

Construction of Cantor Type Sets in $\mathbb{R}^2$ And $\mathbb{R}^3$ and Study of Their Properties

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Abstract
The aim of this paper is to construct general Cantor like sets in two dimensional space $\mathbb{R}^2$ and three dimensional space $\mathbb{R}^3$ and to study their properties related with areas , volumes , Riemann and Lebesgue integrability.
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1. INTRODUCTION

The well-known Cantor set is a subset of closed interval [0, 1]. It is uncountable, closed, compact, nowhere dense and perfect subset of metric space [0, 1] with absolute value metric and is of measure zero. Moreover, it is a subset of all those real numbers in [0, 1] whose ternary expansion contains the digits 0, 2 and not 1. In the year 2008, Smith- Voltera defined Cantor like subsets in [0, 1] of nonzero measures and studied their properties. Author has defined Cantor like sets in [*a*, *b*] and their studied properties, in [4]. In this paper we will construct general Cantor type sets in two dimensional space R^2 and in three dimensional space R^3. We also study their properties related with areas and volume. We will also use these Cantor type sets to construct examples of Lebesgue integrable functions which are not Riemann integrable.

In this paper section 2 is devoted for construction of Cantor type sets $K^9_{[a,b]^2}$ in square and evaluation of its surface area. In section 3, we will construct Cantor type set $K_D^{(ks)}$ in rectangle and evaluate its area. Section 4, is devoted for construction of Cantor type sets $K^{27}_{[a,b]^3}$ in cube and evaluation of its volume. In section 5, we will see construction of Cantor type sets $K_D^{(kst)}$ in parallelepiped and evaluate of its volume.

2. Construction of Cantor type sets $K^9_{[a,b]^2}$ and computation of its area in a closed and bounded square of area $A = (b-a)^2$:

Step 1: Divide square $[a,b]\times[a,b] = [a,b]^2$ of area A in nine equal parts then remove middle open square $D_{1,1}$ of area $\frac{A}{9}$. This will give eight closed squares $S_{1,1}$, $S_{1,2}$, … $S_{1,8}$ each of area $\frac{A}{9}$.

Step 2: Divide each of these eight squares $D_{2,1}$, $D_{2,2}$,... $D_{2,8}$ in nine equal parts, that is in all 72 squares each of area $\frac{A}{9^2}$. This will gives sixty four (8^2) closed squares $S_{2,1}$, $S_{2,2}$, …, $S_{2,64}$ each of area $\frac{A}{9^2}$.

Step 3 : Divide each of these sixty four squares in nine equal parts that is in all $8^2\times 9 = 576$ squares and remove the middle open squares $D_{3,1}$, $D_{3,2}$,... $D_{3,8^2}$ each of area $\frac{A}{9^3}$. This will gives (8^3) closed squares $S_{2,1}$, $S_{2,2}$, …, $S_{2,8^3}$ each of area $\frac{A}{9^3}$.

Step 4: Continue the process. At n^{th} step remove the middle open squares $D_{n,j}$, $j = 1,2,\dots,8^{(n-1)}$ of area $\frac{A}{9^n}$ from $S_{n,j}$, $j = 1,2,\dots,8^{(n-1)}$ respectively then it will gives 8^n closed squares $S_{n,j}$, $j = 1,2,\dots,8^n$ each of area $\frac{A}{9^n}$.

Step 5: Define $P_n = \bigcup_{j=1}^{8^n} S_{n,j}$, $\forall n$

Step 6: Define $K^9_{[a,b]^2} = \bigcap_{n=1}^{\infty} P_n$

This is our Cantor type set in $[a,b]^2$

Property 1: Cantor type set $K^9_{[a,b]^2}$ is of area zero.

Proof: By construction of Cantor type set

$$G = \left(K^9_{[a,b]^2}\right)^c = \bigcup_{n=1}^{\infty}\bigcup_{j=1}^{8^{(n-1)}} D_{n,j}$$

is open set formed by countable number of disjoint open squares where

$$A(I_{nj}) = m(D_{n,j}) = \frac{A}{9^n} \ ,\forall j = 1,2,3,\dots,8^{(n-1)} .$$

Therefore

$$m(G) = \sum_{n=1}^{\infty}\sum_{j=1}^{8^{(n-1)}} m(D_{n,j})$$

$$m(G) = \sum_{n=1}^{\infty}\sum_{j=1}^{8^{(n-1)}} \frac{A}{9^n}$$

$$m(G) = A\sum_{n=1}^{\infty}\frac{8^{(n-1)}}{9^n}$$

$$m(G) = \frac{A}{8}.\frac{8}{9}\sum_{n=0}^{\infty}\frac{8^n}{9^n}$$

$$m(G) = \frac{A}{8}.\frac{8}{9}\frac{1}{1-\frac{8}{9}}$$

$$m(G) = A$$

Now, $m(G^C) = m\,K^9_{[a,b]^2} = m[a,b]^2 - m(G) = A - A = 0$

Thus Cantor type set $K^9_{[a,b]^2}$ is of area zero.

