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On Semi - Symmetric Projective Connection

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Abstract
In this paper we consider a new connection called *Semi-Symmetric Projective Connection* $\wp\Gamma = \left(\bar{G}^i_{jk}, \bar{G}^i_j, 0\right)$. The covariant differentiation with respect to this connection is defined and the commutation formulae for directional differentiation, Berwald covariant differentiation and semi-symmetric projective covariant differentiation have been obtained. Relations between the curvature tensors and torsion tensors arising from Berwald connection $B\Gamma$ and semi-symmetric projective $\wp\Gamma$ connection have also been obtained. Bianchi identities have also been derived.

1. Introduction

Unlike a Riemannian space, a Finsler space possesses various types of connections. Berwald [1] was the first man who introduced the concept of connection in the theory of a Finsler space. He constructed a connection from the standpoint of so-called geometry of paths. He started his theory from the equation of geodesics and applied the theory of general paths to define the connection $B\Gamma = \left(G^i_{jk}, G^i_j, 0\right)$. In 1933, E. Catan [2] produced a connection along the line of his general concept of Euclidean connection. He introduced a system of axioms to give uniquely a Finsler connection $C\Gamma = \left(F^i_{jk}, G^i_j, C^i_{jk}\right)$ from the fundamental function.

In 1951, a young German H. Rund [4] introduced a new process of parallelism from the standpoint of Minkowskian geometry to give a connection $R\Gamma = \left(F^i_{jk}, G^i_j, 0\right)$, while Cartan introduced parallelism from the standpoint of Euclidean geometry.

In 1969, Hashiguchi [3] discussed with Matsumoto about a new connection, called Hashiguchi connection $H\Gamma = \left(G^i_{jk}, G^i_j, C^i_{jk}\right)$.

2. Semi-Symmetric Projective Connection

Let us consider a connection $\wp\Gamma = \left(\bar{G}^i_{jk}, \bar{G}^i_j, 0\right)$ given by

(2.1) $$\bar{G}^i_{jk} \overset{def}{=} G^i_{jk} + p_k \delta^i_j,$$

and $\bar{G}^i_k \overset{def}{=} \bar{G}^i_{jk}\dot{x}^j$, where p_k is a covariant vector which is positively homogeneous of degree zero in $\dot{x}^i$.

Since the $h(h)$ - torsion tensor $\bar{T}^i_{jk}$ of this connection is given by

$$\bar{T}^i_{jk} = \bar{G}^i_{jk} - \bar{G}^i_{kj} = p_k\delta^i_j - p_j\delta^i_k,$$

this connection is semi-symmetric connection.

Transvecting (2.1) with $\dot{x}^j$, we have

(2.2) $$\bar{G}^i_k = G^i_k + p_k\dot{x}^i.$$

Again the transvection by $\dot{x}^k$ gives

$$\bar{G}^i_k\,\dot{x}^k \overset{def}{=} 2\bar{G}^i = 2G^i + p\,\dot{x}^i.$$

i.e.

(2.3) $$\bar{G}^i = G^i + \frac{p}{2}\dot{x}^i,$$

where $p \overset{def}{=} p_k \dot{x}^k$.

Equation (2.3) represents a projective change of the function G^i. Therefore we call the connection $\wp\Gamma$ as a semi-symmetric projective connection.

Differentiating (2.2) partially with respect to $\dot{x}^j$ and using (1.9.3b), we have

$$\dot{\partial}_j \bar{G}^i_k = G^i_{jk} + p_{jk}\dot{x}^i + p_k \delta^i_j,$$

which in view of (2.1), gives

(2.4) $$\dot{\partial}_j \bar{G}^i_k = \bar{G}^i_{jk} + p_{jk}\dot{x}^i,$$

where $p_{jk} \overset{def}{=} \dot{\partial}_j p_k$.

This gives a relation between the connection coffecients of the semi-symmetric projective connection $\wp\Gamma$.

