Some cases of reducible Generalized Hypergeometric Functions

Awadhesh Kumar Pandey, Ramakant Bhardwaj*, Kamal Wadhwa** and Nitesh Singh Thakur***

Department of Mathematics, Patel Institute of Technology, Bhopal, M.P.
*Truba Institute of Technology, Bhopal, M.P.
**Govt. Narmada P.G. College, Hoshangabad, M.P.
***Patel College of Science & Technology, Bhopal, M.P.

Abstract
In this paper we consider the integrals of Generalized hypergeometric function of three variables given by Saran Shanti [4] and obtained functions of two variables of Horn’s list given in ErdelyiA [1]. Our results are also motivated by Singh Pooja & Singh Prof. (Dr.) Harish [3].

Corresponding author E-mail:- pandey1172@gmail.com, pandey.awadhesh1972@gmail.com

Introduction
Horn investigated in particular hypergeometric series of order two and found that, apart from certain series which are either expressible in terms of product of two hypergeometric series in one variable, there are 34 distinct convergent hypergeometric series Erdelyi A [1]. Saran Shanti [4], gave some integral associated with hypergeometric function of three variables. We are using some of them for our investigation such as $F_k$ and $F_p$. Recently the method adopting here has been used by Singh Pooja & Singh Prof. (Dr.) Harish [3]. We are giving here same treatment with some modifications.

Saran Shanti [4], gave the following summations

\[F_k(a_1,a_1,a_1,b_1,b_2,b_2;c_1,c_2,c_3;x,y,z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_m(b_1)_n(c_1)_p}{m!n!p!(c_1)_m(c_2)_n(c_3)_p} x^m y^n z^p \quad \text{.... (1.1)}\]

\[F_k(a_1,a_2,a_2,b_1,b_2,b_1;c_1,c_2,c_3;x,y,z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_m(a_2)_n(b_1)_m(b_2)_n}{m!n!(c_1)_m(c_2)_n(c_3)_p} x^m y^n z^p \quad \text{.... (1.2)}\]

and their integral results are also given by Saran Shanti [4], as

\[F_k(\alpha_1,\alpha_1,\beta_1,\beta_1,\beta_2,\beta_2;\gamma_1,\gamma_1,\gamma_2,\gamma_2,\gamma_3,\gamma_3;x,y,z) = \int_{c}^{d} \left(\frac{e^{-t}}{(t-1)^{\alpha_1}}\right)^{c_2} \left(\frac{e^{-t}}{t}^{c_3} \right) dt \quad \text{.... (1.3)}\]

where $|x| + \frac{y}{1-t} < 1$ along the contour.

\[F_k(a_1,a_2,a_2,b_1,b_2,b_1;c_1,c_2,c_3;x,y,z) = \frac{\Gamma(r_1,\gamma_1)}{\Gamma(r_1)} \int_{c}^{d} \left(\frac{e^{-t}}{t}^{c_3} \right) dt \quad \text{.... (1.4)}\]

Where $b_1 = r + r_1 - 1$, $|t| > |x| \left|\frac{y}{1-t}\right| < 1 - |y|$ along the integral.

Singh Pooja & Singh Prof. (Dr.) Harish[3], used argument as hyperbolic function in (1.3) & (1.4) as

\[F_k(a_1,a_1,a_1,b_1,b_2,b_2;c_1,c_2,c_3,\cosh, \cosh, \cosh); \cosh, \cosh, \cosh) = \frac{\Gamma(r_1,\gamma_1)}{\Gamma(r_1)} \int_{c}^{d} \left(\frac{e^{-t}}{t}^{c_3} \right) dt \quad \text{.... (1.5)}\]

\[F_k(a_1,a_2,a_2,b_1,b_2,b_1;c_1,c_2,c_3,\cosh, \cosh, \cosh), \cosh, \cosh, \cosh, \cosh) = \frac{\Gamma(r_1,\gamma_1)}{\Gamma(r_1)} \int_{c}^{d} \left(\frac{e^{-t}}{t}^{c_3} \right) dt \quad \text{.... (1.6)}\]

