Some cases of reducible Generalized Hypergeometric Functions

Awadhesh Kumar Pandey, Ramakant Bhardwaj*, Kamal Wadhwa** and Nitesh Singh Thakur***
Department of Mathematics, Patel Institute of Technology, Bhopal, M.P.
*Truba Institute of Technology, Bhopal, M.P.
**Govt. Narmda P.G. College, Hoshangabad, M.P.
***Patel College of Science & Technology, Bhopal, M.P.

Abstract
In this paper we consider the integrals of Generalized hypergeometric function of three variables given by Saran Shanti [4] and obtained functions of two variables of Horn's list given in ErdelyiA.[1]. Our results are also motivated by Singh Pooja & Singh Prof. (Dr.) Harish [3].
Corresponding author E.mail:- pandey1172@gmail.com , pandey.awadhesh1972@gmail.com

Introduction
Horn investigated in particular hypergeometric series of order two and found that , apart from certain series which are either expressible in terms of product of two hypergeometric series in one variable, there are 34 distinct convergent hypergeometric series Erdelyi A [1].
Saran Shanti [4], gave some integral associated with hypergeometric function of three variables. We are using some of them for our investigation such as F_E and F_K. Recently the method adopting here has been used by Singh Pooja & Singh Prof. (Dr.) Harish [3]. We are giving here same treatment with some modifications.
Saran Shanti [4], gave the following summations

$F_E(a_1,a_1,a_1,b_1,b_2,b_2;c_1,c_2,c_3;x,y,z)$
$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n+p}(b_1)_m(b_2)_{n+p}}{m!n!p!(c_1)_m(c_2)_n(c_3)_p} x^m y^n z^p \quad \text{.... (1.1)}$$

$F_K(a_1,a_2,a_2,b_1,b_2,b_1;c_1,c_2,c_3;x,y,z)$
$$= \sum_{m,n,p=0}^{\infty} \frac{(a_1)_m(a_2)_{n+p}(b_1)_{m+p}(b_2)_n}{m!n!p!(c_1)_m(c_2)_n(c_3)_p} x^m y^n z^p \quad \text{.... (1.2)}$$

and their integral results are also given by Saran Shanti [4], as

$F_E(a_1,a_1,a_1,b_1,b_2,b_2;c_1,c_2,c_3;x,y,z)$
$$= \frac{\Gamma c_2 \Gamma c_3 \Gamma(2-c_2-c_3)}{(2\pi i)^2} \int_c (-t)^{-c_2}(t-1)^{-c_3}$$
$$\times F_2(a_1,b_1,b_2,c_1,c_2+c_3-1;x,\frac{y}{t}+\frac{z}{1-t})dt \quad \text{....... (1.3)}$$

where $|x| + \left|\frac{y}{t}+\frac{z}{1-t}\right| < 1$ along the contour.

$F_K(a_1,a_2,a_2,b_1,b_2,b_1;c_1,c_2,c_3;x,y,z)$
$$= \frac{\Gamma r \Gamma r_1 \Gamma(2-r-r_1)}{(2\pi i)^2} \int_c (-t)^{-r}(t-1)^{-r_1} \times {}_2F_1(r,a_1;c_1+\tfrac{1}{2};\tfrac{x}{t})$$
$$\times F_2(a_2,b_2,r_1,c_2,c_3;y,\frac{z}{1-t})\, dt \quad \text{..... (1.4)}$$

Where $b_1 = r + r_1 - 1$, $|t| > |x|, \left|\frac{z}{1-t}\right| < 1 - |y|$ along the integral.

