_{n_k})$$

Letting $k \to +\infty$ and using (2.14), we get

$$\lim_{k\to+\infty} d(Tx_{m_k}, Tx_{n_k}) \leq \lim_{k\to+\infty} d(Tx_{m_k-1}, Tx_{n_k-1}) \leq \epsilon \quad 2.22$$

Similarly, we have

$$\lim_{k\to+\infty} d(Ty_{m_k}, Ty_{n_k}) \leq \lim_{k\to+\infty} d(Ty_{m_k-1}, Ty_{n_k-1}) \leq \epsilon \quad 2.23$$
$$\lim_{k\to+\infty} d(Tz_{m_k}, Tz_{n_k}) \leq \lim_{k\to+\infty} d(Tz_{m_k-1}, Tz_{n_k-1}) \leq \epsilon \quad 2.24$$
$$\lim_{k\to+\infty} d(Tw_{m_k}, Tw_{n_k}) \leq \lim_{k\to+\infty} d(Tw_{m_k-1}, Tw_{n_k-1}) \leq \epsilon \quad 2.25$$

Using (2.15) and (2.22) - (2.25), we have

$$\lim_{k\to+\infty} \max\{ d(Tx_{m_k}, Tx_{n_k}), d(Ty_{m_k}, Ty_{n_k}), d(Tz_{m_k}, Tz_{n_k}), d(Tw_{m_k}, Tw_{n_k})\}$$
$$=$$
$$\lim_{k\to+\infty} \max\{ d(Tx_{m_k-1}, Tx_{n_k-1}), d(Ty_{m_k-1}, Ty_{n_k-1}), d(Tz_{m_k-1}, Tz_{n_k-1}), d(Tw_{m_k-1}, Tw_{n_k-1})\}$$
$$= \epsilon \quad 2.26$$

Now, using inequality (2.1) we obtain

$$d(Tx_{m_k}, Tx_{n_k}) = d(TF(x_{m_k-1}, y_{m_k-1}, z_{m_k-1}, w_{m_k-1}), TF(x_{n_k-1}, y_{n_k-1}, z_{n_k-1}, w_{n_k-1})$$
$$\leq \phi\left(\max\{ d(Tx_{m_k-1}, Tx_{n_k-1}), d(Ty_{m_k-1}, Ty_{n_k-1}), d(Tz_{m_k-1}, Tz_{n_k-1}), d(w_{m_k-1}, Tw_{n_k-1})\}\right) \quad 2.27$$
$$d(Ty_{m_k}, Ty_{n_k}) = d(TF(y_{(m_k)-1}, z_{(m_k)-1}, w_{(m_k)-1}, x_{n-1}), TF(y_{n_k-1}, z_{n_k-1}, w_{n_k-1}, x_{n_k-1})$$
$$\leq \phi\left(\max\{ d(Tx_{m_k-1}, Tx_{n_k-1}), d(Ty_{m_k-1}, Ty_{n_k-1}), d(Tz_{m_k-1}, Tz_{n_k-1}), d(w_{m_k-1}, Tw_{n_k-1})\}\right) \quad 2.28$$

$$d(Tz_{m_k}, Tz_{n_k}) = d\left(TF(z_{m_k-1}, w_{m_k-1}, x_{m_k-1}, y_{m_k-1}), TF(z_{n_k-1}, w_{n_k-1}, x_{n_k-1}, y_{n_k-1})\right)$$
$$\leq \phi\left(\max\{ d(Tx_{m_k-1}, Tx_{n_k-1}), d(Ty_{m_k-1}, Ty_{n_k-1}), d(Tz_{m_k-1}, Tz_{n_k-1}), d(w_{m_k-1}, Tw_{n_k-1})\}\right) \quad 2.29$$

and

$$d(Tw_{m_k}, Tw_{n_k}) = d\left(TF(w_{m_k-1}, x_{m_k-1}, y_{m_k-1}, z_{m_k-1}), TF(w_{n_k-1}, x_{n_k-1}, y_{n_k-1}, z_{n_k-1})\right)$$
$$\leq \phi\left(\max\{ d(Tx_{m_k-1}, Tx_{n_k-1}), d(Ty_{m_k-1}, Ty_{n_k-1}), d(Tz_{m_k-1}, Tz_{n_k-1}), d(w_{m_k-1}, Tw_{n_k-1})\}\right) \quad 2.30$$

