

# A Method of Transformation for Generalized Hypergeometric Function ${}_2F_2$

Awadhesh Kumar Pandey, Ramakant Bhardwaj\*, Kamal Wadhwa\*\* and Nitesh Singh Thakur\*\*\*

Department of Mathematics, Patel Institute of Technology, Bhopal, M.P.

\*Truba Institute of Technology, Bhopal, M.P.

\*\*Govt. Narmda P.G. College, Hoshangabad, M.P.

\*\*\*Patel College of Science & Technology, Bhopal, M.P.

## Abstract

By employing an addition theorem for the confluent hypergeometric function, Paris R.B.[3], has obtained a Kummer-type transformation for a  ${}_2F_2(x)$  hypergeometric function with general parameters in the form of a sum of  ${}_2F_2(-x)$  functions. Recently, Choi Junesang and Rathie Arjun K.[1], has obtained the same result without using the addition theorem. The aim of this paper is to derive the result of Paris R.B.[3], with change in the general parameters without using the addition theorem in the line of Choi Junesang and Rathie Arjun K.[1].

**Corresponding author E.mail:-** pandey1172@gmail.com, pandey.awadhesh1972@gmail.com

## 1. Introduction and results required

We start with a Kummer-type transformation for a  ${}_2F_2(x)$  hypergeometric function with general parameters in the form of a sum of  ${}_2F_2(-x)$  functions due to Paris R.B.[3, Eq.(3)]:

$${}_2F_2(a, d; b, c; x) = e^x \sum_{n=0}^{\infty} \frac{(c-d)_n}{(c)_n n!} (-x)^n {}_2F_2(b-a, d; b, c+n; x) \quad \dots\dots (1.1)$$

where  $(a)_n = \frac{\Gamma(a+n)}{\Gamma a}$  ( $n=0,1,2,3, \dots\dots$ ) is the Pochhammer symbol.

Paris R.B.[3], also considered several interesting special cases of (1.1). This result (1.1) was established with the help of the integral representation for  ${}_2F_2$  [5, Eq.(4.8.3.11)]:

$${}_2F_2(a, d; b, c; x) = \frac{\Gamma b}{\Gamma a \Gamma(b-a)} \int_0^1 t^{a-1} (1-t)^{b-a-1} {}_1F_1(d; c; xt) dt \quad \dots\dots (1.2)$$

and

$${}_2F_2(a, d; b, c; x) = \frac{\Gamma b}{\Gamma a \Gamma(b-a)} \int_0^1 t^{b-a-1} (1-t)^{a-1} {}_1F_1(d; c; x-xt) dt \quad \dots\dots (1.3)$$

provided  $R(b) > 0$  and  $R(a) > 0$ , and the addition theorem for the confluent hypergeometric function in the form due to Slater L. J.[4, Eq.(2.3.5)]:

$${}_1F_1(d; c; x-xt) = e^x \sum_{n=0}^{\infty} \frac{(c-d)_n}{(c)_n n!} (-x)^n {}_1F_1(d; c+n; -xt) \quad \dots\dots (1.4)$$

Paris R.B.[3], remarked that the special case of (1.1) when  $c = d$  reduces to the well-known Kummer's first theorem due to [5]:

$${}_1F_1(a; b; x) = e^x {}_1F_1(b-a; b; -x) \quad \dots\dots (1.5)$$

Choi Junesang and Rathie Arjun K.[1], has derived the following result:

$${}_2F_2(d, a; c, b; x) = e^x \sum_{r=0}^{\infty} \frac{(c-d)_r}{(c)_r r!} (-x)^r {}_2F_2(b-a, d; b, c+r; x) \quad \dots\dots (1.6)$$

The aim of this paper is to derive the result of Paris R.B.[3], with change in the general parameters without using the addition theorem in the line of Choi Junesang and Rathie Arjun K.[1].