Singh Pooja & Singh Prof. (Dr.) Harish[3], used argument as hyperbolic function in (1.3) & (1.4) as

$F_E(a_1,a_1,a_1,b_1,b_2,b_2;c_1,c_2,c_3;\cosh x,\cosh y,\cosh z)$
$$= \frac{\Gamma c_2 \Gamma c_3 \Gamma(2-c_2-c_3)}{(2\pi i)^2} \int_c (-t)^{-c_2}(t-1)^{-c_3}$$
$$\times F_2(a_1,b_1,b_2,c_1,c_2+c_3-1;\cosh x,\frac{\cosh y}{t}+\frac{\cosh z}{1-t})dt \quad \text{....... (1.5)}$$

$F_K(a_1,a_2,a_2,b_1,b_2,b_1;c_1,c_2,c_3;\cosh x,\cosh y,z)$
$$= \frac{\Gamma r \Gamma r_1 \Gamma(2-r-r_1)}{(2\pi i)^2} \int_c (-t)^{-r}(t-1)^{-r_1} \times {}_2F_1(r,a_1,c_1+\tfrac{1}{2};\cosh y,\frac{\cosh x}{t})$$
$$\times F_2(a_2;b_2,r_1;c_2,c_3;\cosh y,\frac{\cosh z}{1-t})\, dt \quad \text{..... (1.6)}$$

The following results are due to Singh Pooja & Singh Prof. (Dr.) Harish[3],

$F_E(a_1,a_1,a_1,c_1,c_2+c_3-1,c_2+c_3-1;c_1,c_2,c_3;\cosh x,\cosh y,\cosh z)$
$$= (1-\cosh x-\cosh y)^{-a_1}$$
$$\times H_1(1-c_3,a_1,c_2+c_3-1,c_2;\frac{\cosh y}{\cosh x+\cosh y-1},\frac{\cosh z}{\cosh x+\cosh y-1}) \quad \text{...(1.7)}$$

$F_E(a_1,a_1,a_1,c_1,c_2+c_3-1,c_2+c_3-1;c_1,c_2,c_3;\cosh x,\cosh y,\cosh z)$
$$=(1-\cosh x-\cosh y-\cosh z)^{-a_1}$$

$$\times G_1(a_1, 1-c_2, 1-c_3; \frac{coshy}{1-coshx-coshy-coshz}, \frac{coshz}{1-coshx-coshy-coshz}) \quad \ldots (1.8)$$

$$F_K(a_1, a_2, a_2, c_1+c_3-1, c_2, c_1+c_3-1; c_1, c_2, c_3; coshx, coshy, coshz)$$
$$= (1-coshy-coshz)^{-a_2}$$
$$\times H_2(1-c_1, a_2, a_1, c_1+c_3-1, c_3; \frac{coshx}{coshy+coshz-1}, -coshx) \quad \ldots (1.9)$$

$$F_K(a_1, a_2, a_2, c_1+c_3-1, c_2, c_1+c_3-1; c_1, c_2, c_3; coshx, coshy, coshz)$$
$$= (1-coshx)^{-a_1}(1-coshy)^{-a_2}$$
$$\times H_2(1-c_3, a_1, a_2, c_1+c_3-1, c_1; \frac{coshz}{coshy-1}, \frac{coshz}{coshy-1}) \quad \ldots\ldots (1.10)$$

$$F_K(a_1, a_2, a_2, c_1+c_3-1, c_2, c_1+c_3-1; c_1, c_2, c_3; coshx, coshy, coshz)$$
$$= (1-coshx)^{-a_1}(1-coshy-coshz)^{-a_2}$$
$$\times G_2(a_1, a_2, 1-c_1, 1-c_3; \frac{coshz}{1-coshy}, \frac{coshz}{1-coshy-coshz}) \quad \ldots\ldots (1.11)$$

Where H_1, H_2, & G_1, G_2 are Horn's functions defined by Erdelyi A. [1], as

$$H_1(a, b, c, d; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m-n}(b)_{m+n}(c)_n x^m y^n}{m!n!\,(d)_m} \quad \ldots\ldots (1.12)$$

$$H_2(a, b, c, d, e; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m-n}(b)_m (c)_n (d)_n}{m!n!\,(e)_m} \; x^m y^n \quad \ldots\ldots (1.13)$$

$$G_1(a, b, b'; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m-n}(b')_{m-n} x^m y^n}{m!n!} \quad \ldots\ldots (1.14)$$

$$G_2(a, a', b, b'; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_m (a')_n (b')_{m-n} (b)_{n-m} x^m y^n}{m!n!} \; x^m y^n \quad \ldots\ldots (1.15)$$