We deduce from (2.27) - (2.30) that

$$\max\{ d(Tx_{m_k}, Tx_{n_k}), d(Ty_{m_k}, Ty_{n_k}), d(Tz_{m_k}, Tz_{n_k}), d(Tw_{m_k}, Tw_{n_k})\} \leq$$
$$\phi\left(\max\{ d(Tx_{m_k-1}, Tx_{n_k-1}), d(Ty_{m_k-1}, Ty_{n_k-1}), d(Tz_{m_k-1}, Tz_{n_k-1}), d(w_{m_k-1}, Tw_{n_k-1})\}\right)$$

2.31

Letting $k \to +\infty$ in (2.31) and having in mind (2.16), we get that

$$0 < \epsilon \leq lim_{t\to\epsilon^+} \phi(t) < \epsilon$$

it is a contradiction. Thus $\{Tx_n\}, \{Ty_n\}, \{Tz_n\}$ and $\{Tw_n\}$ are Chauchy sequences in (X, d). Since X is complete metric space, $\{Tx_n\}, \{Ty_n\}, \{Tz_n\}$ and $\{Tw_n\}$ are convergent sequences.

Since T is an ICS mapping, there exist $x, y, z, w \in X$ such that

$$\lim_{n\to+\infty} x_n = x, \lim_{n\to+\infty} y_n = y, \lim_{n\to+\infty} z_n = z, \lim_{n\to+\infty} w_n = w. \quad 2.32$$

Since T is continuous, we have

$$\lim_{n\to+\infty} Tx_n = Tx, \lim_{n\to+\infty} Ty_n = Ty, \lim_{n\to+\infty} Tz_n = Tz, \lim_{n\to+\infty} Tw_n = Tw. \quad 2.33$$

Suppose now the assumption (a) holds, that is, F is continuous. By (2.3), (2.32) and (2.33) we obtain

$$x = \lim_{n\to+\infty} x_{n+1} = \lim_{n\to+\infty} F(x_n, y_n, z_n, w_n)$$
$$= F(\lim_{n\to+\infty} x_n, \lim_{n\to+\infty} y_n, \lim_{n\to+\infty} z_n, \lim_{n\to+\infty} w_n) = F(x, y, z, w)$$
$$y = \lim_{n\to+\infty} y_{n+1} = \lim_{n\to+\infty} F(y_n, z_n, w_n, x_n)$$
$$= F(\lim_{n\to+\infty} y_n, \lim_{n\to+\infty} z_n, \lim_{n\to+\infty} w_n, \lim_{n\to+\infty} x_n) = F(y, z, w, x)$$
$$z = \lim_{n\to+\infty} z_{n+1} = \lim_{n\to+\infty} F(z_n, w_n, x_n, y_n)$$
$$= F(\lim_{n\to+\infty} z_n, \lim_{n\to+\infty} w_n, \lim_{n\to+\infty} x_n, \lim_{n\to+\infty} y_n) = F(z, w, x, y)$$

and

$$w = \lim_{n\to+\infty} w_{n+1} = \lim_{(n\to+\infty)} F(w_n, x_n, y_n, z_n)$$
$$= F(\lim_{n\to+\infty} w_n, \lim_{n\to+\infty} x_n, \lim_{n\to+\infty} y\_n, \lim_{n\to+\infty} z_n = F(w, x, y, z)$$

We have proved that F has a quadrupled fixed point.

Suppose now the assumption (b) holds. Since $\{x_n\}$, $\{z_n\}$ are non- decreasing with $x_n \to x, z_n \to z$ and $\{y_n\}$, $\{w_n\}$ are non- increasing with $y_n \to y, w_n \to w$ then we have

$$x_n \leq x, \ y_n \geq y, z_n \leq z, \ w_n \geq w$$

for all n. Consider now

$$d\big(Tx, TF(x, y, z, w)\big) \leq d(Tx, Tx_{n+1}) + d\big(Tx_{n+1}, TF(x, y, z, w)\big)$$
$$= d(Tx, Tx_{n+1}) + d\big(TF(x_n, y_n, z_n, w_n), TF(x, y, z, w)\big)$$
$$\leq d(Tx, Tx_{n+1}) + \phi\,(\max\{d(Tx_n, Tx), d(Ty_n, Ty), d(Tz_n, Tz), d(Tw_n, Tw)\}) \quad 2.34$$

