Tripled Common Fixed points for weak $(\mu, \varphi, \Psi) -$ Constractions in Partially Ordered Metric Space

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Abstract
In this article, we present tripled coincidence point theorems for F : X^3 → X and g : X → X satisfying weak (\mu, φ, \Psi)- -contractions in partially ordered metric spaces. We also provide non-trivial examples to illustrate our results and new concepts presented herein. Our results unify, generalize and complement various known comparable results from the current literature, Berinde and Borcut [22], Abbas et. al., Aydi et al. [23] and many more previous known results.
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Introduction

Fixed point theorems are very important tools for providing evidence of the existence and uniqueness of solutions to various mathematical models. The literature of the last four decades flourishes with results which discover fixed points of self and nonself nonlinear operators in a metric space. The Banach contraction theorem plays a fundamental role in fixed point theory and has become even more important because being based on iteration, it can be easily implemented on a computer. This theorem provide a technique for solving a variety of applied problems in mathematical science and engineering. There are great number of generalizations of the Banach contraction principle. Bhaskar and Lakshmikantham [1] introduced the notion of coupled fixed point and prove some coupled fixed point results under certain conditions, in a complete metric space endowed with a partial order. Later, Lakshmikantham and Ciric [2] extended these results by defining the mixed g- monotone property. More accurately, they proved coupled coincidence and coupled common fixed point theorems for a mixed g- monotone mapping in a complete metric space endowed with partial order. Karapiner [3] generalized these results on a complete cone metric space endowed with a partial order. For other results on coupled fixed point theory, we refers [4 - 14].

Beside this, in [15] Alber and Guerre - Delabriere presented the generalization of Banach contraction principle by introducing the concept of weak contraction in Hilbert spaces. Rhoades [16] had shown the result of [15] is also valid in complete metric spaces. Khan et.al. [17] introduced the use of control function in metric fixed point problems. This function was referred to as 'Altering distance function' by the authors of [17]. This function and its extensions have been used in several problems of fixed point theory, some of them are noted in [18-21]. In recent times, fixed point theory has developed rapidly in partially ordered metric spaces, that is, in metric spaces endowed with a partial ordering . Using the control functions the weak contraction principle has been generalized in metric spaces [9] and in partially ordered metric spaces in [11].

Recently, Samet and Vetro [14] introduced the notion of fixed point of N- order, as natural extension of the coupled fixed point and established some new coupled fixed point theorems in complete metric spaces, using a new concept of F- invariant set. Later, Berinde and Borcut [22] obtained existence and uniqueness of triplet fixed point results in a complete metric space, endowed with a partial order.

Now we recall come privious known definitions and results which are as follows.

Again, let $(X, \leq)$ be a partially ordered set. The mapping $F: X^3 \to X$ is said to have the mixed monotone property if for any $x, y, z \in X$.

i. $x_1, x_2 \in X,\ x_1 \leq x_2 \Rightarrow F(x_1, y, z) \leq F(x_2, y, z)$,
ii. $y_1, y_2 \in X,\ y_1 \geq y_2 \Rightarrow F(x, y_1, z) \geq F(x, y_2, z)$,
iii. $z_1, z_2 \in X,\ z_1 \leq z_2 \Rightarrow F(x, y, z_1) \leq F(x, y, z_2)$

An element $(x, y, z) \in X^3$ is called a triplet fixed point of F if
$F(x, y, z) = x,\ F(y, x, y) = y,$ and $F(z, y, x) = z.$

Berinde and Borcut [22] proved the following theorem.

Theorem 1.1:- Let $(X, \leq)$ be a partially ordered set and (X,d) be a complete metric space. Let $F: X^3 \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exist constants $a, b, c \in [0,1)$ such that $a + b + c < 1$ for which,

$$d\big(F(x, y, z), F(u, v, w)\big) \leq a\, d(x, u) + b\, d(y, v) + c\, d(z, w) \qquad 1.1$$

For all $x \geq u, y \leq v, z \geq w$. Assume either,

1. F is continuous,
2. X has the following properties:
- if non decreasing sequence $x_n \to x$, then $x_n \leq x$ for all n,
- if non increasing sequence $y_n \to y$, then $y_n \geq x$ for all n,

If there exist $x_0, y_0, z_0 \in X$ such that
$x_0 \leq F(x_0, y_0, z_0)$, $y_0 \geq F(y_0, x_0, y_0)$, and $z_0 \leq F(z_0, y_0, x_0)$
Then there exist $x, y, z \in X$ such that,

$$F(x,y,z) = x, F(y,x,y) = y, \text{and } F(z,y,x) = z$$

In [Abbas, Aydi and Krapinar, Triplet fixed point in partially ordered metric spaces, submitted]. In this respect, let $(X, \leq)$ be a partially ordered set, $F: X^3 \to X$ and $g: X \to X$ two mappings. The mapping F is said to have the mixed g - monotone property if for any x, y, z ∈ X.