Due to Singh Pooja & Singh Prof. (Dr.) Harish[3],

$$(1-coshx-\frac{coshy}{t}-\frac{coshz}{1-t})^{-a_1}$$
$$= (1-coshx-coshy)^{-a_1}$$
$$\times \sum_{m,n=0}^{\infty} \frac{(a_1)_{m+n}}{m!n!} (\frac{coshy}{(1-coshx-coshy)} \cdot \frac{(1-t)}{t})^m (\frac{coshz}{(1-coshx-coshy)} \cdot \frac{1}{(1-t)})^m \quad \ldots (1.16)$$

where $\left|\frac{coshy}{(1-coshx-coshy)} \cdot \frac{(1-t)}{t}\right| < 1$, $\left|\frac{coshz}{(1-coshx-coshy)} \cdot \frac{1}{(1-t)}\right| < 1$, along the contour.

$$(1-coshx-\frac{coshy}{t}-\frac{coshz}{1-t})^{-a_1} = (1-coshx-coshy-coshz)^{-a_1}$$
$$\times \sum_{m,n=0}^{\infty} \frac{(a_1)_{m+n}}{m!n!} \left(\frac{coshy}{(1-coshx-coshy-coshz)} \cdot \frac{(1-t)}{t}\right)^m$$
$$\times (\frac{coshz}{(1-coshx-coshy-coshz)} \cdot \frac{1}{(1-t)})^m \quad \ldots.. (1.17)$$

where $\left|\frac{coshy}{(1-coshx-coshy-coshz)} \cdot \frac{(1-t)}{t}\right| < 1$, $\left|\frac{coshz}{(1-coshx-coshy-coshz)} \cdot \frac{1}{(1-t)}\right| < 1$, along the contour.

Due to Erdelyi A.[1], we have

$$F_2(a_2, b_2, r_1; r_2, r_1; y, \frac{z}{1-t}) = (1-y-\frac{z}{1-t})^{-a_2}$$
$$= (1-y-z)^{-a_2} \sum_{m=0}^{\infty} \frac{(a_2)_m}{m!} (\frac{z}{(1-y-z)} \cdot \frac{t}{(1-t)})^m \quad \ldots.(1.18)$$

$$F_2(a_2, b_2, r_1; r_2, r_1; y, \frac{z}{1-t}) = (1-y-\frac{z}{1-t})^{-a_2}$$
$$= (1-y)^{-a_2} \sum_{m=0}^{\infty} \frac{(a_2)_m}{m!} (\frac{z}{(1-y-z)} \cdot \frac{t}{(1-t)})^m \quad \ldots\ldots (1.19)$$

$${}_2F_1(r, a_1; r; y, \frac{x}{t}) = (1-\frac{x}{t})^{-a_2}$$
$$= (1-x)^{-a_1} \sum_{m=0}^{\infty} \frac{(a_2)_m}{m!} (\frac{x}{(1-x)} \cdot \frac{1}{(1-t)})^m \quad \ldots. (1.20)$$