Example 1: Define $f : [a,b]^2 \to \mathbb{R}$ as

$$f(x) = c\,,\ \text{if } (x,y) \in K^9_{[a,b]^2} = F$$

$$= d\ \text{ if } (x,y) \in \left[K^9_{[a,b]^2}\right]^C = F^C = [a,b]^2 - F$$

This is continuous function at each $(x,y) \in \left[K^9_{[a,b]^2}\right]^C = F^C = [a,b]^2 - F$ and discontinuous at every $(x,y) \in K^9_{[a,b]^2} = F$, where $m\left(K^9_{[a,b]^2} = F\right) = 0$. Therefore f is continuous function i.e. on $[a,b]^2$ and f is also bounded function . Hence, f is Riemann integrable function on $[a,b]$. Since any Riemann integrable function is Lebesgue integrable. Therefore f is also Lebesgue integrable on $[a,b]^2$.

Now,

$$\Re\int_{[a,b]^2} f = \mathcal{L}\int_{[a,b]^2} f = \mathcal{L}\int_F f + \mathcal{L}\int_{F^C} f = \mathcal{L}\int_F c + \mathcal{L}\int_{F^C} d = cm(F) + dm(F^C)$$

$$= c.0 + d(b-a)^2 = d.A\ ,$$

Where, A is area of square.

3. Construction of General Cantor type sets in rectangle $D = [a,b] \times [c,d]$ and computation of its area in a closed and bounded rectangle of area $A = (b-a)(d-c)$:

Step 1: By using middle vertical strip of length $\frac{b-a}{k}$ on X-axis and horizontal strip of length $\frac{d-c}{s}$ on Y-axis where $k,s \geq 3$, divide rectangle D of area A in nine parts and then remove middle open rectangle $D_{1,1}$ of area $\frac{A}{ks}$. This will gives eight closed rectangles $S_{1,1}$, $S_{1,2}$, … $S_{1,8}$ each of area $\leq \frac{A}{9}$.

Step 2 ; Divide each of these eight rectangles in nine parts that is in all 72 rectangles by using middle vertical strip of length $\frac{b-a}{k^2}$ on X-axis and horizontal strip of length on $\frac{d-c}{s^2}$ on Y-axis and remove the middle open rectangles $D_{2,1}$, $D_{2,2}$,... $D_{2,8}$ each of area $\frac{A}{(ks)^2}$. This will gives sixty four (8^2) closed rectangles $S_{2,1}$, $S_{2,2}$, …, $S_{2,64}$ each area $\leq \frac{A}{9^2}$.

Step 3 : Divide each of these sixty four rectangles in nine equal parts that is in all $8^2 \times 9 = 576$ rectangles by using middle vertical strip of length $\frac{b-a}{k^3}$ on X-axis and horizontal strip of length $\frac{d-c}{s^3}$ on Y-axis and remove the middle open rectangles $D_{3,1}$, $D_{3,2}$,... $D_{3,8^2}$ each of area $\frac{A}{(ks)^3}$. This will gives (8^3) closed rectangles $S_{2,1}$, $S_{2,2}$, …, $S_{2,8^3}$ each of area $\leq \frac{A}{9^3}$.

Step 4: Continue the process. At n^{th} step remove the middle open rectangles $D_{n,j}\ , j = 1,2,\dots,8^{(n-1)}$ of area $\frac{A}{(ks)^n}$ from $S_{n,j}\ , j = 1,2,\dots,8^{(n-1)}$ respectively then it will gives 8^n closed rectangles $S_{n,j}\ , j = 1,2,\dots,8^n$ each of area $\leq \frac{A}{9^n}$.

