Mathematical Theory and Modeling www.iiste.org
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.3, No.6, 2013-Selected from International Conference on Recent Trends in Applied Sciences with Engineering Applications

Some Fixed Point Theorems with G-iteration in Normed Linear Spaces

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Abstract
In this paper a new type of one step iteration for self mappings is introduced under certain conditions in normed linear space and studied with a contractive conditions of Rafiq [4].
Key words: Common fixed point , Contractive condition, G-iteration, Mann iteration
Mathematics Subject Classification: 47H10, 54H25.

1. INTRODUCTION

Let X be a nonempty closed convex subset of a normed linear space E and $T: X \to X$ be a self mapping and $\{x_n\}$ be the sequence then for arbitrary $x_0 \in X$ Mann[3] iteration process is defined as

$$x_{n+1} = (1-\lambda_n)x_n + \lambda_n T x_n \quad for\ n \geq 0$$

Similarly Ishikawa[2] iteration process for $\{x_n\}$ is given by

$$x_{n+1} = (1-\lambda_n)x_n + \lambda_n T y_n$$

$$\text{and } y_n = (1-\lambda_n^{'})x_n + \lambda_n^{'} T x_n \ , \quad \text{for } n \geq 0$$

Where $x_0 \in X$ is arbitrary and $\{\lambda_n\}, \{\lambda_n^{'}\}$ are sequences of real numbers such that $0 \leq \lambda_n \leq 1$, $0 \leq \lambda_n^{'} \leq 1$.

By using the concept of Mann iteration process Sahu[5] introduced a new G-iteration process:-

Let T be a self mapping of Banach space then G-iteration process associated by T is defined in the following manner,

Let $x_0, x_1 \in X$ and

$$x_{n+2} = (\mu_n - \lambda_n)x_{n+1} + \lambda_n T x_{n+1} + (1-\mu_n) T x_n \qquad \text{for } n \geq 0$$

Where $\{\mu_n\}$ and $\{\lambda_n\}$ satisfy

(i) $\lambda_0 = \mu_0 = 1$

(ii) $0 < \lambda_n < 1, n > 0$ and $\mu_n \geq \lambda_n$ for $n \geq 0$

(iii) $\lim_{n\to\infty} \lambda_n = h > 0$

(iv) $\lim_{n\to\infty} \mu_n = 1$

Das and Debata [1] generalized the Ishikawa iteration processes from the case of one self mapping to the case of two self mappings S and T of X given by

$$x_{n+1} = (1-\lambda_n)x_n + \lambda_n S y_n$$

$$\text{and } y_n = (1-\lambda_n^{'})x_n + \lambda_n^{'} T x_n \ , \quad \text{for } n \geq 0$$

By using above iteration Das and Debata[1] established the common fixed points of quasi-nonexpansive mappings in a uniformly convex Banach space.Several other researchers such as Takahashi and Tamura[6] investigated iteration in a strictly convex Banach space,for the case of two nonexpansive mappings under different assumptions and contractive conditions.

Our aim in this paper is to establish some fixed point theorems by using a more general contractive condition than those of Rafiq [4].We shall use a new type of one step iteration for self mappings and employ the following contractive definition:

Let X be a closed convex subset of normed linear space N Suppose that T be a self mapping of X .There exist a constant $L \geq 0$ such that $\forall x, y \in X$,we have

$$\|Tx - Ty\| \leq e^{L\|x-Tx\|}\left(2\delta\|x-Tx\| + \delta\|x-y\|\right) \qquad \ldots(1.1)$$

Where $0 \le \delta < 1$ and e^x denotes the exponential function of $x \in X$.

After this we extend the above contractive definition for a pair of mapping in following manner:

Let X be a closed convex subset of normed linear space N Suppose that S and P are two self mappings of X satisfying the following contractive condition

$$\|Sx - Py\| \le e^{L\|x-Sx\|}\left(2\delta\|x - Sx\| + \delta\|x - Py\|\right) \quad \forall x, y \in X,\ L \ge 0 \quad ...(1.2)$$

Where $0 \le \delta < 1$ and e^x denotes the exponential function of $x \in X$.

2. MAIN RESULTS

Theorem 2.1: Let X be a closed convex subset of normed linear space N and let T be a self mapping of X satisfying the contractive condition of (1.1) and $\{x_n\}$ be the sequence of G-iterates associated with T then G-iteration process is defined in the following manner:

Let $x_0, x_1 \in X$ and

$$x_{n+2} = (\mu_n - \lambda_n)x_{n+1} + \lambda_n Tx_{n+1} + (1 - \mu_n - \lambda_n + k_n)Tx_n + (\lambda_n - k_n)x_n \qquad \text{for } n \ge 0$$

where $\{\mu_n\}, \{\lambda_n\}$ and $\{k_n\}$ satisfying

(i) $\mu_n = \lambda_n = k_n = 1$ if $n = 0$

(ii) $0 < \lambda_n < 1$, $0 < k_n < 1$ for $n > 0$.