2. Main results

Reduction of F_E & F_K into Horn's function (Modification)

$$F_E(c_2+c_3, c_2+c_3, c_2+c_3, c_1, c_2+c_3-1, c_2+c_3-1; c_1, c_2, c_3; x, y, z)$$
$$= (1-x-y)^{-c_2-c_3}$$
$$\times H_1(1-c_3, c_2+c_3, c_2+c_3-1, c_2; \frac{y}{x+y-1}, \frac{z}{x+y-1}) \quad \ldots.. (2.1)$$

$$F_E(c_2+c_3, c_2+c_3, c_2+c_3, c_1, c_2+c_3-1, c_2+c_3-1; c_1, c_2, c_3; x, y, z)$$
$$= (1-x-y-z)^{-c_2-c_3}$$
$$\times G_1(c_2+c_3, 1-c_2, 1-c_3; \frac{y}{1-x-y-z}, \frac{z}{1-x-y-z}) \quad \ldots.. (2.2)$$

$$F_K(a_1, c_1+c_3, c_1+c_3, c_1+c_3-1, c_2, c_1+c_3-1; c_1, c_2, c_3; x, y, z)$$
$$= (1-y-z)^{-c_1-c_3} \; H_2(1-c_1, c_1+c_3, a_1, c_1+c_3-1, c_3; \frac{x}{y+z-1}, -x) \;. \quad \ldots\ldots.. (2.3)$$

$$F_K(a_1, c_1+c_3, c_1+c_3, c_1+c_3-1, c_2, c_1+c_3-1; c_1, c_2, c_3; x, y, z)$$
$$= (1-x)^{-a_1}(1-y)^{-c_1-c_3} \; H_2(1-c_3, a_1, c_1+c_3, c_1+c_3-1, c_1; \frac{z}{y-1}, \frac{z}{y-1}) \quad \ldots\ldots(2.4)$$

$F_K(a_1, c_1+c_3, c_1+c_3, c_1+c_3-1, c_2, c_1+c_3-1; c_1, c_2, c_3; x, y, z)$

$$= (1-x)^{-a_1}(1-y-z)^{-c_1-c_3}\ G_2\left(a_1, c_1+c_3, 1-c_1, 1-c_3; \frac{z}{1-y}, \frac{z}{1-y-z}\right) \quad \ldots\ldots (2.5)$$

Proof of (2.a) & (2.b) :-

From equation (1.3), we have

$$F_E(a_1,a_1,a_1,b_1,b_2,b_2; c_1,c_2,c_3; x,y,z) = \frac{\Gamma c_2 \Gamma c_3 \Gamma(2-c_2-c_3)}{(2\pi i)^2}\int_c (-t)^{-c_2}(t-1)^{-c_3}$$

$$\times F_2\left(a_1,b_1,b_2,c_1,c_2+c_3-1; x, \frac{y}{t}+\frac{z}{1-t}\right)dt \quad \ldots\ldots (2.6)$$

where $|x| + \left|\frac{y}{t}+\frac{z}{1-t}\right| < 1$ along the contour.

Putting $a_1 = c_2+c_3$, $b_1 = c_1$ and $b_2 = c_2+c_3-1$, in equation (2.6), we have

$F_E(c_2+c_3, c_2+c_3, c_2+c_3, c_1, c_2+c_3-1, c_2+c_3-1; c_1,c_2,c_3; x,y,z)$

$$= \frac{\Gamma c_2 \Gamma c_3 \Gamma(2-c_2-c_3)}{(2\pi i)^2}\int_c (-t)^{-c_2}(t-1)^{-c_3}\left(1-x-\frac{y}{t}-\frac{z}{1-t}\right)^{-c_2-c_3}dt \quad \ldots (2.7)$$

on expansion, we get

$$\left(1-x-\frac{y}{t}-\frac{z}{1-t}\right)^{-c_2-c_3} = (1-x-y)^{-c_2-c_3}$$

$$\times \sum_{m,n=0}^{\infty}\frac{(c_2+c_3)_{m+n}}{m!n!}\left(\frac{y}{(1-x-y)}\cdot\frac{(1-t)}{t}\right)^m\left(\frac{z}{(1-x-y)}\cdot\frac{1}{(1-t)}\right)^m \quad \ldots\ldots (2.8)$$

where $\left|\frac{y}{(1-x-y)}\cdot\frac{(1-t)}{t}\right| < 1, \left|\frac{z}{(1-x-y)}\cdot\frac{1}{(1-t)}\right| < 1$ along the contour.