Step 5: Define $P_n = \bigcup_{j=1}^{8^n} S_{n,j}\ \ ,\forall n$

Step 6: Define $K_D^{ks} = \bigcap_{n=1}^{\infty} P_n$

This is our Cantor type set in the rectangle$D$.
Property 1: Cantor type set K_D^{ks} is of area $A - \frac{A}{ks-8}$.
Proof: By construction of Cantor type set

$$G = (K_D^{ks})^C = \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{8^{(n-1)}} D_{n,j}$$

is open set formed by countable number of disjoint open rectangles where

$$A(D_{n,j}) = m(D_{n,j}) = \frac{A}{(ks)^n} \quad \forall j = 1,2,\dots,8^{(n-1)} \ .$$

Therefore,

$$m(G) = \sum_{n=1}^{\infty} \sum_{j=1}^{8^{(n-1)}} m(D_{n,j})$$

$$m(G) = \sum_{n=1}^{\infty} \sum_{j=1}^{8^{(n-1)}} \frac{A}{(ks)^n}$$

$$m(G) = A \sum_{n=1}^{\infty} \frac{8^{(n-1)}}{(ks)^n}$$

$$m(G) = \frac{A}{8} \cdot \frac{8}{(ks)} \sum_{n=0}^{\infty} \frac{8^n}{(ks)^n}$$

$$m(G) = \frac{A}{8} \cdot \frac{8}{(ks)} \frac{1}{1 - \frac{8}{(ks)}}$$

$$m(G) = \frac{A}{ks - 8}$$

Now, $m(G^C) = m\,(K_D^{ks}) = m(D) - m(G) = A - \frac{A}{ks-8}$
Thus, Cantor type set K_D^{ks} is of area $A - \frac{A}{ks-8}$.
Remark: If $k = s = 3$ then $m\,(K_D^{ks}) = 0$ and if $k \text{ or } s > 3$ then $(K_D^{ks}) > 0$.
Example 1: Define $f : D \to \mathbb{R}$ as

$$f(x\,,y) = p\,, \text{ if } (x,y) \in K_D^{ks} = F$$
$$= q \text{ if } (x,y) \in [K_D^{ks}]^C = F^C = D - F$$

Case i) if $k = s = 3$ then f is a continuous function at each $(x,y) \in F^C = D - F$ as F^C is open set containing open rectangles and discontinuous at every $(x,y) \in F$, as F contains no open set. Therefore f is continuous a.e. on D as $m(F) = 0$, if $k = s = 3$ and f is also bounded function. Hence, f is Riemann integrable function on D. Since, any Riemann integrable function is Lebesgue integrable. Therefore, f is also Lebesgue integrable function on D.
Now,

$$\Re \int_D f = \mathcal{L} \int_D f = \mathcal{L} \int_F f + \mathcal{L} \int_{F^C} f = \mathcal{L} \int_F p + \mathcal{L} \int_{F^C} q = pm(F) + qm(F^C) = p.0 + qA = q.A \ ,$$

Where, A is area of rectangle.
Case ii) If $k \text{ or } s > 3$ from $k, s \geq 3$ then f is continuous function at each $(x,y) \in F^C$ as F^C is open set containing open rectangles and discontinuous at every$(x,y) \in F$ as F contains no open set. Therefore f is not continuous a.e. on D as $m(F) > 0$ and hence f is not Riemann integrable function on D.Moreover, f is bounded function as range of f is $\{p,q\}$.

Now, we prove f is measurable function.
Let r be arbitrary real number and without loss of generality suppose $p < q$, then
Sub case i) If $r < p$ then $f^{-1}\{(r,\infty)\} = f^{-1}\{p,q\} = D$ is measurable.
Sub case ii) If $p \leq r < q$ then $f^{-1}\{(r,\infty)\} = f^{-1}\{q\} = F^C$ is measurable as any open subset of D is measurable.
Sub case iii) If $r \geq q$ then $f^{-1}\{(r,\infty)\} = f^{-1}\{\phi\} = \phi$ is measurable.
Thus, for any $r \in \mathbb{R}$ the set $f^{-1}\{(r,\infty)\}$ is measurable subset of D. Therefore, f is measurable and bounded function on D.
Therefore, f is Lebesgue integrable on D.
Now,

$\mathcal{L}\int_D f = \mathcal{L}\int_F f + \mathcal{L}\int_{F^C} f = \mathcal{L}\int_F p + \mathcal{L}\int_{F^C} q = pm(F) + qm(F^C) = p\left[A - \frac{A}{ks-8}\right] + q\left[\frac{A}{ks-8}\right]$,
Where, A is area of rectangle D.