Taking as $n \to \infty$ and using (2.33), the right hand side of (2.34) tends to 0, so we get that $d(Tx, TF(x, y, z, w)) = 0$. Thus $Tx = TF(x, y, z, w)$ and T is injective, we get that x = F(x,y,z,w). Similarly we find that

$$y = F(y, z, w, x), \ z = F(z, w, x, y) \text{ and } w = F(w, x, y, z)$$

Thus we proved that F has a quadruple fixed point. This complete proof of the Theorem 2.2.

Corollary 2.3:- Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d in X such that (X, d) is a complete metric space. Suppose $T: X \to X$ is a ICS mapping and $F: X^4 \to X$ is such that F has the mixed monotone property. Assume that there exists $\phi \in \phi$ such that

$$d\big(TF(x, y, z, w), TF(u, v, p, q)\big) \leq \phi\left(\frac{d(Tx,Tu)+ d(Ty,Tv)+ d(Tz,Tp)+ d(w,Tq)}{4}\right) \quad 2.35$$

for any $x, y, z, w \in X$ for which $x \leq u, v \leq y, z \leq p, \ q \leq w$. Suppose either

i. F is continuous, or
ii. X has the following property:
(a) if non decreasing sequence $x_n \to x$ (respectively, $z_n \to z$), then $x_n \leq x$, (respectively, $z_n \leq z$) for all n,
(b) if non increasing sequence $y_n \to y$ (respectively, $w_n \to w$), then $y_n \leq y$, (respectively, $w_n \geq w$) for all n.

If there exists $x_0, y_0, z_0, w_0 \in X$ such that $x_0 \geq F(x_0, y_0, z_0, w_0), y_0 \leq F(y_0, z_0, w_0, x_0), z_0 \leq F(z_0, w_0, x_0, y_0)$ and w_0 $\geq F(w_0, Sx_0, y_0, z_0)$, then there exist $x, y, z, w \in X$ such that

$$x = F(x, y, z, w), \ y = F(y, z, w, x), \ z = F(z, w, x, y) \text{ and } w = F(w, x, y, z)$$

that is, F has a quadrupled fixed point.

Proof:- It suffices to remark that

$$\frac{d(Tx,Tu)+ d(Ty,Tv)+ d(Tz,Tp)+ d(w,Tq)}{4} \leq \max\{d(Tx, Tu), d(Ty, Tv), d(Tz, Tp), d(w, Tq)\} \quad 2.36$$

Then, we apply Theorem 2.2 because that ϕ us non decreasing.

Corollary 2.4:- Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d in X such that (X, d) is a complete metric space. Suppose $T: X \to X$ is a ICS mapping and F: X^4 → X is such that F has the mixed monotone property. Assume that there exists $k \in [0,1)$ such that

$$d\big(TF(x, y, z, w), TF(u, v, p, q)\big) \leq k \max\{d(Tx, Tu), d(Ty, Tv), d(Tz, Tp), d(w, Tq)\} \quad 2.37$$

for any $x, y, z, w \in X$ for which $x \leq u, v \leq y, z \leq p, \ q \leq w$. Suppose either

i. F is continuous, or
ii. X has the following property:

(a) if non decreasing sequence $x_n \to x$ (respectively, $z_n \to z$), then $x_n \le x$, (respectively, $z_n \le z$) for all n,
(b) if non increasing sequence $y_n \to y$ (respectively, $w_n \to w$), then $y_n \le y$, (respectively, $w_n \ge w$) for all n.

If there exists $x_0, y_0, z_0, w_0 \in X$ such that $x_0 \ge F(x_0, y_0, z_0, w_0), y_0 \le F(y_0, z_0, w_0, x_0), z_0 \le F(z_0, w_0, x_0, y_0)$ and $w_0 \ge F(w_0, Sx_0, y_0, z_0)$, then there exist $x, y, z, w \in X$ such that

$$x = F(x, y, z, w), \quad y = F(y, z, w, x), \quad z = F(z, w, x, y) \text{ and } w = F(w, x, y, z)$$

that is, F has a quadrupled fixed point.

Proof:- It suffices if we take $\phi(t) = kt$ in Theorem 2.2.