Differentiating (2.1) partially with respect to $\dot{x}^r$ and using (1.9.4a), we have

(2.5) $$\dot{\partial}_r \bar{G}^i_{jk} \overset{def}{=} \bar{G}^i_{rjk} = G^i_{rjk} + p_{rk}\delta^i_j.$$

Transvecting (2.5) by $\dot{x}^r$ and using (1.9.4c), we get

$$\dot{x}^r \bar{G}^i_{rjk} = 0.$$

Again transvecting (2.5) by $\dot{x}^j$, we get

(2.6) $$\dot{x}^j \bar{G}^i_{rjk} = p_{rk}\dot{x}^i.$$

Using this, the equation (2.4) becomes

(2.7) $$\dot{\partial}_j \bar{G}^i_k = \bar{G}^i_{jk} + \bar{G}^i_{jrk}\dot{x}^r.$$

Contracting *i* and *j* in (2.1), we get

(2.8) $$\bar{G}^r_{rk} = G^r_{rk} + np_k.$$

By eliminating p_k in (2.1) and (2.8), we have

$$\bar{G}^i_{jk} = G^i_{jk} + \frac{1}{n}\delta^i_j(\bar{G}^r_{rk} - G^r_{rk}),$$

i.e.

$$\bar{G}^i_{jk} - \frac{1}{n}\delta^i_j \bar{G}^r_{rk} = G^i_{jk} - \frac{1}{n}\delta^i_j G^r_{rk}.$$

Therefore we get n^3 quantities θ^i_{jk} defined by

(2.9) $$\theta^i_{jk} = G^i_{jk} - \frac{1}{n}\delta^i_j G^r_{rk},$$

which are invariant under the semi-symmetric projective change (2.1).

Similarly the contraction of *i* and *k* in (2.1) gives

(2.10) $$\bar{G}^r_{jr} = G^r_{jr} + p_j.$$

Eliminating p_j in (2.1) and (2.10), we get

$$\bar{G}^i_{jk} = G^i_{jk} + \delta^i_j(\bar{G}^r_{kr} - G^r_{kr}),$$

i.e.

$$\bar{G}^i_{jk} - \delta^i_j \bar{G}^r_{kr} = G^i_{jk} - \delta^i_j G^r_{kr}.$$

Again we get n^3 quantities η^i_{jk} defined by

(2.11) $$\eta^i_{jk} = G^i_{jk} - \delta^i_j G^r_{kr},$$

which are invariant under the semi-symmetric projective change (2.1).

3. Semi-Symmetric Projective Covariant Differentiation

The semi-symmetric projective covariant derivative of an arbitrary tensor T^i_j is defined as

(3.1) $$\wp_k T^i_j = \partial_k T^i_j - (\dot{\partial}_r T^i_j)\bar{G}^r_k + T^r_j \bar{G}^i_{rk} - T^i_r \bar{G}^r_{jk}.$$

Taking semi-symmetric projective covariant differentiation of y^i, we get

$$\wp_k y^i = \partial_k y^i - (\dot{\partial}_r y^i)\bar{G}^r_k + y^r \bar{G}^i_{rk}$$

$$= -\delta^i_r \bar{G}^r_k + \bar{G}^i_k = 0.$$

Therefore the tangential vector y^i is covariant constant with respect to semi-symmetric projective connection $\wp\Gamma$.

Semi-symmetric projective covariant derivative of g_{ij} is given by

$$\wp_k g_{ij} = \partial_k g_{ij} - (\dot{\partial}_r g_{ij})\bar{G}^r_k - g_{rj}\bar{G}^r_{ik} - g_{ir}\bar{G}^r_{jk}.$$

In view of (2.1), (2.2), (1.5.2) and (1.9.5) above equation may be written as

$$\wp_k g_{ij} = B_k g_{ij} - 2 p_k g_{ij},$$

which in view of (1.9.9), can be written as

$$\wp_k g_{ij} = y_r G^r_{ijk} - 2 g_{ij} p_k .$$

This shows that the semi-symmetric projective connection is not metrical.