Example 2: Define $f: D \to \mathbb{R}$ as

$$f(x,y) = xy, \text{ if } (x,y) \in F^C = D - F$$
$$= x^2 - y^2 \text{ if } (x,y) \in K_D^{ks} = F$$

Case i) If $k = s = 3$ then f is continuous function at each $(x,y) \in F^C = D - F$ as F^C is open set containing open rectangles and discontinuous at every $(x,y) \in F$, if $k = s = 3$ as F contains no open set . Therefore f is continuous a.e. on D as $m(F) = 0$ and f is also bounded function. Hence f is Riemann integrable function on D. Since, any Riemann integrable function is Lebesgue integrable. Therefore, f is also Lebesgue integrable function on$D$.
Now,

$$\Re\int_D f = \mathcal{L}\int_D f = \mathcal{L}\int_F f + \mathcal{L}\int_{F^C} f = \mathcal{L}\int_D f\chi_F + \mathcal{L}\int_D f\chi_{F^C}$$
$$= \int_a^b\int_c^d xy\,dxdy + \int_a^b\int_c^d 0\,dxdy \text{ , as } m(F) = 0 \text{ and } m(F^C) = A$$
$$= \frac{(b-a)^2(d-c)^2}{4} + 0 = \frac{(b-a)^2(d-c)^2}{4}$$

Case ii) If k or $s > 3$ from $k, s \geq 3$ then f is continuous function at each $(x,y) \in F^C = D - F$ as F^C is open set containing open rectangles and discontinuous at every $(x,y) \in F$, as F contains no open set. Therefore f is not continuous a.e. on D as $m(F) > 0$ and hence f is not Riemann integrable function on D.
Now $g(x,y) = xy$ and $h(x,y) = x^2 - y^2$ are continuous and bounded functions on D. Moreover, F and F^C are measurable subsets of D. Therefore $f = g\chi_F + h\chi_{F^C}$ is bounded and measurable function and hence, f is Lebesgue integrable function on D.

4. Construction of General Cantor type sets $K^{27}_{[a,b]^3}$ in a closed and bounded cube of volume $V = (b-a)^3$:

Step 1: Divide square $[a,b] \times [a,b] \times [a,b] = [a,b]^3$ of volume $V = (b-a)^3$ in twenty seven $(3^3 = 27)$ equal parts then remove middle open cube $D_{1,1}$ of volume $\frac{V}{27}$. This will give twenty six closed cubes $S_{1,1}$, $S_{1,2}$, … $S_{1,26}$ each of volume $\frac{V}{27}$.

Step 2: Divide each of these twenty six closed cubes in 27 equal parts i.e.in all 26×27 cubes and remove the middle open cubes $D_{2,j}$, $j = 1,2,3,\ldots,26 \times 27$, each of volume$\frac{V}{27^2}$. This will gives 26^2 closed cubes $S_{2,1}$, $S_{2,2}$, …, $S_{2,26^2}$ each of volume $\frac{V}{27^2}$.

Step 3: Divide each of these 26^2 cubes in 27 equal parts that is in all $26^2 \times 27$ cubes and remove the middle open cubes $D_{3,1}$, $D_{3,2}$,… $D_{3,26^2}$ each of volume $\frac{V}{27^3}$. This will gives(26^3) closed cubes $S_{2,1}$, $S_{2,2}$, …, $S_{2,26^3}$ each of volume $\frac{V}{27^3}$.

Step 4: Continue the process. At n^{th} step remove the middle open cubes $D_{n,j}$, $j = 1,2,\ldots,26^{(n-1)}$ of volume $\frac{V}{27^n}$.from $S_{n,j}$, $j = 1,2,\ldots,26^{(n-1)}$ respectively then it will gives 26^n closed cubes $S_{n,j}$, $j = 1,2,\ldots,26^n$ each of volume $\frac{V}{27^n}$.

Step 5: Define $P_n = \bigcup_{j=1}^{26^n} S_{n,j}$, $\forall n$

Step 6: Define $K^{27}_{[a,b]^2} = \bigcap_{n=1}^{\infty} P_n$

This is our Cantor type set in $[a,b]^3$.

Property 1: Cantor type set $K^{27}_{[a,b]^3}$ is of volume zero.