i. $x_1, x_2 \in X$, $gx_1 \leq gx_2 \Rightarrow F(x_1, y, z) \leq F(x_2, y, z)$,
ii. $y_1, y_2 \in X$, $gy_1 \geq gy_2 \Rightarrow F(x, y_1, z) \geq F(x, y_2, z)$,
iii. $z_1, z_2 \in X$, $gz_1 \leq gz_2 \Rightarrow F(x, y, z_1) \leq F(x, y, z_2)$

An element $(x, y, z) \in X^3$ is called a triplet coincidence point of F and g if

$$F(x,y,z) = gx, \ F(y,x,y) = gy, \ \text{and } F(z,y,x) = gz.$$

while (gx, gy, gz) is said a triplet point of coincidence of mappings F and g. Moreover (x, y, z) is called a triplet common fixed point of F and g if

$$F(x,y,z) = gx, \ F(y,x,y) = gy, \ \text{and } F(z,y,x) = gz.$$

At last mappings F and g are called commutative if

$$g(F(x,y,z)) = F(gx, gy, gz), \ \forall\, x,y,z \in X$$

In the same paper, they proved the following result.

Theorem 1.2:- Let $(X, \leq)$ be a partially ordered set and (X, d) be a complete metric space. Assume there is a function $\varphi: [0, \infty) \to [0, \infty)$ such that $\varphi(t) < t$ for each $t > 0$. Also suppose that $F: X^3 \to X$ and $g: X \to X$ are such that F having the mixed $g -$ monotone property on X. Assume that there exist constants $a, b, c \in [0,1)$ such that $a + 2b + c < 1$ such that,

$$d(F(x,y,z), F(u,v,w)) \leq \varphi\left(a\, d(gx,gu) + b\, d(gy,gv) + c\, d(gz,gw)\right) \qquad 1.2$$

for all $gx \geq gu, gy \leq gv, gz \geq gw$.
Suppose $(X^3) \subset g(X)$, g is continuous and commutes with F. Suppose either,

1. F is continuous,
2. X has the following properties:
- if non decreasing sequence $gx_n \to x$, then $gx_n \leq x$ for all n,
- if non increasing sequence $gy_n \to y$, then $gy_n \geq y$ for all n,

If there exist $x_0, y_0, z_0 \in X$ such that

$$gx_0 \leq F(x_0, y_0, z_0), \ gy_0 \geq F(y_0, x_0, y_0), \ \text{and } gz_0 \leq F(z_0, y_0, x_0).$$

Then there exist $x, y, z \in X$ such that,

$$F(x,y,z) = gx, F(y,x,y) = gy, \text{and } F(z,y,x) = gz$$

that is, F and g have a triplet coincidence point.

In [23] Aydi et.al. prove the following theorem

Theorem 1.3:- Let $(X, \leq)$ be a partially ordered set and (X, d) be a complete metric space. Assume there is a function $\varphi: [0, \infty) \to [0, \infty)$ such that $\varphi(t) < t$ for each $t > 0$. Also suppose that $F: X^3 \to X$ and $g: X \to X$ are such that F having the mixed g - monotone property on X. Assume that there exist constants $a, b, c \in [0,1)$ such that $a + 2b + c < 1$ such that,

$$d(F(x,y,z), F(u,v,w)) + d(F(y,x,y), F(v,u,v)) + d(F(z,y,x), F(w,v,u))$$
$$\leq 3\,\varphi\left(\frac{d(gx,gu) + d(gy,gv) + d(gz,gw)}{3}\right) \qquad 1.3$$

For all $gx \geq gu, gy \leq gv, gz \geq gw$.
Suppose $F(X^3) \subset g(X)$, g is continuous and commutes with F. Suppose either,

1. F is continuous,
2. X has the following properties:
- if non decreasing sequence $gx_n \to x$, then $gx_n \leq x$ for all n,
- if non increasing sequence $gy_n \to y$, then $gy_n \geq y$ for all n,

If there exist $x_0, y_0, z_0 \in X$ such that

$$gx_0 \leq F(x_0, y_0, z_0), \ gy_0 \geq F(y_0, x_0, y_0), \ \text{and } gz_0 \leq F(z_0, y_0, x_0)$$

Then there exist $x, y, z \in X$ such that,

$$F(x,y,z) = gx, F(y,x,y) = gy, \text{and } F(z,y,x) = gz$$

that is, F and g have a triplet coincidence point.

The purpose of this paper is to present some triplet fixed point theorems for a g - monotone mapping in partially ordered metric space which are generalization of the results of Berinde and Borcut [22] and many more privious known results.

Main Results

First we give some definitions, which are use to prove of the main theorem.

Definition 2.1 :- Let Φ denote all functions $\varphi: [0,\infty) \to [0,\infty)$ which satisfy

i. φ is continuous and non decreasing,

ii. $\varphi(t) = 0$ iff $t = 0$,

iii. $\varphi(r+s+t) \le \varphi(r) + \varphi(s) + \varphi(t)\ \forall\ r,s,t \in [0,\infty)$

For example, functions $\varphi_1(t) = kt$ where $k > 0, \varphi_2(t) = \frac{t}{t+1}, \varphi_3(t) = \mathrm{In}(t+1)$, and $\varphi_4(t) = \min\{t,1\}$ are in Φ.