and

$$\left(1-x-\frac{y}{t}-\frac{z}{1-t}\right)^{-c_2-c_3} = (1-x-y-z)^{-c_2-c_3}$$

$$\times \sum_{m,n=0}^{\infty}\frac{(c_2+c_3)_{m+n}}{m!n!}\left(\frac{y}{(1-x-y-z)}\cdot\frac{(1-t)}{t}\right)^m\left(\frac{z}{(1-x-y-z)}\cdot\frac{1}{(1-t)}\right)^m \quad \ldots (2.9)$$

where $\left|\frac{y}{(1-x-y-z)}\cdot\frac{(1-t)}{t}\right| < 1, \left|\frac{z}{(1-x-y-z)}\cdot\frac{1}{(1-t)}\right| < 1$ along the contour.

Using equation (2.8), (2.9) and then evaluating the integral after changing the order of integration and summation, keeping

$$\frac{(2\pi i)^2}{\Gamma(1-a)\,\Gamma(1-b)\,\Gamma(a+b)} = \int_c (-t)^{a-1}(t-1)^{b-1}dt \quad \ldots (2.10)$$

$F_E(c_2+c_3, c_2+c_3, c_2+c_3, c_1, c_2+c_3-1, c_2+c_3-1; c_1,c_2,c_3; x,y,z)$

$$= (1-x-y)^{-c_2-c_3}$$

$$\times H_1\left(1-c_3, c_2+c_3, c_2+c_3-1, c_2; \frac{y}{x+y-1}, \frac{z}{x+y-1}\right) \quad \ldots (2.11)$$

Which is new reduction **result in F_E** and

$F_E(c_2+c_3, c_2+c_3, c_2+c_3, c_1, c_2+c_3-1, c_2+c_3-1; c_1,c_2,c_3; x,y,z)$

$$= (1-x-y-z)^{-c_2-c_3}$$

$$\times G_1\left(c_2+c_3, 1-c_2, 1-c_3; \frac{y}{1-x-y-z}, \frac{z}{1-x-y-z}\right) \quad \ldots (2.12)$$

Which is also new reduction **result** in $\mathbf{F_E}$.

In this way equations (2.1) & (2.2) are proved.

Proof of (2.3), (2.4) & (2.5) :-

From equation (1.2), we have

$F_K(a_1,a_2,a_2,b_1,b_2,b_1; c_1,c_2,c_3; x,y,z)$

$$= \sum_{m,n,p=0}^{\infty}\frac{(a_1)_m (a_2)_{n+p}(b_1)_{m+p}(b_2)_n}{m!n!p!(c_1)_m(c_2)_n(c_3)_p}x^m y^n z^p \quad \ldots (2.13)$$

absolutely convergent if p=(1-m)(1-n), where $|x| < 1$, $|y| < 1$ and $|z| < 1$.

From equation (1.4), we have

$F_K(a_1,a_2,a_2,b_1,b_2,b_1; c_1,c_2,c_3; x,y,z)$

$$= \frac{\Gamma r \Gamma r_1 \Gamma(2-r-r_1)}{(2\pi i)^2}\int_c (-t)^{-r}(t-1)^{-r_1} \times {}_2F_1\left(r, a_1; c_1+\frac{1}{2}; \frac{x}{t}\right)$$

$$\times F_2\left(a_2, b_2, r_1, c_2, c_3; y, \frac{z}{1-t}\right)dt \quad \ldots (2.14)$$

$|x| < 1$ and $\left|\frac{z}{1-t}\right| < 1 - |y|$ along the contour.