Proof: By construction of Cantor type set

$$G = \left[K^{27}_{[a,b]^3}\right]^C = \bigcup_{n=1}^{\infty}\bigcup_{j=1}^{26^{(n-1)}} D_{n,j}$$

is open set formed by countable number of disjoint open cubes where

$$V(I_{nj}) = m(D_{n,j}) = \frac{V}{27^n} \text{ , } \forall j = 1,2,3,\ldots,26^{(n-1)}.$$

Therefore

$$m(G) = \sum_{n=1}^{\infty}\sum_{j=1}^{26^{(n-1)}} m(D_{n,j})$$

$$m(G) = \sum_{n=1}^{\infty} \sum_{j=1}^{26^{(n-1)}} \frac{V}{27^n}$$

$$m(G) = V \sum_{n=1}^{\infty} \frac{26^{(n-1)}}{27^n}$$

$$m(G) = \frac{V}{26} \cdot \frac{26}{27} \sum_{n=0}^{\infty} \frac{26^n}{27^n}$$

$$m(G) = \frac{V}{26} \cdot \frac{26}{27} \frac{1}{1 - \frac{26}{27}}$$

$$m(G) = V$$

Now, $m(G^C) = m\, K^{27}_{[a,b]^3} = m[a,b]^3 - m(G) = V - V = 0$

Thus, Cantor type set $K^{27}_{[a,b]^3}$ is of area zero.

5. **Construction of General Cantor type sets in parellelopiped $\boldsymbol{P = [a,b] \times [c,d] \times [e,f]}$ and computation of its volume in a closed and bounded parellelopiped of volume $\boldsymbol{V = (b-a)(d-c)(f-e)}$:**

Step 1 : By using middle vertical strip of length $\frac{b-a}{k}$ on X-axis and horizontal strip of length $\frac{d-c}{s}$ on Y-axis and strip of length $\frac{f-e}{t}$ on Z-axis where $\boldsymbol{k, s, t \geq 3}$, divide parellelopiped P of volume V in 27 parts and then remove middle open parellelopiped $P_{1,1}$ of volume $\frac{V}{kst}$. This will gives 27 closed parellelopiped $S_{1,1}$, $S_{1,2}$, … $S_{1,26}$ each of volume $\leq \frac{V}{27}$.

Step 2 : Divide each of these 26 parellelopiped in 27 parts that is in all 26×27 parellelopiped by using middle vertical strip of length $\frac{b-a}{k^2}$ on X-axis and horizontal strip of length on $\frac{d-c}{s^2}$ on Y-axis and strip of length on $\frac{f-e}{t^2}$ on Z-axis and remove the middle open parellelopiped $P_{2,1}$, $P_{2,2}$,… $P_{2,26}$ each of volume $\frac{V}{(kst)^2}$. This will gives 26^2 closed parellelopiped $S_{2,1}$, $S_{2,2}$, …, $S_{2,26^2}$ each volume $\leq \frac{V}{27^2}$.

Step 3 : Divide each of these 26^2 parellelopiped in 27 equal parts that is in all $26^2 \times 27$ parellelopiped by using middle vertical strip of length $\frac{b-a}{k^3}$ on X-axis and horizontal strip of length $\frac{d-c}{s^3}$ on Y-axis and strip of length on $\frac{f-e}{t^2}$ on Z-axis and remove the middle open parellelopiped $P_{3,1}$, $P_{3,2}$,… $P_{3,26^2}$ each of volume $\frac{V}{(kst)^3}$. This leaves (26^3) closed parellelopiped $S_{2,1}$, $S_{2,2}$, …, $S_{2,26^3}$ each of volume $\leq \frac{V}{27^3}$.

Step 4: Continue the process. At n^{th} step remove the middle open parellelopiped $P_{n,j}$, $j = 1,2,\ldots,26^{(n-1)}$ of volume $\frac{V}{(kst)^n}$ from $S_{n,j}$, $j = 1,2,\ldots,26^{(n-1)}$ respectively then it will gives 26^n closed parellelopiped $S_{n,j}$, $j = 1,2,\ldots,26^n$ each of volume $\leq \frac{V}{27^n}$.

Step 5 : Define $P_n = \bigcup_{j=1}^{26^n} S_{n,j}$, $\forall n$

Step 6 : Define $K_P^{kst} = \bigcap_{n=1}^{\infty} P_n$

This is our Cantor type set in parellelopiped P.

Property 1: Cantor type set K_P^{kst} is of volume $V - \frac{V}{kst-27}$.