Definition 2.2:- Let Ψ be the set of all functions $\psi : [0,\infty) \to [0,\infty)$ which satisfy $\lim_{t\to q}\psi(t) > 0$ for all $q > 0$ and $\lim_{t\to 0} \psi(t) = 0$

For example, functions $\psi_1(t) = kt$ where $k > 0$, $\psi_2(t) = \frac{\ln(2t+1)}{2}$ are in Ψ.

Now we prove our main results.

Theorem 2.3:- Let $(X,\le)$ be a partially ordered set and (X,d) be a complete metric space. Let $F : X^3 \to X$ be a continuous mapping having the mixed g - monotone property on X and $F(X^3) \subset g(X)$. Suppose there exist $\mu, \varphi \in \Phi$, $\psi \in \Psi$ for which,

$$\mu\left(d(F(x,y,z),F(u,v,w))\right) \le \frac{1}{3}\varphi\left(d(gx,gu) + d(gy,gv) + d(gz,gw)\right) - \frac{1}{3}\psi\left(d(gx,gu) + d(gy,gv) + d(gz,gw)\right) \qquad 2.1$$

For all $gx \ge gu$, $gy \le gv$ and $gz \ge gw$.

Assume that F is continuous, g is continuous and commutes with F. If there exist $x_0, y_0, z_0 \in X$ such that

$$gx_0 \le F(x_0,y_0,z_0),\ gy_0 \ge F(y_0,x_0,y_0),\ \text{and}\ gz_0 \le F(z_0,y_0,x_0)$$

Then there exist $x,y,z \in X$ such that,

$$F(x,y,z) = gx, F(y,x,y) = gy, \text{and}\ F(z,y,x) = gz.$$

That is, F and g have a triplet coincidence point.

Proof: Let $x_0, y_0, z_0 \in X$ such that

$$gx_0 \le F(x_0,y_0,z_0),\ gy_0 \ge F(y_0,x_0,y_0),\ \text{and}\ gz_0 \le F(z_0,y_0,x_0)$$

We can choose $x_1, y_1, z_1 \in X$ such that

$$gx_1 = F(x_0,y_0,z_0),\ gy_1 = F(y_0,x_0,y_0),\ \text{and}\ gz_1 = F(z_0,y_0,x_0) \qquad 2.2$$

This can be done because $F(X^3) \subset g(X)$. Continuing this process, we construct a sequence $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X such that

$$gx_{n+1} = F(x_n,y_n,z_n),\ gy_{n+1} = F(y_n,x_n,y_n),\ \text{and}\ gz_{n+1} = F(z_n,y_n,x_n) \qquad 2.3$$

By induction, we will prove that

$$gx_n \le gx_{n+1},\ gy_n \ge gy_{n+1}\ \text{and}\ gz_n \le gz_{n+1} \qquad 2.4$$

Since,

$$gx_0 \le F(x_0,y_0,z_0),\ gy_0 \ge F(y_0,x_0,y_0),\ \text{and}\ gz_0 \le F(z_0,y_0,x_0) \qquad 2.5$$

therefore by (2.2) we have

$$gx_0 \le gx_1,\ gy_0 \ge gy_1\ \text{and}\ gz_0 \le gz_1$$

Thus (2.4) is true for n = 0. We suppose that (2.4) is true for some $n > 0$. Since F has the mixed g - monotone property, by (2.4) we have that

$$\begin{aligned} gx_{n+1} &= F(x_n,y_n,z_n) \le F(x_{n+1},y_n,z_n) \\ &\le F(x_{n+1},y_n,z_{n+1}) \le F(x_{n+1},y_{n+1},z_{n+1}) = gx_{n+2} \\ gy_{n+2} &= F(y_{n+1},x_{n+1},y_{n+1}) \ge F(y_{n+1},x_n,y_{n+1}) \\ &\ge F(y_n,x_n,y_{n+1}) \ge F(y_n,x_n,y_n) = gy_{n+1} \end{aligned}$$

and

$$\begin{aligned} gz_{n+1} &= F(z_n,y_n,x_n) \le F(z_{n+1},y_n,x_n) \\ &\le F(z_{n+1},y_{n+1},x_n) \le F(z_{n+1},y_{n+1},x_{n+1}) = gz_{n+2} \end{aligned}$$

That is (2.4) is true for any $n \in N$. If for some $k \in N$,

$$gx_k = gx_{k+1},\ gy_k = gy_{k+1}\ \text{and}\ gz_k = gz_{k+1}$$

then, by(2.3) (x_k, y_k, z_k) is a triplet coincidence point of F and g. From now on, we assume that at least

$$gx_n \ne gx_{n+1},\ gy_n \ne gy_{n+1}\ \text{and}\ gz_n \ne gz_{n+1} \qquad 2.5$$