The following results are due to Singh Pooja & Singh Prof. (Dr.) Harish[3],

\[F_k(a_1, a_1, a_1, c_1, c_2 + c_3 - 1, c_2 + c_3 - 1; c_1, c_2, c_3, \cosh, \cosh, \cosh) = \left(1 - \cosh x - \cosh y \right)^{-a_1} \]

\[H_k(1-c_3,a_1, c_2 + c_3 - 1; c_2,\frac{\cosh x}{\cosh x + \cosh y - 1},\frac{\cosh y}{\cosh x + \cosh y - 1}) \quad \text{.... (1.7)}\]

\[F_k(a_1, a_1, a_1, c_1, c_2 + c_3 - 1, c_2 + c_3 - 1; c_1, c_2, c_3, \cosh, \cosh, \cosh) = \left(1 - \cosh x - \cosh y - \cosh z \right)^{-a_1} \]
Where $H_1, H_2$ & $G_1, G_2$ are Horn’s functions defined by Erdelyi A. [1], as

\[
H_1(a, b, c, d; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_m (b)_m (c)_n (d)_n}{m! n!} x^m y^n 
\]

\[
H_2(a, b, c; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_m (b)_m (c)_n}{m! n!} x^m y^n 
\]

\[
G_1(a, b, b'; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_m (b)_m (b')_n}{m! n!} x^m y^n 
\]

\[
G_2(a, a', b'; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_m (a')_m (b')_n}{m! n!} x^m y^n 
\]

Due to Singh Pooja & Singh Prof (Dr. Harish)[3],

\[
1 - \cosh x = \cosh \left( \frac{t}{1 - t} \right) = \sum_{m,n=0}^{\infty} \frac{(1 - \cosh x)(1 - \cosh y)}{1 - t} \left( \frac{1}{1 - t} \right)^m \left( \frac{\cosh x}{\cosh (1 - \cosh x)} \right)^{m-1} \left( 1 - \cosh y \right)^{-1} 
\]

where $\frac{1 - \cosh x}{(1 - \cosh x) \cosh y} < 1$, along the contour.

\[
1 - \cosh x = \cosh \left( \frac{t}{1 - t} \right) = \sum_{m,n=0}^{\infty} \frac{(1 - \cosh x)(1 - \cosh y)}{1 - t} \left( \frac{1}{1 - t} \right)^m \left( \frac{\cosh x}{\cosh (1 - \cosh x)} \right)^{m-1} \left( 1 - \cosh y \right)^{-1} 
\]

where $\frac{1 - \cosh x}{(1 - \cosh x) \cosh y} < 1$, along the contour.

Due to Erdelyi A.[1], we have

\[
F_2(a_2, b_2, r_1, r_2, r_3; x, y, z) = (1 - \frac{r_1}{1 - r_1})^{-a_2} \sum_{m=0}^{\infty} \frac{(a_2)_m}{m!} \left( \frac{x}{1 - y} \right)^{t} (1 - t)^m
\]

\[
F_2(a_2, b_2, r_1, r_2, r_3; x, y, z) = (1 - \frac{r_1}{1 - r_1})^{-a_2} \sum_{m=0}^{\infty} \frac{(a_2)_m}{m!} \left( \frac{x}{1 - y} \right)^{t} (1 - t)^m
\]

\[
F_1(r, a_2, r_1, y, z) = (1 - \frac{r_1}{1 - r_1})^{-a_2} \sum_{m=0}^{\infty} \frac{(a_2)_m}{m!} \left( \frac{x}{1 - y} \right)^{t} (1 - t)^m
\]

\[
F_1(r, a_2, r_1, y, z) = (1 - \frac{r_1}{1 - r_1})^{-a_2} \sum_{m=0}^{\infty} \frac{(a_2)_m}{m!} \left( \frac{x}{1 - y} \right)^{t} (1 - t)^m
\]

2. Main results

Reduction of $F_k$ & $F_k$ into Horn’s function (Modification)

\[
F_k(c_2 + c_3, c_2 + c_3, c_2 + c_3, c_1, c_2, c_3 - 1 ; c_3, c_1, c_2, c_3 ; x, y, z)
\]

\[
F_k(c_2 + c_3, c_2 + c_3, c_2 + c_3, c_1, c_2, c_3 - 1 ; c_3, c_1, c_2, c_3 ; x, y, z)
\]

\[
F_k(a_4, a_1, c_1 + c_3, c_1 + c_3 - 1, c_2, c_1, c_3 - 1 ; c_1, c_2, c_3 ; x, y, z)
\]

\[
F_k(a_4, a_1, c_1 + c_3, c_1 + c_3 - 1, c_2, c_1, c_3 - 1 ; c_1, c_2, c_3 ; x, y, z)
\]
Using equation (2.8), (2.9) and then evaluating the integral after changing the order of integration and
which is new reduction