(iii) $\mu_n \ge \lambda_n, \mu_n \ge k_n$ for $n \ge 0$.

(iv) $\lim_{n\to\infty} \lambda_n = h = \lim_{n\to\infty} k_n$ where $h > 0$.

(v) $\lim_{n\to\infty} \mu_n = 1$.

If $\{x_n\}$ converges in X then it converges to a fixed point of T.

Proof:- If $\{x_n\}$ converges on $z \in X$ then $\lim\limits_{n\to\infty} x_n = z$.

Now we shall show that z is the fixed point of T .

Consider

$$\begin{aligned}
\|z - Tz\| &\le \|z - x_{n+2}\| + \|x_{n+2} - Tz\| \\
&\le \|z - x_{n+2}\| + \left\|(\mu_n - \lambda_n)x_{n+1} + \lambda_n Tx_{n+1} + (1 - \mu_n - \lambda_n + k_n)Tx_n + (\lambda_n - k_n)x_n - Tz\right\| \\
&\le \|z - x_{n+2}\| + (\mu_n - \lambda_n)\|x_{n+1} - Tz\| + \lambda_n\|Tx_{n+1} - Tz\| + (1 - \mu_n - \lambda_n + k_n)\|Tx_n - Tz\| \\
&\qquad + (\lambda_n - k_n)\|x_n - Tz\| \\
&\le \|z - x_{n+2}\| + (\mu_n - \lambda_n)\|x_{n+1} - Tz\| \\
&\qquad + \lambda_n\left\{e^{L\|x_{n+1} - Tx_{n+1}\|}\left(2\delta\|x_{n+1} - Tx_{n+1}\| + \delta\|x_{n+1} - z\|\right)\right\} \\
&\qquad + (1 - \mu_n - \lambda_n + k_n)\|Tx_n - Tz\| + (\lambda_n - k_n)\|x_n - Tz\|
\end{aligned} \qquad ...(2.1.1)$$

We observe by the definition of G-iteration that

$$\|x_{n+1} - Tx_{n+1}\| \le \frac{1}{\lambda_n}\|x_{n+1} - x_{n+2}\| + \frac{(1 - \mu_n)}{\lambda_n}\|x_{n+1} - Tx_n\| + \frac{(\lambda_n - k_n)}{\lambda_n}\|x_n - Tx_n\|$$

Now putting above value in (2.1.1) then we have

$$\|z-Tz\| \le \|z-x_{n+2}\| + (\mu_n-\lambda_n)\|x_{n+1}-Tz\|$$
$$+\lambda_n e^{L\left(\frac{1}{\lambda_n}\|x_{n+1}-x_{n+2}\|+\frac{(1-\mu_n)}{\lambda_n}\|x_{n+1}-Tx_n\|+\frac{(\lambda_n-k_n)}{\lambda_n}\|x_n-Tx_n\|\right)}$$
$$\left[2\delta\left(\frac{1}{\lambda_n}\|x_{n+1}-x_{n+2}\|+\frac{(1-\mu_n)}{\lambda_n}\|x_{n+1}-Tx_n\|+\frac{(\lambda_n-k_n)}{\lambda_n}\|x_n-Tx_n\|\right)+\delta\|x_{n+1}-z\|\right]$$
$$+(1-\mu_n-\lambda_n+k_n)\|Tx_n-Tz\|+(\lambda_n-k_n)\|x_n-Tz\|$$

Letting $n\to\infty$ then we have

$$\|z-Tz\| \le (1-h)\|z-Tz\|$$

which is a contradiction.

Hence $z=Tz$ is a fixed point of T.

Remark: When $\{\mu_n\}=\{1\}$ and $\{\lambda_n\}=\{k_n\}$ then above G-iterative process reduces to Mann iteration.

Theorem 2.2: Let X be a closed convex subset of normed linear space N and let S and P be a pair of self mappings of X satisfying the contractive condition of(1.2) and $\{x_n\}$ be the sequence of G-iterates associated with S and P then G-iteration process is defined in the following manner:

Let $x_0, x_1 \in X$ and

$$x_{2n+2} = (\mu_n-\lambda_n)x_{2n+1} + \lambda_n Sx_{2n+1} + (1-\mu_n-\lambda_n+k_n)\mathrm{P}\,x_{2n} + (\lambda_n-k_n)x_{2n}$$

and $x_{2n+3} = (\mu_n-\lambda_n)x_{2n+2} + \lambda_n Px_{2n+2} + (1-\mu_n-\lambda_n+k_n)\mathrm{S}\,x_{2n+1} + (\lambda_n-k_n)x_{2n+1}$, for $n\ge 0$

where $\{\mu_n\}$, $\{\lambda_n\}$ and $\{k_n\}$ satisfying

(i) $\mu_n=\lambda_n=k_n=1$ if $n=0$.