for any $n \in N$. From (2.4) and the inequality (2.1), we have

$$d(gx_{n+1}, gx_n) = d\left(F(x_n,y_n,z_n), F(x_{n-1},y_{n-1},z_{n-1})\right)$$

$$\mu\left(d\big(F(x_n,y_n,z_n),F(x_{n-1},y_{n-1},z_{n-1})\big)\right)$$
$$\le \tfrac{1}{3}\varphi\big(d(gx_n,gx_{n-1})+d(gy_n,gy_{n-1})+d(gz_n,gz_{n-1})\big)$$
$$-\tfrac{1}{3}\psi\big(d(gx_n,gx_{n-1})+d(gy_n,gy_{n-1})+$$

dgzn,gzn−1

$$\mu\left(d(gx_{n+1},gx_n)\right) \le \tfrac{1}{3}\varphi\big(d(gx_n,gx_{n-1})+d(gy_n,gy_{n-1})+d(gz_n,gz_{n-1})\big)$$
$$-\tfrac{1}{3}\psi\big(d(gx_n,gx_{n-1})+d(gy_n,gy_{n-1})+d(gz_n,gz_{n-1})\big) \qquad 2.6$$

Similarly we get

$$\mu\big(d(gy_{n+1},gy_n)\big) \le \tfrac{1}{3}\varphi\big(d(gx_n,gx_{n-1})+d(gy_n,gy_{n-1})+d(gy_n,gy_{n-1})\big)$$
$$-\tfrac{1}{3}\psi\big(d(gx_n,gx_{n-1})+d(gy_n,gy_{n-1})+d(gy_n,gy_{n-1})\big) \qquad 2.7$$
$$\mu\left(d(gz_{n+1},gz_n)\right) \le \tfrac{1}{3}\varphi\big(d(gx_n,gx_{n-1})+d(gy_n,gy_{n-1})+d(gz_n,gz_{n-1})\big)$$
$$-\tfrac{1}{3}\psi\big(d(gx_n,gx_{n-1})+d(gy_n,gy_{n-1})+d(gz_n,gz_{n-1})\big) \qquad 2.8$$

For each $n \ge 1$.

By adding (2.6), (2.7) and (2.8) and from the property of \mu we get

$$\mu\left(H(x_n,y_n,z_n)\right) \le \varphi\big(H(x_n,y_n,z_n)\big) - \psi\left(H(x_{n-1},y_{n-1},z_{n-1})\right) \qquad 2.9$$

where

$$H(x_n,y_n,z_n) = d\big(g(x_n),g(x_{n+1})\big) + d\big(g(y_n),g(y_{n+1})\big) + d\big(g(z_n),g(z_{n+1})\big)$$

or

$$H(x_n,y_n,z_n) = d\big(F(x_n,y_n,z_n),F(x_{n-1},y_{n-1},z_{n-1})\big)$$
$$+d\big(F(y_n,x_n,y_n),F(y_{n-1},x_{n-1},y_{n-1})\big)$$
$$+d\big(F(z_n,y_n,x_n),F(z_{n-1},y_{n-1},x_{n-1})\big)$$

Using the fact of μ,φ are non decreasing, we get

$$H(x_n,y_n,z_n) \le H(x_{n-1},y_{n-1},z_{n-1})$$

We set,

$$\delta_n = H(x_{n-1},y_{n-1},z_{n-1}) = d(gx_n,gx_{n-1}) + d(gy_n,gy_{n-1}) + d(gz_n,gz_{n-1}) \qquad 2.10$$

then the sequence $\{\delta_n\}$ is decreasing. Therefore, there is some $\delta \ge 0$ such that

$$\lim_{n\to\infty}\delta_n = \lim_{n\to\infty}\left(d(gx_n,gx_{n-1})+d(gy_n,gy_{n-1})+d(gz_n,gz_{n-1})\right) = \delta \qquad 2.11$$

We shall show that $\delta = 0$. Suppose, to the contrary, that $\delta > 0$. Then taking the limit as $n \to \infty$ of both sides of (2.9) and have in mind that we suppose $\lim_{n\to q}\psi(t) > 0$ for all $q > 0$ and μ,φ are continuous, we have

$$\mu(\delta) = \lim_{n\to\infty}\varphi(\delta_n)$$
$$\lim_{n\to\infty}\mu\left(\delta_n\right) \le \lim_{n\to\infty}\left(\varphi(\delta_{n-1}) - \psi(\delta_{n-1})\right) \le \mu(\delta)$$

a contradiction. Thus $\delta = 0$, that is

$$\lim_{n\to\infty}\delta_n = \lim_{n\to\infty}\left(d(gx_n,gx_{n-1})+d(gy_n,gy_{n-1})+d(gz_n,gz_{n-1})\right) = 0 \qquad 2.12$$

In what follows, we shall prove that $\{gx_n\},\{gy_n\}$ and $\{gz_n\}$ are Cauchy sequences. Suppose, to the contrary, that atleast one of $\{gx_n\}\,\{gy_n\},\{gz_n\}$ in not Cauchy sequence. Then there exists an $\epsilon > 0$ for which we can find subsequence $\{gx_{n(k)}\},\{gx_{m(k)}\}$ of $\{gx_n\}$ and $\{gy_{n(k)}\},\{gy_{m(k)}\}$ of $\{gy_n\}$ and $\{gz_{n(k)}\},\{gz_{m(k)}\}$ of $\{gz_n\}$ with $n(k) > m(k) \ge k$ such that

$$d\big(gx_{n(k)},gx_{m(k)}\big) + d\big(gy_{n(k)},gy_{m(k)}\big) + d\big(gz_{n(k)},gz_{m(k)}\big) \ge \epsilon \qquad 2.13$$