Proof: By construction of Cantor type set

$$G = (K_P^{kst})^C = \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{27^{(n-1)}} P_{n,j}$$

is open set formed by countable number of disjoint open parellelopiped where

$$V(P_{n,j}) = m(P_{n,j}) = \frac{V}{(kst)^n} \quad \forall j = 1,2,\ldots,26^{(n-1)} .$$

Therefore,

$$m(G) = \sum_{n=1}^{\infty} \sum_{j=1}^{26^{(n-1)}} m(P_{n,j})$$

$$m(G) = \sum_{n=1}^{\infty} \sum_{j=1}^{26^{(n-1)}} \frac{V}{(kst)^n}$$

$$m(G) = V\sum_{n=1}^{\infty}\frac{26^{(n-1)}}{(kst)^n}$$
$$m(G) = \frac{V}{26}.\frac{26}{(kst)}\sum_{n=0}^{\infty}\frac{26^n}{(kst)^n}$$
$$m(G) = \frac{V}{26}.\frac{26}{(kst)}\frac{1}{1-\frac{26}{(kst)}}$$
$$m(G) = \frac{V}{kst-26}$$

Now, $m(G^C) = m\,(K_P^{kst}) = m(P) - m(G) = V - \frac{V}{kst-26}$

Thus, Cantor type set K_P^{kst} is of volume $V - \frac{V}{kst-26}$.

Remark: If $k = s = t = 3$ then $m[K_P^{kst}] = 0$ and if $k \; or \; s \; or \; t > 3$ then $m[K_P^{kst}] > 0$.

Example 1 : Define $f : P \to \mathbb{R}$ as

$$f(x\,,y,z) = p\,,\ \text{if}\ (x\,,y,z) \in K_P^{kst} = F$$
$$= q\ \text{ if}\ (x\,,y,z) \in [K_P^{kst}]^C = F^C = P - F$$

Case i) If $k = s = t = 3$ then f continuous function at each $(x\,,y,z) \in F^C = P - F$ as F^C is open set containing open parellelopiped and discontinuous at every $(x\,,y,z) \in F$, as F contains no open set. Therefore f is continuous a.e. on P as $m(F) = 0$, if $k = s = t = 3$ and f is also bounded function. Hence f is Riemann integrable function on P. Since, any Riemann integrable function is Lebesgue integrable. Therefore, f is also Lebesgue integrable function on P.

Now,

$$\Re\int_P f = \mathcal{L}\int_P f = \mathcal{L}\int_F f + \mathcal{L}\int_{F^C} f = \mathcal{L}\int_F p + \mathcal{L}\int_{F^C} q = pm(F) + qm(F^C) = p.0 + qV = q.V\ ,$$

Where V is volume of parellelopiped P.

Case ii) If $k \; or \; s \; or \; t > 3$ among $k, s\; t \geq 3$ then f is continuous function at each $(x\,,y,z) \in F^C = P - F$ as F^C is open set containing open parellelopiped and discontinuous at every $(x\,,y,z) \in F$, as F contains no open set. Therefore f is not continuous a.e. on P, as $m(F) > 0$ and hence f is not Riemann integrable function on P. As range of f is $\{p, q\}$ implies f is bounded function.

Now, we prove f is measurable function.

Let r be arbitrary real number and without loss of generality suppose $p < q$, then

Sub case i) If $r < p$ then $f^{-1}\{(r,\infty)\} = f^{-1}\{p, q\} = P$ is measurable.

Sub case ii) If $p \leq r < q$ then $f^{-1}\{(r,\infty)\} = f^{-1}\{q\} = F^C$ is measurable as any open subset of P is measurable.

Sub case iii) If $r \geq q$ then $f^{-1}\{(r,\infty)\} = f^{-1}\{\phi\} = \phi$ is measurable.

Thus, for any $r \in \mathbb{R}$ the set $f^{-1}\{(r,\infty)\}$ is measurable subset of P. Therefore, f is measurable and bounded function on P.

Therefore, f is Lebesgue integrable on P.

Now,

$$\mathcal{L}\int_P f = \mathcal{L}\int_F f + \mathcal{L}\int_{F^C} f = \mathcal{L}\int_F p + \mathcal{L}\int_{F^C} q = pm(F) + qm(F^C)$$
$$= p\left[V - \frac{V}{kst-26}\right] + \left[q\frac{V}{kst-26}\right]\ ,$$

Where V is volume of parellelopiped P.

Conclusions

(i) We construct Cantor type sets in $\mathbb{R}^2$ and $\mathbb{R}^3$ of zero measure as well as of nonzero measures.

(ii) We study their properties related with Riemann and Lebesgue integration.

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