Using equation (1.18) and(1.19) & by replacing $c_2 = r_2$, $c_3 = r_1$, we have

$$F_2\left(a_2, b_2, r_1, c_2, c_3; y, \frac{z}{1-t}\right) = F_2\left(a_2, b_2, r_1, r_2, r_1; y, \frac{z}{1-t}\right)$$

$$= \left(1-y-\frac{z}{1-t}\right)^{-a_2}$$

$$= (1-y-z)^{-a_2}\sum_{m=0}^{\infty}\frac{(a_2)_m}{m!}\left(\frac{z}{(1-y-z)}\cdot\frac{t}{(1-t)}\right)^m \quad \ldots (2.15)$$

and

$F_2(a_2, b_2, r_1; r_2, r_1; y, \frac{z}{1-t}) = (1-y-\frac{z}{1-t})^{-a_2}$

$= (1-y)^{-a_2} \sum_{m=0}^{\infty} \frac{(a_2)_m}{m!} (\frac{z}{(1-y-z)} \cdot \frac{t}{(1-t)})^m$ ...... (2.16)

putting $c_1 + \frac{1}{2} =$ r and using equation (1.20), we get

${}_2F_1(r, a_1; c_1 + \frac{1}{2}; \frac{x}{t}) = {}_2F_1(r, a_1; r; \frac{x}{t})$

$= (1-\frac{x}{t})^{-a_2}$

$= (1-x)^{-a_1} \sum_{m=0}^{\infty} \frac{(a_2)_m}{m!} (\frac{x}{(1-x)} \cdot \frac{1}{(1-t)})^m$ .... (2.17)

putting $a_2 = c_1 + c_3$, $b_1 = c_1 + c_3 - 1$ and $b_2 = c_2$ in equation (2.14) & also using equation (2.15) and equation (2.17), we get

$F_K(a_1, c_1 + c_3, c_1 + c_3, c_1 + c_3 - 1, c_2, c_1 + c_3 - 1; c_1, c_2, c_3; x, y, z)$

$= \frac{\Gamma c_1 \Gamma c_3 \Gamma(2-c_1-c_3)}{(2\pi i)^2} \int_c (-t)^{-c_1} (t-1)^{-c_3} (1-\frac{x}{t})^{-a_1} (1-y-\frac{z}{1-t})^{-c_1-c_3} dt$ ... (2.18)

Now using (2.15) and $(1-\frac{x}{t})^{-a_1} = \sum_{m,=0}^{\infty} \frac{(a_1)_m}{m!} (\frac{x}{t})^m$ and integrating term by term we will have

$F_K(a_1, c_1 + c_3, c_1 + c_3, c_1 + c_3 - 1, c_2, c_1 + c_3 - 1; c_1, c_2, c_3; x, y, z)$

$= (1-y-z)^{-a_2} H_2(1-c_1, c_1 + c_3, a_1, c_1 + c_3 - 1, c_3; \frac{x}{y+z-1}, -x)$. .... (2.19)

Which is new reduction **result** in $\mathbf{F_K}$.

$F_K(a_1, c_1 + c_3, c_1 + c_3, c_1 + c_3 - 1, c_2, c_1 + c_3 - 1; c_1, c_2, c_3; x, y, z)$

$= (1-x)^{-a_1} (1-y)^{-c_1-c_3} H_2(1-c_3, a_1, c_1 + c_3, c_1 + c_3 - 1, c_1; \frac{z}{y-1}, \frac{z}{y-1})$ ......(2.20)

Which is also new reduction **result** in $\mathbf{F_K}$.

$F_K(a_1, c_1 + c_3, c_1 + c_3, c_1 + c_3 - 1, c_2, c_1 + c_3 - 1; c_1, c_2, c_3; x, y, z)$

$= (1-x)^{-a_1} (1-y-z)^{-c_1-c_3} G_2(a_1, c_1 + c_3, 1-c_1, 1-c_3; \frac{z}{1-y}, \frac{z}{1-y-z})$ ..... (2.21)

Which is also new reduction **result** in $\mathbf{F_K}$.

In this way equations (2.3), (2.4) & (2.5) are proved.

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