Additionally correspondence to $m(k)$. we may choose $n(k)$ such that it is the smallest integer satisfying (2.13) and $n(k) > m(k) \ge k$. Thus

$$d\big(gx_{n(k)-1},gx_{m(k)}\big) + d\big(gy_{n(k)-1},gy_{m(k)}\big) + d\big(gz_{n(k)-1},gz_{m(k)}\big) < \epsilon \qquad 2.14$$

By using triangle inequality and having in mind of (2.13) and (2.14)

$$\epsilon \le p_k = d\big(gx_{n(k)},gx_{m(k)}\big) + d\big(gy_{n(k)},gy_{m(k)}\big) + d\big(gz_{n(k)},gz_{m(k)}\big)$$
$$\le d\big(gx_{n(k)},gx_{n(k)-1}\big) + d\big(gy_{n(k)-1},gy_{m(k)}\big) + d\big(gz_{n(k)},gz_{n(k)-1}\big)$$
$$+d\big(gy_{n(k)-1},gy_{m(k)}\big) + d\big(gz_{n(k)},gz_{n(k)-1}\big) + d\big(gz_{n(k)-1},gz_{m(k)}\big)$$
$$< d\big(gx_{n(k)},gx_{n(k)-1}\big) + d\big(gy_{n(k)},gy_{n(k)-1}\big) + d\big(gz_{n(k)},gz_{n(k)-1}\big) + \epsilon \qquad 2.15$$

letting $k \to \infty$ in (2.15) and using (2.12)

$$\lim_{k\to\infty}p_k = \lim_{k\to\infty}\Big(d\big(gx_{n(k)},gx_{m(k)}\big) + d\big(gy_{n(k)},gy_{m(k)}\big) + d\big(gz_{n(k)},gz_{m(k)}\big)\Big)$$
$$\lim_{k\to\infty}p_k = \epsilon \qquad 2.16$$

Again by triangular inequality,

$$p_k = d\big(gx_{n(k)},gx_{m(k)}\big) + d\big(gy_{n(k)},gy_{m(k)}\big) + d\big(gz_{n(k)},gz_{m(k)}\big)$$

$$\leq d(gx_{n(k)}, gx_{n(k)+1}) + d(gx_{n(k)+1}, gx_{m(k)+1}) + d(gx_{m(k)+1}, gx_{m(k)})$$
$$d(gy_{n(k)}, gy_{n(k)+1}) + d(gy_{n(k)+1}, gy_{m(k)+1}) + d(gy_{m(k)+1}, gy_{m(k)})$$
$$d(gz_{n(k)}, gz_{n(k)+1}) + d(gz_{n(k)+1}, gz_{m(k)+1}) + d(gz_{m(k)+1}, gz_{m(k)})$$
$$\leq \delta_{n(k)+1} + \delta_{m(k)+1} + d(gx_{n(k)+1}, gx_{m(k)+1})$$
$$+ d(gy_{n(k)+1}, gy_{m(k)+1}) + d(gz_{n(k)+1}, gz_{m(k)+1}) \qquad 2.17$$

Since $n(k) > m(k)$, then

$$gx_{n(k)} \geq gx_{m(k)}, \quad gy_{n(k)} \leq gy_{m(k)}, \quad gz_{n(k)} \geq gz_{m(k)} \qquad 2.18$$

Take (2.18) in (2.1) to get,

$$d(gx_{n(k)+1}, gx_{m(k)+1}) + d(gy_{n(k)+1}, gy_{m(k)+1}) + d(gz_{n(k)+1}, gz_{m(k)+1})$$
$$= d\left(F(x_{n(k)}, y_{n(k)}, z_{n(k)}), F(x_{m(k)}, y_{m(k)}, z_{m(k)})\right)$$
$$+ d\left(F(y_{n(k)}, x_{n(k)}, y_{n(k)}), F(y_{m(k)}, x_{m(k)}, y_{m(k)})\right)$$
$$+ d\left(F(z_{n(k)}, y_{n(k)}, x_{n(k)}), F(z_{m(k)}, y_{m(k)}, x_{m(k)})\right)$$

This implies,and using the property of φ we get,

$\mu(p_k) \leq \varphi(\delta_{n(k)+1}) + \varphi(\delta_{m(k)+1}) + \varphi(p_k) - \psi(p_k)$

Letting $k \to \infty$ and having in mind (2.10) and (2.14), we get

$\mu(\epsilon) \leq \varphi(0) + \varphi(\epsilon) - \lim_{k\to\infty} \psi(p_k) < \varphi(\epsilon)$

Which contradiction. This shows that $\{gx_n\}$, $\{gy_n\}$and $\{gz_n\}$are Cauchy sequences. Since X is a complete metric space, there exist $x, y, z \in X$ such that

$$\lim_{n\to\infty}\{gx_n\} = x, \quad \lim_{n\to\infty}\{gy_n\} = y, \quad \lim_{n\to\infty}\{gz_n\} = z \qquad 2.19$$