Corollary 2.5:- Let $(X, \le)$ be a partially ordered set and suppose there is a metric d in X such that (X, d) is a complete metric space. Suppose $T: X \to X$ is a ICS mapping and $F: X^4 \to X$ is such that F has the mixed monotone property. Assume that there exists $k \in [0,1)$ such that

$$d\big(TF(x, y, z, w), TF(u, v, p, q)\big) \le \frac{k}{4}\big(d(Tx, Tu) + d(Ty, Tv) + d(Tz, Tp) + d(w, Tq)\big) \quad 2.38$$

for any $x, y, z, w \in X$ for which $x \le u, v \le y, z \le p, q \le w$. Suppose either

i. F is continuous, or
ii. X has the following property:
(a) if non decreasing sequence $x_n \to x$ (respectively, $z_n \to z$), then $x_n \le x$, (respectively, $z_n \le z$) for all n,
(b) if non increasing sequence $y_n \to y$ (respectively, $w_n \to w$), then $y_n \le y$, (respectively, $w_n \ge w$) for all n.

If there exists $x_0, y_0, z_0, w_0 \in X$ such that $x_0 \ge F(x_0, y_0, z_0, w_0), y_0 \le F(y_0, z_0, w_0, x_0), z_0 \le F(z_0, w_0, x_0, y_0)$ and $w_0 \ge F(w_0, Sx_0, y_0, z_0)$, then there exist $x, y, z, w \in X$ such that

$$x = F(x, y, z, w), \ y = F(y, z, w, x), \ z = F(z, w, x, y) \text{ and } w = F(w, x, y, z)$$

that is, F has a quadrupled fixed point.

Proof:- It follows by taking $\phi(t) = kt$ in Corollary 2.3.

Now, we shall prove the existence and uniqueness of a quadruple fixed point, for a product X^4 of a partially ordered set $(X, \le)$, we define a partial ordering in the following way: for all $(x, y, z, w), (u, v, p, q) \in X^4$

$$(x, y, z, w) \le (u, v, p, q) \to x \le u, \ y \ge v, \ z \le p \text{ and } w \ge q \quad 2.39$$

We say that $(x, y, z, w), (u, v, p, q) \in X^4$ are comparable if

$$(x, y, z, w) \le (u, v, p, q) \text{ or } (x, y, z, w) \ge (u, v, p, q) \quad 2.40$$

Also we say that (x, y, z, w) is equal to (u, v, p, q) if and only if $x = u, y = v, z = p, w = q$.

Theorem 2.6:- In addition to hypothesis of Theorem 2.2., suppose that for all $(x, y, z, w), (u, v, p, q) \in X^4$, there exists $(a, b, c, e) \in X^4$ such that

$$F(a, b, c, e), F(b, c, e, a), F(c, e, a, b), F(e, a, b, c))$$

is comparable to

$$(F(x, y, z, w), F(y, z, w, x), F(z, w, x, y), F(w, x, y, z))$$

and

$$\big(F(u, v, p, q), F(v, p, q, u), F(p, q, u, v), F(q, u, v, p)\big).$$

Then, F has a unique quadruple fixed point (x, y, z, w).

Proof:- The set of quadruple fixed points of F is non empty due to Theorem --. Assume, now, $(x, y, z, w), (u, v, p, q) \in X^4$ are two quadruple fixed points of F, that is,

$$F(x, y, z, w) = x, \ F(u, v, p, q) = u, F(y, z, w, x) = y, \ F(v, p, q, u) = v$$
$$F(z, w, x, y) = z, \ F(p, q, u, v) = p, F(w, x, y, z) = w, \ F(q, u, v, p) = q \quad 2.41$$