4. Commutation Formula for Semi-Symmetric Projective Covariant Differential Operator and Directional Differential Operator

Let X^i be an arbitrary contravariant vector. Then

(4.1) $$\wp_k X^i = \partial_k X^i - (\dot{\partial}_r X^i)\bar{G}^r_k + X^r \bar{G}^i_{rk}.$$

Differentiating (4.1) partially with respect to $\dot{x}^j$, we have

$$\dot{\partial}_j(\wp_k X^i) = \dot{\partial}_j(\partial_k X^i) - \dot{\partial}_j(\dot{\partial}_r X^i)\bar{G}^r_k - (\dot{\partial}_r X^i)(\dot{\partial}_j \bar{G}^r_k) + (\dot{\partial}_j X^r)\bar{G}^i_{rk} + X^r(\dot{\partial}_j \bar{G}^i_{rk}).$$

Using (2.5) and (2.7), we get

(4.2) $$\dot{\partial}_j(\wp_k X^i) = \dot{\partial}_j(\partial_k X^i) - \dot{\partial}_j(\dot{\partial}_r X^i)\bar{G}^r_k - (\dot{\partial}_r X^i)(\bar{G}^r_{jk} + \bar{G}^r_{jsk}\dot{x}^s) + (\dot{\partial}_j X^r)\bar{G}^i_{rk} + X^r \bar{G}^i_{jrk}.$$

The semi-symmetric projective covariant derivative of $\dot{\partial}_j X^i$ is given by

(4.3) $$\wp_k(\dot{\partial}_j X^i) = \partial_k(\dot{\partial}_j X^i) - \dot{\partial}_r(\dot{\partial}_j X^i)\bar{G}^r_k + (\dot{\partial}_j X^r)\bar{G}^i_{rk} - (\dot{\partial}_j X^i)\bar{G}^r_{jk}.$$

From (4.2) and (4.3), we get

(4.4) $$\dot{\partial}_j(\wp_k X^i) - \wp_k(\dot{\partial}_j X^i) = -(\dot{\partial}_r X^i)\bar{G}^r_{jsk}\dot{x}^s + X^r \bar{G}^i_{jrk}.$$

Similarly for a covariant vector and a tensor, we have

$$\dot{\partial}_j(\wp_k X_i) - \wp_k(\dot{\partial}_j X_i) = -(\dot{\partial}_r X_i)\bar{G}^r_{jsk}\dot{x}^s - X_r \bar{G}^r_{jik},$$

and

$$\dot{\partial}_h(\wp_k T^i_j) - \wp_k(\dot{\partial}_h T^i_j) = -(\dot{\partial}_r T^i_j)\bar{G}^r_{hsk} y^s + T^r_j \bar{G}^i_{hrk} - T^i_r \bar{G}^r_{hjk}.$$

The tensor $\bar{G}^r_{hjk}$ is called *hv*-curvature tensor with respect to the semi-symmetric projective connection.

5. Ricci Commutation Formula

Applying the semi-symmetric projective covariant differentiation to $\wp_k X^i$, we get

(5.1) $$\wp_h(\wp_k X^i) = \partial_h(\wp_k X^i) - \dot\partial_r(\wp_k X^i)\bar G^r_h + (\wp_k X^r)\bar G^i_{rh} - (\wp_r X^i)\bar G^r_{kh}.$$

In view of (4.1), above equation can be written as

(5.2) $$\begin{aligned}\wp_h\wp_k X^i &= \partial_h\left\{\partial_k X^i - (\dot\partial_r X^i)\bar G^r_k + X^r\bar G^i_{rk}\right\}\\ &-\dot\partial_r\left\{\partial_k X^i - (\dot\partial_s X^i)\bar G^s_k + X^s\bar G^i_{sk}\right\}\bar G^r_h\\ &+\left\{\partial_k X^r - (\dot\partial_s X^r)\bar G^s_k + X^s\bar G^r_{sk}\right\}\bar G^i_{rh}\\ &-\left\{\partial_r X^i - (\dot\partial_s X^i)\bar G^s_r + X^s\bar G^i_{sr}\right\}\bar G^r_{kh}.\end{aligned}$$