From (2.19) and the continuity of g,

$$\lim_{n\to\infty}\{g(gx_n)\} = gx, \ im_{n\to\infty}\{g(gy_n)\} = gy, \ \lim_{n\to\infty}\{g(gz_n)\} = gz \qquad 2.20$$

From the commutativity of F and g, we have

$$g(gx_{n+1}) = g(F(x_n, y_n, z_n)) = F(gx_n, gy_n, gz_n)$$
$$g(gy_{n+1}) = g(F(y_n, x_n, y_n)) = F(gy_n, gx_n, gy_n)$$
$$g(gz_{n+1}) = g(F(z_n, y_n, x_n)) = F(gz_n, gy_n, gx_n) \qquad 2.21$$

Now we shall show that

$$gx = F(x, y, z), \ gy = F(y, x, y), \text{ and } gz = F(z, y, x)$$

Suppose that F is continuous. Letting $n \to \infty$ in (2.21), therefore by (2.19) and (2.20) we obtain

$$gx = \lim_{n\to\infty}\{g(gx_n)\} = \lim_{n\to\infty} F(gx_n, gy_n, gz_n) = F(x, y, z)$$
$$gy = \lim_{n\to\infty}\{g(gy_n)\} = \lim_{n\to\infty} F(gy_n, gx_n, gy_n) = F(y, x, y)$$
$$gz = \lim_{n\to\infty}\{g(gz_n)\} = \lim_{n\to\infty} F(gz_n, gy_n, gy_n) = F(z, y, x)$$

We have proved that F and g have a tripled coincidence point.

Corollary 2.4:- Let $(X, \leq)$ be a partially ordered set and (X,d) be a complete metric space. Let $F: X^3 \to X$ be a continuous mapping having the mixed g - monotone property on X and $(X^3) \subset g(X)$. Suppose there exist $\alpha \in [0,1)$ for which,

$$d(F(x, y, z), F(u, v, w)) \leq \alpha\,(d(gx, gu) + d(gy, gv) + d(gz, gw)) \qquad 2.22$$

For all $gx \geq gu$, $gy \leq gv$ and $gz \geq gw$.

Assume that F is continuous, g is continuous and commutes with F. If there exist $x_0, y_0, z_0 \in X$ such that

$$gx_0 \leq F(x_0, y_0, z_0), \ gy_0 \geq F(y_0, x_0, y_0), \text{ and } gz_0 \leq F(z_0, y_0, x_0)$$

Then there exist $x, y, z \in X$ such that,

$$F(x, y, z) = gx, F(y, x, y) = gy, \text{and } F(z, y, x) = gz.$$

That is, F and g have a triplet coincidence point.

Proof:- It follows by taking $\mu(t) = \alpha(t), \varphi(t) = 3\alpha^2(t)$ and $\psi(t) = \frac{\frac{3\alpha^2}{2}}{t}$ in Theorem 2.3.

Corollary 2.5:- Let $(X, \leq)$ be a partially ordered set and (X, d) be a complete metric space. Let $F: X^3 \to X$ be a continuous mapping having the mixed g - monotone property on X and $F(X^3) \subset g(X)$. Suppose there exist $\varphi \in \Phi$, $\psi \in \Psi$ for which,

$$d(F(x, y, z), F(u, v, w)) \leq (d(gx, gu) + d(gy, gv) + d(gz, gw))$$
$$-\psi(d(gx, gu) + d(gy, gv) + d(gz, gw)) \ 2.23$$

For all $gx \geq gu$, $gy \leq gv$ and $gz \geq gw$.

Assume that F is continuous, g is continuous and commutes with F. If there exist $x_0, y_0, z_0 \in X$ such that

$$gx_0 \leq F(x_0, y_0, z_0), \ gy_0 \geq F(y_0, x_0, y_0), \text{ and } gz_0 \leq F(z_0, y_0, x_0)$$

Then there exist $x, y, z \in X$ such that,

$$F(x, y, z) = gx, F(y, x, y) = gy, \text{and } F(z, y, x) = gz.$$

That is, F and g have a triplet coincidence point.

Proof:- In Theorem 2.3, taking $\mu(t), = \varphi(t) = t$ we get corollary 2.5.

Theorem 2.6:- Let $(X, \leq)$ be a partially ordered set and (X, d) be a complete metric space. Let $F : X^3 \to X$ be a continuous mapping having the mixed g - monotone property on X and $(X^3) \subset g(X)$. Suppose there exist $\mu, \varphi \in \Phi$, $\psi \in \Psi$ for which,

$$\mu\Big(d\big(F(x,y,z),F(u,v,w)\big) + d\big(F(y,x,y),F(v,u,v)\big) + d\big(F(z,y,x),F(w,v,u)\big)\Big)$$
$$\leq \varphi\big(d(gx,gu) + d(gy,gv) + d(gz,gw)\big) - \psi\big(d(gx,gu) + d(gy,gv) + d(gz,gw)\big) \quad 2.24$$

For all $gx \geq gu$, $gy \leq gv$ and $gz \geq gw$.