We shall show that (x, y, z, w) and (u, v, p, q) are equal. By assumption, there exists $(a, b, c, d) \in X^4$ such that

$$F(a, b, c, e), F(b, c, e, a), F(c, e, a, b), F(e, a, b, c))$$

is comparable to

$$F(x, y, z, w), F(y, z, w, x), F(z, w, x, y), F(w, x, y, z))$$

and

$$(F(u, v, p, q), F(v, p, q, u), F(p, q, u, v), F(q, u, v, p).$$

Define sequences $\{a_n\}, \{b_n\}, \{c_n\}$ and $\{e_n\}$ such that

$$a_0 = a, \quad b_0 = b, \quad c_0 = c \text{ and } e_0 = e$$

and for any $n \ge 1$

$$a_n = F(a_{n-1}, b_{n-1}, c_{n-1}, e_{n-1}), b_n = F(b_{n-1}, c_{n-1}, e_{n-1}, a_{n-1}),$$
$$c_n = F(c_{n-1}, e_{n-1}, a_{n-1}, b_{n-1}), e_n = F(e_{n-1}, a_{n-1}, b_{n-1}, c_{n-1}) \quad 2.42$$

for all n. Further, set $x_0 = x, y_0 = y, z_0 = z, w_0 = w$ and $u_0 = u, v_0 = v, p_0 = p, q_0 = q$, and on the same way define the sequences $\{x_n\}, \{y_n\}, \{z_n\}, \{w_n\}$ and $\{u_n\}, \{v_n\}, \{p_n\}, \{q_n\}$. Then it is easy that

$$x_n = F(x,y,z,w),\ u_n = F(u,v,p,q), y_n = F(y,z,w,x),\ v_n = F(v,p,q,u),$$
$$z_n = F(z,w,x,y),\ p_n = F(p,q,u,v), w_n = F(w,x,y,z),\ q_n = F(q,u,v,p) \quad 2.43$$

for all $n \geq 1$. Since $\{x_n\}, \{y_n\}, \{z_n\}, \{w_n\} = (x_1, y_1, z_1, w_1) = (x, y, z, w)$ is comparable to $\big(F(a,b,c,e), F(b,c,e,a), F(c,e,a,b), F(e,a,b,c)\big) = (a_1, b_1, c_1, e_1)$, then it is easy to show $(x,y,z,w) \leq (a_1, b_1, c_1, e_1)$. Recursively, we get that

$$(x,y,z,w) \leq (a_n, b_n, c_n, e_n) \text{ for all } n \geq 1 \quad 2.44$$

By (2.44) and (2.1) we have

$$d(Tx, Ta_{n+1}) = d\big(TF(x,y,z,w), TF(a_n,b_n,c_n,e_n)\big)$$
$$\leq \phi\,(\max\{d(Tx,Ta_n), d(Ty,Tb_n), d(Tz,Tc_n), d(Tw,Te_n)\}) \quad 2.45$$
$$d(Ty, Tb_{n+1}) = d\big(TF(y,z,w,x), TF(b_n,c_n,e_n,a_n)\big)$$
$$\leq \phi\,(\max\{d(Tx,Ta_n), d(Ty,Tb_n), d(Tz,Tc_n), d(Tw,Te_n)\}) \quad 2.46$$
$$d(Tz, Tc_{n+1}) = d\big(TF(z,w,x,y), TF(c_n,e_n,a_n,b_n)\big)$$
$$\leq \phi\,(\max\{d(Tx,Ta_n), d(Ty,Tb_n), d(Tz,Tc_n), d(Tw,Te_n)\}) \quad 2.47$$

and

$$d(Tw, Te_{n+1}) = d\big(TF(w,x,y,z), TF(e_n,a_n,b_n,c_n)\big)$$
$$\leq \phi\,(\max\{d(Tx,Ta_n), d(Ty,Tb_n), d(Tz,Tc_n), d(Tw,Te_n)\}) \quad 2.48$$

It follows from (2.45)- (2.48) that

$$\max\{d(Tx,Ta_{n+1}), d(Ty,Tb_{n+1}), d(Tz,Tc_{n+1}), d(Tw,Te_{n+1})\}$$
$$\leq \phi\,(\max\{d(Tx,Ta_n), d(Ty,Tb_n), d(Tz,Tc_n), d(Tw,Te_n)\})$$

Therefore, for each $n \geq 1$,

$$\max\{d(Tx,Ta_n), d(Ty,Tb_n), d(Tz,Tc_n), d(Tw,Te_n)\}$$
$$\leq \phi^n\,(\max\{d(Tx,Ta_0), d(Ty,Tb_0), d(Tz,Tc_0), d(Tw,Te_0)\}) \quad 2.49$$