On simplifying, we get

(5.3) $$\begin{aligned}\wp_h\wp_k X^i &= \partial_h\partial_k X^i - \partial_h(\dot\partial_r X^i)\bar G^r_k - (\dot\partial_r X^i)(\partial_h\bar G^r_k) + (\partial_h X^r)\bar G^i_{rk}\\ &+ X^r(\partial_h\bar G^i_{rk}) - \dot\partial_r(\partial_k X^i)\bar G^r_h + (\dot\partial_r\dot\partial_s X^i)\bar G^s_k\bar G^r_h\\ &+ (\dot\partial_s X^i)(\bar G^s_{rk} + (\dot\partial_r\bar G^s_k)\bar G^r_h - (\dot\partial_r X^s)\bar G^i_{sk}\bar G^r_h - X^s\bar G^i_{rsk}\bar G^r_h\\ &+ (\partial_k X^r)\bar G^i_{rh} - (\dot\partial_s X^r)\bar G^s_k\bar G^i_{rh} + X^s\bar G^r_{sk}\bar G^i_{rh}\\ &- (\partial_r X^i)\bar G^r_{kh} + (\dot\partial_s X^i)\bar G^s_r\bar G^r_{kh} - X^s\bar G^i_{sr}\bar G^r_{kh}.\end{aligned}$$

Interchanging the indices *h* and *k*, we obtain

(5.4) $$\begin{aligned}\wp_k(\wp_h X^i) &= \partial_k\partial_h X^i - \partial_k(\dot\partial_r X^i)\bar G^r_h - (\dot\partial_r X^i)(\partial_k\bar G^r_h) + (\partial_k X^r)\bar G^i_{rh} + X^r(\partial_k\bar G^i_{rh})\\ &- \dot\partial_r(\partial_h X^i)\bar G^r_k + (\dot\partial_r\dot\partial_s X^i)\bar G^s_h\bar G^r_k + (\dot\partial_s X^i)(\dot\partial_r\bar G^s_h)\bar G^r_k\\ &- (\dot\partial_r X^s)\bar G^i_{sh}\bar G^r_k - X^s\bar G^i_{rsh}\bar G^r_k + (\partial_h X^r)\bar G^i_{rk} - (\dot\partial_s X^r)\bar G^s_h\bar G^i_{rk}\\ &+ X^s\bar G^r_{sh}\bar G^i_{rk} - (\partial_r X^i)\bar G^r_{hk} + (\dot\partial_s X^i)\bar G^s_r\bar G^r_{hk} - X^s\bar G^i_{sr}\bar G^r_{hk}.\end{aligned}$$

From (5.3) and (5.4) we get

(5.5) $$\begin{aligned}\wp_k(\wp_h X^i) - \wp_h(\wp_k X^i) &= X^r\{\partial_k\bar G^i_{rh} + \bar G^i_{srk}\bar G^s_h + \bar G^s_{rh}\bar G^i_{sk} + \bar G^i_{rs}\bar G^s_{kh} - k/h\}\\ &- \dot\partial_r X^i\{\partial_k\bar G^r_h + \bar G^r_{sk}\bar G^s_h + p_{sk}y^r\bar G^s_h + \bar G^r_s\bar G^s_{kh} - k/h\}\\ &- \partial_r X^i\{\bar G^r_{hk} - k/h\}.\end{aligned}$$