Assume that F is continuous, g is continuous and commutes with F. If there exist $x_0, y_0, z_0 \in X$ such that

$$gx_0 \leq F(x_0,y_0,z_0),\ gy_0 \geq F(y_0,x_0,y_0),\ \text{and}\ gz_0 \leq F(z_0,y_0,x_0)$$

Then there exist $x, y, z \in X$ such that,

$$F(x,y,z) = gx, F(y,x,y) = gy, \text{and}\ F(z,y,x) = gz.$$

That is, F and g have a triplet coincidence point.

Proof:- From the Theorem 2.3 we have,

$$\mu\Big(d\big(F(x,y,z),F(u,v,w)\big)\Big) \leq \frac{1}{3}\varphi\big(d(gx,gu) + d(gy,gv) + d(gz,gw)\big) - \frac{1}{3}\psi\big(d(gx,gu) + d(gy,gv) + d(gz,gw)\big) \quad 2.25$$

Similarly we get,

$$\mu\Big(d\big(F(y,x,y),F(v,u,v)\big)\Big) \leq \frac{1}{3}\varphi\big(d(gx,gu) + d(gy,gv) + d(gy,gv)\big) - \frac{1}{3}\psi\big(d(gx,gu) + d(gy,gv) + d(gy,gv)\big) \quad 2.26$$

and

$$\mu\Big(d\big(F(z,y,x),F(w,v,u)\big)\Big) \leq \frac{1}{3}\varphi\big(d(gx,gu) + d(gy,gv) + d(gz,gw)\big) - \frac{1}{3}\psi\big(d(gx,gu) + d(gy,gv) + d(gz,gw)\big) \quad 2.27$$

by adding (2.25), (2.26) and (2.27) and property of μ then the result is follows similarly to the prove of Theorem 2.3 and nothing to remain prove in Theorem 2.6.

Remark 2.7:- If we take $\varphi(t) = \frac{1}{3}$ t and $\Psi(t) = \frac{2}{3}t$ in Theorem 2.6 then we get special case of Theorem 1.3.

Remark 2.8:- If we take $\mu = 3t$, $\varphi(t) = t$ and $\Psi(t) = (1 - k)\,t$ in Theorem 2.3 then we get special case of Theorem 1.1 for $a = b = c = \frac{k}{3}$ where $k < 3$.

Theorem 2.9:- In addition to hypothesis of Theorem 2.3 suppose that for all (x,y,z) and (u,v,w) in X^3, there exists (a,b,c) in X^3 such that $(F(a,b,c),F(b,a,b),F(c,b,a))$ is comparable to $(F(x,y,z),F(y,x,y),F(z,y,x))$ and $\big(F(u,v,w),F(v,u,v),F(w,v,u)\big)$. Also assume that μ, φ are non decreasing. Then F and g have unique tripled common fixed point (x,y,z) that is

$$x = gx = F(x,y,z),\ y = gy = F(y,x,y)\ \text{and}\ z = gz = F(z,y,x).$$

Proof:- Due to Theorem 2.3, the set of tripled coincidence points of F and g is not empty. Assume now, that (x,y,z) and (u,v,z) are two tripled coincidence points of F and g that is

$$F(x,y,z) = gx, F(y,x,y) = gy\ \text{and}\ F(z,y,x) = gz,$$
$$F(u,v,w) = gu, F(v,u,v) = gv\ \text{and}\ F(w,v,u) = gw$$

We will show that (gx,gy,gz) and (gu,gv,gw) are equal.

By assumption, there is (a,b,c) in X^3 such that $\big(F(a,b,c),F(b,a,b),F(c,b,a)\big)$ is comparable to $(F(x,y,z),F(y,x,y), F(z,y,x))$ and $(F(u,v,w),F(v,u,v),F(w,v,u))$.

Define the sequence $\{ga_n\},\{gb_n\}$and $\{gc_n\}$such that $a = a_0, b = b_0, c = c_0$ and

$$ga_n = F(a_{n-1},b_{n-1},c_{n-1})$$
$$gb_n = F(b_{n-1},a_{n-1},b_{n-1})$$
$$gc_n = F(c_{n-1},b_{n-1},a_{n-1})$$

for all n. Further, set $x = x_0, y = y_0, z = z_0$ and $u = u_0, v = v_0, w = w_0$ and similarly define the sequences $\{gx_n\},\{gy_n\},\{gz_n\}$and $\{gu_n\},\{gv_n\},\{gw_n\}$. Then,

$$gx_n = F(x,y,z)\quad gu_n = F(u,v,w)$$
$$gy_n = F(y,x,y)\quad gv_n = F(v,u,v) \quad 2.28$$
$$gz_n = F(z,y,x)\quad gw_n = F(w,v,u)$$

for all $n \geq 1$. Since $(F(x,y,z),F(y,x,y),F(z,y,x)) = (gx_1,gy_1,gz_1) = (gx,gy,gz)$ is comparable to $\big(F(a,b,c),F(b,a,b),F(c,b,a)\big) = (ga_1,gb_1,gc_1)$, then it is easy to see that $(gx,gy,gz) \geq (ga_1,gb_1,gc_1)$. Recursively, we get that