It is known that $\phi(t) < t$ and $\lim_{r \to t^+} \phi(r) < t$ imply $\lim_{n \to \infty} \phi^{n(t)} = 0$ for each $t > 0$.
Thus, from (2.49)

$$\lim_{n \to \infty} \max\{d(Tx,Ta_n), d(Ty,Tb_n), d(Tz,Tc_n), d(Tw,Te_n)\} = 0$$

This yield that

$$\lim_{n \to \infty} d(Tx,Ta_n) = 0,\ \lim_{n \to \infty} d(Ty,Tb_n) = 0$$
$$\lim_{n \to \infty} d(Tz,Tc_n) = 0,\ \lim_{n \to \infty} d(Tw,Te_n) = 0 \quad 2.50$$

Analogously, we show that

$$\lim_{n \to \infty} d(Tu,Ta_n) = 0,\ \lim_{n \to \infty} d(Tv,Tb_n) = 0$$
$$\lim_{n \to \infty} d(Tp,Tc_n) = 0,\ \lim_{n \to \infty} d(Tq,Te_n) = 0 \quad 2.51$$

Combining (2.50) and (2.51) yields that (Tx,Ty,Tz,Tw) and (Tu,Tv,Tp.Tq) are equal. The fact that T is injective gives us x = u, y = v, z = p and w = q.
This complete prove of the Theorem 2.7.

Examples

Now we state some examples showing that our results are effective.

Example 3.1:- Let $X = [\frac{1}{2}, 64]$ with the metric $d(x,y) = |x - y|$ for all $x, y \in X$ and the usual ordering $\leq$.
Clearly, (X, d) is complete metric space.
Let $T: X \to X$ and $F: X^4 \to X$ be defined by

$$Tx = \ln x + 1 \text{ and } F(x,y,z,w) = 8\left(\sqrt{\left(\frac{xz}{yw}\right)}\right)^{\frac{1}{6}},\ \forall\, x,y,z,w \in X$$

It is clear that T is an ICS mapping, F has mixed monotone property and continuous.
Set $k = \frac{1}{2}$. Taking $x,y,z,w,u,v,p,q \in X$ for which $x \leq u, y \geq v, z \leq p$ and $w \geq q$, we have

$$d\big(TF(x,y,z,w), TF(u,v,p,q)\big) = \frac{1}{12}|(\ln x + \ln z - 2\ln y - 2\ln w) - (\ln u + \ln p - 2\ln v - 2\ln q)|$$
$$\leq \frac{1}{12}|\ln x - \ln u| + \frac{1}{6}|\ln y - \ln v| + \frac{1}{12}|\ln z - \ln p| + \frac{1}{6}|\ln w - \ln q|$$

$$\leq \frac{1}{6}(|\ln x - \ln u| + |\ln y - \ln v| + |\ln z - \ln p| + |\ln w - \ln q|)$$
$$= \frac{k}{3}\big(d(Tx,Tu) + d(Ty,Tv) + d(Tz,Tp) + d(Tw,Tq)\big)$$

which is the contractive condition (2.1). Moreover, taking $x_0 = z_0 = 1$ and $y_0 = w_0 = 64$, we have

$$x_0 \leq F(x_0,y_0,z_0,w_0),\ y_0 \geq F(y_0,z_0,w_0,x_0),\ z_0 \leq F(z_0,w_0,x_0,y_0),\ w_0 \geq F(w_0,Sx_0,y_0,z_0)$$

Therefore all the conditions of Corollary (2.5) hold and (8,8,8,8) is the unique quadruple fixed point of F, since also the hypotheses of Theorem 2.7 hold.

Finally following example shows that if T is not an ICS mapping then the conclusion of the Theorem 2.2 fails.

Example 3.2:- Let $X = R$ with the usual metric and the usual ordering. Let $F: X^4 \rightarrow X$ be defined by $F(x,y,z,w) = 2x - y + 2z - w + 1$, for all $x,y,z,w \in X$ then F has the mixed monotone property and F is continuous. Also, there exists $x_0 = 1, y_0 = 0, z_0 = 1$ and $w_0 = 0$ such that

$$x_0 \leq F(x_0,y_0,z_0,w_0),\ y_0 \geq F(y_0,z_0,w_0,x_0),\ z_0 \leq F(z_0,w_0,x_0,y_0),\ w_0 \geq F(w_0,Sx_0,y_0,z_0)$$

Let $T: X \rightarrow X$ be defined by $T(x) = 1$ for all $x \in X$, then T is not an ICS mapping. It is obvious that the condition (2.1) holds for $\phi \in \phi$. However, F has no quadruple fixed point.