(ii) $0<\lambda_n<1$, $0<k_n<1$ for $n>0$.

(iii) $\mu_n\ge\lambda_n, \mu_n\ge k_n$ for $n\ge 0$.

(iv) $\lim_{n\to\infty}\lambda_n = h = \lim_{n\to\infty}k_n$ where $h>0$.

(v) $\lim_{n\to\infty}\mu_n = 1$.

If $\{x_n\}$ converges to z in X then z is the common fixed point S and P.

Proof:-If $\{x_n\}$ converges on $z\in X$ then $\lim_{n\to\infty} x_n = z$.

Now we shall show that z is the common fixed point of S and P.

Consider

$$\|z-Pz\| \le \|z-x_{2n+3}\| + \|x_{2n+3}-Pz\|$$
$$\le \|z-x_{2n+3}\| + \|(\mu_n-\lambda_n)x_{2n+2} + \lambda_n T_2 x_{2n+2} + (1-\mu_n-\lambda_n+k_n)T_1 x_{2n+1} + (\lambda_n-k_n)x_{2n+1} - T_1 z\|$$
$$\le \|z-x_{2n+3}\| + (\mu_n-\lambda_n)\|x_{2n+2}-Pz\| + \lambda_n\|Sx_{2n+2}-Pz\| + (1-\mu_n-\lambda_n+k_n)\|Px_{2n+1}-Pz\|$$
$$+(\lambda_n-k_n)\|x_{2n+1}-Pz\|$$
$$\le \|z-x_{2n+3}\| + (\mu_n-\lambda_n)\|x_{2n+2}-Pz\| + \lambda_n\left\{e^{L\|x_{2n+2}-Sx_{2n+2}\|}\left(2\delta\|x_{2n+2}-Sx_{2n+2}\|+\delta\|x_{2n+2}-Pz\|\right)\right\}$$
$$+(1-\mu_n-\lambda_n+k_n)\|Px_{2n+1}-Pz\| + (\lambda_n-k_n)\|x_{2n+1}-Pz\|$$

...(2.2.1)

We observe by the definition of G-iteration that

$$\|x_{2n+2}-Sx_{2n+2}\|\le\frac{1}{\lambda_n}\|x_{2n+2}-x_{2n+3}\|+\frac{(1-\mu_n)}{\lambda_n}\|x_{2n+2}-Px_{2n+1}\|+\frac{(\lambda_n-k_n)}{\lambda_n}\|x_{2n+1}-Px_{2n+1}\|$$

Now putting above values in (2.2.1) then we have

$$\|z-Pz\|\le\|z-x_{2n+3}\|+(\mu_n-\lambda_n)\|x_{2n+2}-Pz\|$$

$$+\lambda_n e^{L\left\{\frac{1}{\lambda_n}\|x_{2n+2}-x_{2n+3}\|+\frac{(1-\mu_n)}{\lambda_n}\|x_{2n+2}-Px_{2n+1}\|+\frac{(\lambda_n-k_n)}{\lambda_n}\|x_{2n+1}-Px_{2n+1}\|\right\}}$$

$$\left[2\delta\left(\frac{1}{\lambda_n}\|x_{2n+2}-x_{2n+3}\|+\frac{(1-\mu_n)}{\lambda_n}\|x_{2n+2}-Px_{2n+1}\|+\frac{(\lambda_n-k_n)}{\lambda_n}\|x_{2n+1}-Px_{2n+1}\|\right)+\delta\|x_{2n+2}-Pz\|\right]$$

$$+(1-\mu_n-\lambda_n+k_n)\|Px_{2n+1}-Pz\|+(\lambda_n-k_n)\|x_{2n+1}-Pz\|$$

Letting $n\to\infty$ then we have

$$\|z-Pz\|\le(1-h+h\delta)\|z-Pz\|$$

Which is a contradiction.

Hence $z=Pz$ i.e. z is a fixed point of P.

Similarly we can show that

$$\|z-Sz\|\le(1-h+h\delta)\|z-Sz\|$$

Hence $z=Sz$ i.e. z is a fixed point of S.

Finally we can say that z is a common fixed point of S & P.

This completes the proof of theorem.

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