$$(gx, gy, gz) \geq (ga_n, gb_n, gc_n) \quad \forall\, n \geq 0. \tag{2.29}$$

By using (2.29) and (2.1), we have

$$\mu\left(d\big(F(x,y,z), F(a_n,b_n,c_n)\big)\right) \leq \frac{1}{3}\varphi\big(d(gx,ga_n) + d(gy,gb_n) + d(gz,gc_n)\big)$$
$$-\frac{1}{3}\psi\big(d(gx,ga_n) + d(gy,gb_n) + d(gz,gc_n)\big) \tag{2.30}$$

From (2.30), we deduce that $\gamma_{n+1} \leq \varphi(\gamma_n)$, where $\gamma_n = d\big(F(x,y,z), F(a_{n-1},b_{n-1},c_{n-1})\big)$.

$$\gamma_n \leq \varphi^n(\gamma_0)$$

That is the sequence $d\big(F(x,y,z), F(a_{n-1},b_{n-1},c_{n-1})\big)$ is decreasing. Therefore, there exists $\alpha \geq 0$ such that

$$\lim_{n\to\infty} [d(gx,ga_n) + d(gy,gb_n) + d(gz,gc_n)] = \alpha. \tag{2.31}$$

We shall show that $\alpha = 0$. Suppose, to the contrary, that $\alpha > 0$. Taking the limit as $n \to \infty$ in (2.30), we have

$$\mu(\alpha) \leq \varphi(\alpha) - \lim_{n\to\infty} \psi\big(d(gx,ga_n) + d(gy,gb_n) + d(gz,gc_n)\big) < \varphi(\alpha)$$

a contradiction. Thus, $\alpha = 0$, that is

$$\lim_{n\to\infty} [d(gx,ga_n) + d(gy,gb_n) + d(gz,gc_n)] = 0. \tag{2.32}$$

It implies

$$\lim_{n\to\infty} [d(gx,ga_n)] = \lim_{n\to\infty} [d(gy,gb_n)] = \lim_{n\to\infty} [d(gz,gc_n)] = 0$$

Similarly we show that

$$\lim_{n\to\infty} [d(gu,ga_n)] = \lim_{n\to\infty} [d(gv,gb_n)] = \lim_{n\to\infty} [d(gw,gc_n)] = 0 \tag{2.33}$$

Combining (2.32) and (2.33) yields that (gx, gy, gz) and (gu, gv, gw) are equal.

Since $F(x,y,z) = gx, F(y,x,y) = gy$ and $F(z,y,x) = gz$ by commutativity of F and g, we have

$$g(F(x,y,z)) = g(gx) = F(gx,gy,gz)$$
$$g(F(y,x,y)) = g(gy) = F(gy,gx,gy)$$
$$g(F(z,y,x)) = g(gz) = F(gz,gy,gx),$$

Denote $gx = x', gy = y'$ and $gz = z'$. From the precedent identities,

$$F(x',y',z') = gx', F(y',x',y') = gy' \text{ and } F(z',y',x') = gz'$$

That is, (x',y',z') is a tripled coincidence point of F and g. Consequently, (gx',gy',gz') and (gx,gy,gz) are equal, that is $gx = gx', gy = gy'$ and $gz = gz'$.

We deduce $gx = gx' = x, gy = gy' = y$ and $gz = gz' = z$. Therefore, (x',y',z') is a tripled common fixed point of F and g. Its uniqueness follows from Theorem 2.3.

Remark that Theorem 2.3 is more general than Theorem 1.1, since the contractive condition (2.1) is weaker than (1.1), also Theorem 2.3 is generalization of the Theorem 1.3. A fact which clearly illustrated by the following example.

Example 2.10:- Let $X = R$ with $d(x,y) = |\, x - y\,|$ and natural ordering and let $g: X \to X$, and $F: X^3 \to X$ be given by

$$g(x) = \frac{n+1}{n}x, \quad n = 1,2,3\ldots\ldots x \in X$$

and

$$F(x,y,z) = \frac{x+y+z}{2} \quad \forall\ (x,y,z) \in X^3$$

It is clear that F is continuous and the mixed $g-$ monotone property. We now take $\mu(t) = t, \varphi(t) = \frac{n+1}{n}t$ and $\psi(t) = \frac{n(n+2)}{n+1}t$. Then it is easy to see that all the hypotheses of Theorem 2.3 are satisfied and (0,0,0) is tripled coincidence point of F and g.

Now for $x = u, z = w$ and $v > y$, we have

$$d\big(F(x,y,z), F(u,v,w)\big) = \frac{1}{2}|\, v - y\,| > \frac{1}{3}|\, v - y\,| \geq \frac{k}{3}[d(x,u) + d(y,v) + d(z,w)]$$

for any $k \in [0,1)$ that is the conition (1.1) given in Theorem 1.1 is not applicable for, $a = b = c = \frac{k}{3}$.

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