References

1. Agarwal R.P., El-Gebeily M. A. and O'Regan D., Generalized contractions in partially ordered metric spaces, Applicable Analysis 87 2008, 109 - 116.
2. Aydi H., Mujahid A. and Postolache M., Coupled coincidence points for hybrid pair of mappings via mixed monotone property. J. Adv. Math. Studies 51, 118-126 2012.
3. Aydi H., Some coupled fixed point results on partial metric spaces. Int. J. Math. Math. Sci., 2011/, Article ID 647091, 11 2011.
4. Aydi H., Samet B. and Vetro C., Coupled fixed point results in cone metric spaces for ~ w-compatible mappings. Fixed Point Theory Appl. 2011/, 27 2011.
5. Aydi H., Damjanovic B., Samet B. and Shatanawi W., Coupled fixed point theorems for nonlinear contractions in partially ordered G-metric spaces. Math. Comput. Model. 54, 2443-2450 2011.
6. Aydi H., Shatanawi W. and Postolache M. Coupled fixed point results for \varphi, \psi-weakly contractive mappings in ordered G-metric spaces. Comput. Math. Appl. 63/, 298-309 2012.
7. Aydi H., Karapinar E., and Postolache M., Tripled coincidence point theorems for weak \varphi-contractions in partially ordered metric spaces., Fixed Point Theory and Applications 2012, 2012:44 doi: 10.1186/1687-1812-2012-44.
8. Bhaskar T.G. and Lakshmikantham V., Fixed point theory in partially ordered metric spaces and applications. Nonlinear Anal. 65,/ 1379 - 1393 2006.
9. Berinde V. and Borcut M., Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces. Nonlinear Anal. 7415, 4889-4897 2011.
10. Choudhury B.S., Metiya N. and Kundu A., Coupled coincidence point theorems in ordered metric spaces. Ann. Univ. Ferrara 57,/ 1-16 2011.
11. Choudhury, B.S. and Kundu A., A coupled coincidence point result in partially ordered metric spaces for compatible mappings. Nonlinear Anal. 73,/ 2524-2531 2010.
12. Karapinar E. Couple fixed point on cone metric spaces. Gazi Univ. J. Sci. 241/, 51-58 2011.
13. Karapinar E., Coupled fixed point theorems for nonlinear contractions in cone metric spaces. Comput. Math. Appl. 5912 3656-3668 2010.
14. Lakshmikantham V., Ciric Lj. B., "Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces". Nonlinear Anal. 70,/ 4341-4349 2009.
15. Luong N.V. and Thuan N.X., Coupled fixed points in partially ordered metric spaces and application. Nonlinear Anal. 74,/ 983-992 2011.
16. Nieto, J.J., Lopez, R.R., Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations. Acta Math. Sinica Engl. Ser. 23, 2205–2212 2007
17. Ran, A.C.M., Reurings, M.C.B., A fixed point theorem in partially ordered sets and some applications to matrix equations. Proc. Am. Math. Soc. 132, 1435 - 1443 2004.
18. Samet B., Coupled fixed point theorems for a generalized Meir- Keeler contraction in partially ordered metric spaces. Nonlinear Anal. 7412/, 4508-4517 2010.
19. Samet B. and Vetro C., Coupled fixed point, f-invariant set and fixed point of N-order. Ann. Funct. Anal. 12, 46-56 2010.
20. M. Abbas, H. Aydi and E. Karapinar, Tripled fixed points of multi-valued nonlinear contraction mappings in partially ordered metric spaces, Abstract and Applied Analysis, 2011, Article ID 812690, 12 pages.
21. K.P. Chi , On a fixed point theorem for certain class of maps satisfying a contractive condition depended on an another function, Lobachevskii J. Math. 30 4 2009 289-291.
22. S. Moradi and M. Omid, A fixed-point theorem for integral type inequality depending on another function, Int. J. Math. Anal. Ruse 4, 2010, 1491-1499 .
23. Karapiner E., Quadruple Fixed Point Theorems for Weak ϕ- Contraction, ISRN Math. Anal. 2011, ID 989423, 15 pages, doi:10.5402/2011/989423.