From equation (1.2), we have

\[ F_k(a_1, a_1 + c_1, b_1, b_2, c_1, c_2, c_3; x, y, z) \]

\[ = (1 - x)^{-a_2} (1 - y - z)^{-c_1 - c_3} F_2(a_1, a_1 + c_1, 1 - c_1; 1 - c_3; \frac{x}{1 - y}, \frac{z}{1 - y - z}) \]

\[ \cdots (2.5) \]

**Proof of (2.a) & (2.b):**

From equation (1.3), we have

\[ F_k(a_1, a_1, a_1 + b_1, b_2, b_2, c_1, c_2, c_3; x, y, z) \]

\[ = \sum_{m,n=0}^{\infty} \frac{(c_1 + c_2 + c_3)_{m+n}}{m! n!} \left( \frac{y}{(1-x-y)} \right)^m \left( \frac{z}{1-t} \right)^n \]

\[ \cdots (2.6) \]

where \(|x| + \frac{|y|}{t} + \frac{|z|}{1-t} < 1\) along the contour.

Putting \(a_1 = c_2 + c_3, b_1 = c_1\) and \(b_2 = c_2 + c_3 - 1\), in equation (2.6), we have

\[ F_k(c_2 + c_3, c_2 + c_3, c_2 + c_3 + c_2, c_2 + c_3 - 1, c_2 + c_3 - 1; c_1, c_2, c_3; x, y, z) \]

\[ = \sum_{m,n=0}^{\infty} \frac{(c_2 + c_3)_{m+n}}{m! n!} \left( \frac{y}{(1-x-y)} \right)^m \left( \frac{z}{1-t} \right)^n \]

\[ \cdots (2.7) \]

on expansion, we get

\[ (1 - x - y - z)^{-c_2 - c_3} \]

\[ \times \sum_{m,n=0}^{\infty} \frac{(c_2 + c_3)_{m+n}}{m! n!} \left( \frac{y}{(1-x-y)} \right)^m \left( \frac{z}{1-t} \right)^n \]

\[ \cdots (2.8) \]

where \(\frac{y}{(1-x-y)} \cdot \frac{(1-t)}{t} < 1\) and \(\frac{z}{1-t} < 1\) along the contour.

Using equation (2.6), (2.9) and then evaluating the integral after changing the order of integration and summation, keeping

\[ \frac{(2n)!^2}{(1-a) \cdot (1-b) \cdot (a+b)} = \int_c^{(t - a)^{-1} (t - b)^{-1} dt} \]

\[ \cdots (2.10) \]

\[ F_k(c_2 + c_3, c_2 + c_3, c_2 + c_3 + c_2, c_2 + c_3 - 1, c_2 + c_3 - 1; c_1, c_2, c_3; x, y, z) \]

\[ = (1 - x - y - z)^{-c_2 - c_3} \]

\[ \times G_1(c_2 + c_3, 1 - c_2, 1 - c_3; \frac{y}{1-x-y-z}, \frac{z}{1-x-y-z}) \]

\[ \cdots (2.11) \]

Which is new reduction result in \(F_k\) and

\[ F_k(c_2 + c_3, c_2 + c_3, c_2 + c_3 + c_2, c_2 + c_3 - 1, c_2 + c_3 - 1; c_1, c_2, c_3; x, y, z) \]

\[ = (1 - x - y - z)^{-c_2 - c_3} \]

\[ \times G_1(c_2 + c_3, 1 - c_2, 1 - c_3; \frac{y}{1-x-y-z}, \frac{z}{1-x-y-z}) \]

\[ \cdots (2.12) \]

Which is also new reduction result in \(F_k\).

In this way equations (2.1) & (2.2) are proved.

**Proof of (2.3), (2.4) & (2.5):**

From equation (1.2), we have

\[ F_k(a_1, a_2, a_2, b_1, b_2, b_1, c_1, c_2, c_3; x, y, z) \]

\[ = \sum_{m,n,p=0}^{\infty} \frac{(a_1 m (a_2)_{m+p} (b_1)_{m+p} (b_2)_{m+p})}{m! n! p!} x^m y^n z^p \]

\[ \cdots (2.13) \]

absolutely convergent if \(p = (1-m)(1-n)\), where \(|x| < 1, |y| < 1\) and \(|z| < 1\).