Using (4.1), above equation may be written as

(5.7) $$\wp_k(\wp_h X^i) - \wp_h(\wp_k X^i) = X^r\bar R^i_{rhk} - (\dot\partial_r X^i)\bar R^r_{hk} - (\wp_r X^i)\bar T^r_{hk},$$

where

(5.8) $$\bar R^i_{rhk} = \partial_k\bar G^i_{rh} + \bar G^s_{rh}\bar G^i_{sk} - \bar G^i_{srh}\bar G^s_k - h/k.$$

(5.9)
$$\overline{R}^r_{hk} = \partial_k \overline{G}^r_h - (\dot{\partial}_s \overline{G}^r_h)\overline{G}^s_k - h/k,$$
$$= \partial_k \overline{G}^r_h - \overline{G}^r_{sh}\overline{G}^s_k - \overline{G}^r_{sjh}\dot{x}^j \overline{G}^s_k - h/k,$$

(5.10)
$$\overline{T}^r_{hk} = \overline{G}^r_{hk} - \overline{G}^r_{kh}.$$

The tensors $\overline{R}^i_{rhk}$, $\overline{R}^r_{hk}$ and $\overline{T}^r_{hk}$ are called *h*-curvature tensor, *(v)h*- torsion tensor and *(h)h*- torsion tensor respectively with respect to semi-symmetric projective connection (2.1).

Transvecting (5.8) by $\dot{x}^r$, we get
$$\dot{x}^r \overline{R}^i_{rhk} = \partial_k \overline{G}^i_{rh}\dot{x}^r - \overline{G}^i_{srh}\overline{G}^s_k\dot{x}^r + \overline{G}^i_{sk}\overline{G}^s_{rh}\dot{x}^r - h/k$$
$$= \partial_k \overline{G}^i_h - \overline{G}^s_k \overline{G}^i_{srh}\dot{x}^r + \overline{G}^i_{sk}\overline{G}^s_h - h/k$$
$$= \overline{R}^i_{hk}.$$

Therefore

(5.11)
$$\overline{R}^i_{rhk}\dot{x}^r = \overline{R}^i_{hk}.$$

Differentiating $\overline{R}^i_{hk}$ partially with respect to $\dot{x}^r$, we obtain
$$\dot{\partial}_r \overline{R}^i_{hk} = \dot{\partial}_r\{\partial_k \overline{G}^i_h - \overline{G}^s_k \overline{G}^i_{sjh}\dot{x}^j + \overline{G}^i_{sk}\overline{G}^s_h - h/k\}$$
$$= \dot{\partial}_k \partial_k \overline{G}^i_h - (\dot{\partial}_r \overline{G}^s_k)\overline{G}^i_{sjh}\dot{x}^j - \overline{G}^s_k(\dot{\partial}_r \overline{G}^i_{sjh})\dot{x}^j$$
$$- \overline{G}^s_k \overline{G}^i_{sjh}\delta^j_r + (\dot{\partial}_r \overline{G}^i_{sk})\overline{G}^s_h + \overline{G}^i_{sk}(\dot{\partial}_r \overline{G}^s_h) - h/k$$
which is in view of (2.5)and (2.7) can be written as

(5.12)
$$\dot{\partial}_r \overline{R}^i_{hk} = \overline{R}^i_{rhk} + \overline{E}^i_{rhk},$$
where

(5.13)
$$\overline{E}^i_{rhk} = \partial_k \overline{G}^i_{sjh}\dot{x}^j - \overline{G}^s_{rk}\overline{G}^i_{sjh}\dot{x}^j - \overline{G}^s_{rtk}\dot{x}^t \overline{G}^i_{sjh}\dot{x}^j$$
$$- (\dot{\partial}_r \overline{G}^i_{sjh})\overline{G}^s_k \dot{x}^j \overline{G}^i_{rsk}\overline{G}^s_h + \overline{G}^i_{sk}\overline{G}^s_{rjh}\dot{x}^j - h/k.$$

6. Relation between Curvature Tensors and Torsion Tensors arising from Berwald Connection and Semi - Symmetric Projective Connection