## 2. Main Result

$${}_2F_2(b, a; a, b; x+y) = e^{x+y} \sum_{u=0}^{\infty} \frac{(a-b)_u}{(a)_u u!} (-x-y)^u \quad \dots\dots (2.1)$$

**Proof:-**

Start with the left-hand side of (2.1) and use (1.2), it becomes

$${}_2F_2(b, a; a, b; x+y) = \frac{\Gamma a}{\Gamma b \Gamma(a-b)} \int_0^1 t^{b-1} (1-t)^{a-b-1} \times {}_1F_1(b; a; xt+yt) dt \quad \dots\dots (2.2)$$

which can be written as

$${}_2F_2(b, a; a, b; x+y) = \frac{\Gamma a}{\Gamma b \Gamma(a-b)} e^{x+y} \int_0^1 t^{b-1} (1-t)^{a-b-1} e^{-x-y} \times {}_1F_1(b; a; xt+yt) dt \quad \dots\dots (2.3)$$

Using equation (1.5) in the integrand of the integral in equation (2.3), we have

$${}_2F_2(b, a; a, b; x+y) = \frac{\Gamma a}{\Gamma b \Gamma(a-b)} e^{x+y} \int_0^1 t^{b-1} (1-t)^{a-b-1} e^{-x(1-t)-y(1-t)} \times {}_1F_1(a-b; a; -xt-yt) dt \quad \dots\dots (2.4)$$

Now expand  $e^{-x(1-t)-y(1-t)}$  in equation (2.4) as the Maclaurin series, after a little simplification, we obtain

$${}_2F_2(b, a; a, b; x+y) = \frac{\Gamma a}{\Gamma b \Gamma(a-b)} e^{x+y} \sum_{u=0}^{\infty} \frac{(-x-y)^u}{u!} \int_0^1 t^{b-1} (1-t)^{a-b+u-1} e^{-x(1-t)-y(1-t)} \times {}_1F_1(a-b; a; -xt-yt) dt \quad \dots\dots (2.5)$$

Substituting  $1-t = z$  in equation (2.5) and simplifying, we have

$${}_2F_2(b, a; a, b; x+y) = \frac{\Gamma a}{\Gamma b \Gamma(a-b)} e^{x+y} \cdot \sum_{u=0}^{\infty} \frac{(-x-y)^u}{u!} \int_0^1 (1-z)^{b-1} z^{a-b+u-1} e^{-x(1-t)-y(1-t)} \times {}_1F_1(a-b; a; -x(1-z)-y(1-z)) dz \quad \dots\dots (2.6)$$

Finally, applying (1.3) to the integral part in the last identity, we have

$${}_2F_2(b, a; a, b; x+y) = e^{x+y} \sum_{u=0}^{\infty} \frac{(a-b)_u}{(a)_u} \frac{(-x-y)^u}{u!} \times {}_2F_2(b-a, b; b, a+u; -x-y) \quad \dots\dots (2.7)$$

This completes the proof of (2.1).

**Acknowledgement**

Corresponding Author (A.P.) is thankful to all reviewers and Authors whose references are taken to prepare this article.

**References**

[1] Choi Junesang and Rathie Arjun K., “Another method for a kummer-type transformation for a  ${}_2F_2$  hypergeometric function”, Commun. Korean Mathematical Society, Vol. 22, No. 3, July (2007) pp. 369-371.  
 [2] Erd’elyi A., “Higher transcendental functions”, McGraw Hill, New York, Vol-1 (1953).  
 [3] Paris R. B., “A Kummer-type transformation for a  ${}_2F_2$  hypergeometric function”, J. Comput. Appl. Math. 173 (2005) pp. 379-382.  
 [4] Slater L. J., “Confluent Hypergeometric Functions”, Cambridge University Press, Cambridge, 1960.  
 [5] “Generalized Hypergeometric Functions”, Cambridge University Press, Cambridge,(1966).

This academic article was published by The International Institute for Science, Technology and Education (IISTE). The IISTE is a pioneer in the Open Access Publishing service based in the U.S. and Europe. The aim of the institute is Accelerating Global Knowledge Sharing.

More information about the publisher can be found in the IISTE's homepage:

<http://www.iiste.org>

## CALL FOR PAPERS

The IISTE is currently hosting more than 30 peer-reviewed academic journals and collaborating with academic institutions around the world. There's no deadline for submission. **Prospective authors of IISTE journals can find the submission instruction on the following page:** <http://www.iiste.org/Journals/>

The IISTE editorial team promises to review and publish all the qualified submissions in a **fast** manner. All the journals articles are available online to the readers all over the world without financial, legal, or technical barriers other than those inseparable from gaining access to the internet itself. Printed version of the journals is also available upon request of readers and authors.

### IISTE Knowledge Sharing Partners

EBSCO, Index Copernicus, Ulrich's Periodicals Directory, JournalTOCS, PKP Open Archives Harvester, Bielefeld Academic Search Engine, Elektronische Zeitschriftenbibliothek EZB, Open J-Gate, OCLC WorldCat, Universe Digital Library, NewJour, Google Scholar