From equation (1.4), we have

\[ F_k(a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_2, c_3; x, y, z) \]

\[ = \int_{(1-a) \cdot (1-b) \cdot (a+b)}^{(t - a)^{-1} (t - b)^{-1} dt} \]

\[ \cdots (2.14) \]

\[ \left| z \left/ \left(1-t \right) \right| < 1 \right| y \mid \text{along the contour.} \]

Using equation (1.18) & (1.19) & by replacing \(c_2, r_2, c_3 = r_1\), we have

\[ F_2(a_1, b_2, r_1, c_2, c_3, y; z) = F_2(a_1, b_2, r_1, r_2, r_3; y, z) \]

\[ = (1 - y - z)^{-a_2} \sum_{m=0}^{\infty} \frac{(a_2)_{m}}{m!} \left( \frac{y}{1-y-z} \right)^m \left( \frac{z}{1-t} \right)^m \]

\[ \cdots (2.15) \]

and
\[ F_2(a_2, b_2, r_1, r_2, r_1 : y, \frac{x}{1-t}) = (1 - y - \frac{z}{1-t})^{-a_2} = (1 - y)^{-a_2} \sum_{m=0}^{\infty} \frac{(a_2)_m}{m!} \left( \frac{x}{1-y-z} \right)^m (1-t)^m \]  

\[ \ldots (2.16) \]

Putting \( c_1 + \frac{1}{2} = r \) and using equation (1.20), we get

\[ 2F_1(r, a_1; c_1 : \frac{1}{2}) = 2F_1(r, a_1; \frac{r}{2}) = (1 - \frac{r}{2})^{-a_2} \sum_{m=0}^{\infty} \frac{(a_2)_m}{m!} \left( \frac{x}{1-y-z} \right)^m (1-t)^m \]  

\[ \ldots (2.17) \]

Putting \( a_2 = c_1 + c_3, b_1 = c_1 + c_3 - 1 \) and \( b_2 = c_2 \) in equation (2.14) & also using equation (2.15) and equation (2.17), we get

\[ F_K(a_1, c_1 + c_3, c_1 + c_3 - 1, c_2, c_1 + c_3 - 1; c_1, c_2, c_3; x, y, z) = \frac{\Gamma(c_1) \Gamma(c_3) \Gamma(2-c_1-c_3)}{\Gamma(c_1-c_3)} \int_0^1 (1-t)^{-c_1} (t-1)^{-c_3} (1 - \frac{x}{t})^{-a_2} \sum_{m=0}^{\infty} \frac{(a_2)_m}{m!} \left( \frac{x}{1-y-z} \right)^m (1-t)^m \]  

\[ \ldots (2.18) \]

Now using (2.15) and \( (1 - \frac{r}{2})^{-a_1} = \sum_{m=0}^{\infty} \frac{(a_1)_m}{m!} \left( \frac{x}{r} \right)^m \) and integrating term by term we will have

\[ F_K(a_1, c_1 + c_3, c_1 + c_3 - 1, c_2, c_1 + c_3 - 1; c_1, c_2, c_3; x, y, z) = (1 - y - z)^{-a_1} H_2(1-c_1, c_1 + c_3, d_1, c_1 + c_3 - 1, c_3; \frac{x}{y^2-1}, -x) \]  

\[ \ldots (2.19) \]

Which is new reduction result in \( F_K \).

\[ F_K(a_1, c_1 + c_3, c_1 + c_3 - 1, c_2, c_1 + c_3 - 1; c_1, c_2, c_3; x, y, z) = (1 - x)^{-a_1} (1 - y)^{-c_1-c_3} H_2(1-c_1, a_1, c_1 + c_3, d_1 + c_3 - 1, c_1; \frac{x}{y-1}, \frac{x}{y-1}) \]  

\[ \ldots (2.20) \]

Which is also new reduction result in \( F_K \).

\[ F_K(a_1, c_1 + c_3, c_1 + c_3 - 1, c_2, c_1 + c_3 - 1; c_1, c_2, c_3; x, y, z) = (1 - x)^{-a_1} (1 - y - z)^{-c_1-c_3} G_2(a_1, c_1 + c_3, 1-c_1, 1-c_3; \frac{x}{1-y-z}, \frac{x}{1-y-z}) \]  

\[ \ldots (2.21) \]

Which is also new reduction result in \( F_K \).

In this way equations (2.3), (2.4) & (2.5) are proved.

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**References**


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