In view of (2.1), (2.2) and (2.5) equation (5.8) may be written as
$$R^i_{rhk} = \partial_k(G^i_{rh} + p_h\delta^i_r) + (G^s_{rh} + p_h\delta^s_r)(G^i_{sk} + p_k\delta^i_s)$$
$$- (G^i_{srh} + p_{sh}\delta^i_r)(G^s_k + p_k\dot{x}^s) - h/k$$
$$= (\partial_k G^i_{rh} - G^i_{srh}G^s_k + G^s_{rh}G^i_{sk}) + [(\partial_k p_h - G^s_k p_{sh})\delta^i_r$$
$$+ (G^s_{rh}p_k\delta^i_s + p_h\delta^s_r G^i_{sk} + p_k p_h \delta^i_s \delta^s_r) - h/k],$$
which in view of (1.10.2), (1.9.4c) and and $p_{sh}\dot{x}^s = 0$ can be written as

(6.1)
$$R^i_{rhk} = H^i_{rhk} + (\partial_k p_h - G^s_k p_{sh} - h/k)\delta^i_r.$$

The semi-symmetric projective covariant derivative of p_h is given by

(6.2)
$$\wp_k p_h = \partial_k p_h - (\dot{\partial}_r p_h)\overline{G}^r_k - p_r \overline{G}^r_{hk}.$$

In view of (6.2), equation (6.1) becomes

$$\bar{R}^i_{rhk} = H^i_{rhk} + \delta^i_r \left\{ \wp_k p_h + p_{rh}\bar{G}^r_k + p_r \bar{G}^r_{hk} - G^s_k p_{sh} - h/k \right\}$$

$$= H^i_{rhk} + \delta^i_r \left\{ \wp_k p_h + p_{rh}(\bar{G}^r_k - G^r_k) + p_r \bar{G}^r_{hk} - h/k \right\}$$

Using (2.1) and (2.2), we get

$$\bar{R}^i_{rhk} = H^i_{rhk} + \delta^i_r \left\{ \wp_k p_h + p_{rh} p_k \dot{x}^r + p_r G^r_{hk} + p_r p_k \delta^r_h - h/k \right\}$$

i.e.

(6.3) $$\bar{R}^i_{rhk} = H^i_{rhk} + \delta^i_r \left\{ \wp_k p_h - \wp_h p_k \right\}.$$

Transvecting (6.3) by $\dot{x}^r$, we get

(6.4) $$\bar{R}^i_{hk} = H^i_{hk} + \dot{x}^i \left\{ \wp_k p_h - \wp_h p_k \right\}.$$

7. Bianchi Identities

From (6.3), we have

$$\bar{R}^i_{jkh} + \bar{R}^i_{khj} + \bar{R}^i_{hjk} = H^i_{jkh} + H^i_{khj} + H^i_{hjk} + \delta^i_j \left\{ \wp_h p_k - \wp_k p_h \right\}$$

$$+ \delta^i_k \left\{ \wp_j p_h - \wp_h p_j \right\} + \delta^i_h \left\{ \wp_k p_j - \wp_j p_k \right\}.$$

Using (1.10.10), we get

(7.1) $$\bar{R}^i_{jkh} + \bar{R}^i_{khj} + \bar{R}^i_{hjk} = \delta^i_j \left\{ \wp_h p_k - \wp_k p_h \right\} + \delta^i_k \left\{ \wp_j p_h - \wp_h p_j \right\} + \delta^i_h \left\{ \wp_k p_j - \wp_j p_k \right\}.$$

The $(h)h$-torsion tensor is given by

(7.2) $$\bar{T}^i_{jk} = \bar{G}^i_{jk} - \bar{G}^i_{kj} = (G^i_{jk} + p_k \delta^i_j) - (G^i_{kj} + p_j \delta^i_k) = p_k \delta^i_j - p_j \delta^i_k$$

