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A Method of Transformation for Generalized Hypergeometric Function $_2F_2$

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Abstract
By employing an addition theorem for the confluent hypergeometric function, Paris R.B.[3], has obtained a Kummer-type transformation for a $_2F_2(x)$ hypergeometric function with general parameters in the form of a sum of $_2F_2(-x)$ functions. Recently, Choi Junesang and Rathie Arjun K.[1], has obtained the same result without using the addition theorem. The aim of this paper is to derive the result of Paris R.B.[3], with change in the general parameters without using the addition theorem in the line of Choi Junesang and Rathie Arjun K.[1].
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1. Introduction and results required

We start with a Kummer-type transformation for a $_2F_2(x)$ hypergeometric function with general parameters in the form of a sum of $_2F_2(-x)$ functions due to Paris R.B.[3, Eq.(3)]:

$$_2F_2(a , d ; b, c ; x) = e^x \sum_{n=0}^{\infty} \frac{(c-d)_n}{(c)_n \; n!}(-x)^n \, _2F_2(b-a , d ; b, c+n ; x) \qquad \ldots\ldots (1.1)$$

where $(a)_n = \frac{\Gamma(a+n)}{\Gamma a}$ (n =0,1,2,3, …….) is the Pochhammer symbol.

Paris R.B.[3], also considered several interesting special cases of (1.1).This result (1.1) was established with the help of the integral representation for $_2F_2$ [5, Eq.(4.8.3.11)]:

$$_2F_2(a , d ; b, c ; x)= \frac{\Gamma b}{\Gamma a \, \Gamma(b-a)} \int_0^1 t^{a-1} (1-t)^{b-a-1} \, _1F_1(d ; c ; xt) \, dt \qquad \ldots\ldots (1.2)$$

and

$$_2F_2(a , d ; b, c ; x)= \frac{\Gamma b}{\Gamma a \, \Gamma(b-a)} \int_0^1 t^{b-a-1} (1-t)^{a-1} \, _1F_1(d ; c ; x- xt) \, dt \qquad \ldots\ldots (1.3)$$

provided $R(b) > 0$ and $R(a) > 0$, and the addition theorem for the confluent hypergeometric function in the form due to Slater L. J.[4, Eq.(2.3.5)]:

$$_1F_1(d ; c ; x- xt)= e^x \sum_{n=0}^{\infty} \frac{(c-d)_n}{(c)_n \; n!}(-x)^n \, _1F_1(d ; c+n ; -xt) \qquad \ldots\ldots (1.4)$$

Paris R.B.[3], remarked that the special case of (1.1) when $c = d$ reduces to the well-known Kummer's first theorem due to [5]:

$$_1F_1(a ; b ; x) = e^x \, _1F_1(b-a ; b ; -x) \qquad \ldots\ldots (1.5)$$

Choi Junesang and Rathie Arjun K.[1], has derived the following result:

$$_2F_2(d , a ; c, b ; x) = e^x \sum_{r=0}^{\infty} \frac{(c-d)_r}{(c)_r \; r!}(-x)^r \, _2F_2(b-a , d ; b, c+r ; x) \qquad \ldots\ldots (1.6)$$

The aim of this paper is to derive the result of Paris R.B.[3], with change in the general parameters without using the addition theorem in the line of Choi Junesang and Rathie Arjun K.[1].

2. Main Result

$$_2F_2(b , a ; a, b ; x+y) = e^{x+y} \sum_{u=0}^{\infty} \frac{(a-b)_u}{(a)_u \; u!}(-x-y)^u$$
$$\times \, _2F_2(b-a, b ; b, a+u ; -x-y) \qquad \ldots\ldots (2.1)$$

Proof:-
Start with the left-hand side of (2.1) and use (1.2), it becomes

$${}_2F_2(b, a ; a, b ; x+y) = \frac{\Gamma a}{\Gamma b\, \Gamma(a-b)} \int_0^1 t^{b-1} (1-t)^{a-b-1} \times {}_1F_1(b ; a ; xt+yt)\, dt \qquad \ldots\ldots. (2.2)$$

which can be written as

$${}_2F_2(b, a ; a, b ; x+y) = \frac{\Gamma a}{\Gamma b\, \Gamma(a-b)} e^{x+y} \int_0^1 t^{b-1} (1-t)^{a-b-1} e^{-x-y} \times {}_1F_1(b ; a ; xt+yt)\, dt \qquad \ldots\ldots. (2.3)$$

Using equation (1.5) in the integrand of the integral in equation (2.3), we have

$${}_2F_2(b , a ; a, b ; x+y) = \frac{\Gamma a}{\Gamma b\, \Gamma(a-b)} e^{x+y} \int_0^1 t^{b-1} (1-t)^{a-b-1} e^{-x(1-t)-y(1-t)} \times {}_1F_1(a-b ; a ; -xt-yt)\, dt \qquad \ldots\ldots. (2.4)$$

Now expand $e^{-x(1-t)-y(1-t)}$ in equation (2.4) as the Maclaurin series, after a little simplification, we obtain

$${}_2F_2(b , a ; a, b ; x+y) = \frac{\Gamma a}{\Gamma b\, \Gamma(a-b)} e^{x+y} \sum_{u=0}^{\infty} \frac{(-x-y)^u}{u!} \int_0^1 t^{b-1} (1-t)^{a-b+u-1} e^{-x(1-t)-y(1-t)} \times {}_1F_1(a-b ; a ; -xt-yt)\, dt \qquad \ldots\ldots. (2.5)$$

Substituting $1\text{-}t = z$ in equation (2.5) and simplifying, we have

$${}_2F_2(b , a ; a, b ; x+y) = \frac{\Gamma a}{\Gamma b\, \Gamma(a-b)} e^{x+y} . \sum_{u=0}^{\infty} \frac{(-x-y)^u}{u!} \int_0^1 (1-z)^{b-1} z^{a-b+u-1} e^{-x(1-t)-y(1-t)} \times {}_1F_1(a-b ; a ; -x(1-z) - y(1-z))\, dz \qquad \ldots\ldots. (2.6)$$

Finally, applying (1.3) to the integral part in the last identity, we have

$${}_2F_2(b , a ; a, b ; x+y) = e^{x+y} \sum_{u=0}^{\infty} \frac{(a-b)_u}{(a)_u\, u!} (-x-y)^u \times {}_2F_2(b-a, b ; b, a+u ; -x-y) \qquad \ldots\ldots. (2.7)$$

This completes the proof of (2.1).

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References
[1] Choi Junesang and Rathie Arjun K., “Another method for a kummer-type transformation for a ${}_2F_2$ hypergeometric function”, Commun. Korean Mathematical Society, Vol. 22, No. 3, July (2007) pp. 369-371.
[2] Erd´elyi A., “*Higher transcendental functions”,* McGraw Hill, New York, Vol-1 (1953).
[3] Paris R. B., “*A Kummer-type transformation for a ${}_2F_2$ hypergeometric function”*, J. Comput. Appl. Math. 173 (2005) pp. 379-382.
[4] Slater L. J., “*Confluent Hypergeometric Functions”*, Cambridge University Press, Cambridge, 1960.
[5] “*Generalized Hypergeometric Functions”*, Cambridge University Press, Cambridge,(1966).