This implies

$$\wp_h \bar{T}^i_{jk} = \delta^i_j \wp_h p_k - \delta^i_k \wp_h p_j.$$

Therefore

(7.3) $$\wp_h \bar{T}^i_{jk} + \wp_j \bar{T}^i_{kh} + \wp_k \bar{T}^i_{hj} = \delta^i_j \wp_h p_k - \delta^i_k \wp_h p_j + \delta^i_k \wp_j p_h - \delta^i_h \wp_j p_k + \delta^i_h \wp_k p_j - \delta^i_j \wp_k p_h.$$

In view of (7.3), equation (7.1) becomes

(7.4) $$\bar{R}^i_{jkh} + \bar{R}^i_{khj} + \bar{R}^i_{hjk} = \wp_h \bar{T}^i_{jk} + \wp_j \bar{T}^i_{kh} + \wp_k \bar{T}^i_{hj}.$$

This is the first Bianchi identity .

Let X^i be an arbitrary contravariant vector. By Ricci commutation formula (5.7) , we have

$$\wp_j \wp_k X^i - \wp_k \wp_j X^i = X^r \bar{R}^i_{rkj} - (\dot{\partial}_r X^i) \bar{R}^r_{kj} - (\wp_r X^i) \bar{T}^r_{kj}.$$

Applying semi-symmetric projective covariant differentiation with respect to x^m, we get

$$\wp_m \wp_j \wp_k X^i - \wp_m \wp_k \wp_j X^i = \wp_m X^r . \bar{R}^i_{rkj} + X^r . \wp_m \bar{R}^i_{rkj} - \wp_m \dot{\partial}_r X^i . \bar{R}^r_{kj}$$

$$- \dot{\partial}_r X^i . \wp_m \bar{R}^r_{kj} - \wp_m \wp_r X^i . \bar{T}^r_{kj} - \wp_r X^i . \wp_m \bar{T}^r_{kj}.$$

Similarly

$$\wp_j\wp_k\wp_m X^i - \wp_j\wp_m\wp_k X^i = \wp_j X^r.\overline{R}^i_{rmk} + X^r.\wp_j\overline{R}^i_{rmk} - \wp_j\dot{\partial}_r X^i.\overline{R}^r_{mk}$$
$$-\dot{\partial}_r X^i.\wp_j\overline{R}^r_{mk} - \wp_j\wp_r X^i.\overline{T}^r_{mk} - \wp_r X^i.\wp_j\overline{T}^r_{mk},$$

and

$$\wp_k\wp_m\wp_j X^i - \wp_k\wp_j\wp_m X^i = \wp_k X^r.\overline{R}^i_{rjm} + X^r.\wp_k\overline{R}^i_{rjm} - \wp_k\dot{\partial}_r X^i.\overline{R}^r_{jm}$$
$$-\dot{\partial}_r X^i.\wp_k\overline{R}^r_{jm} - \wp_k\wp_r X^i.\overline{T}^r_{jm} - \wp_r X^i.\wp_k\overline{T}^r_{jm}.$$

Adding these three equations, we get

$$\begin{aligned}(7.5)\quad & \wp_m\wp_j\wp_k X^i - \wp_m\wp_k\wp_j X^i + \wp_j\wp_k\wp_m X^i \\ & -\wp_j\wp_m\wp_k X^i + \wp_k\wp_m\wp_j X^i - \wp_k\wp_j\wp_m X^i \\ & = \left[\wp_m X^r.\overline{R}^i_{rkj} + \wp_j X^r.\overline{R}^i_{rmk} + \wp_k X^r.\overline{R}^i_{rjm}\right] \\ & + X^r\left[\wp_m\overline{R}^i_{rkj} + \wp_j\overline{R}^i_{rmk} + \wp_k\overline{R}^i_{rjm}\right] \\ & -\dot{\partial}_r X^i\left[\wp_m\overline{R}^r_{kj} + \wp_j\overline{R}^r_{mk} + \wp_k\overline{R}^r_{jm}\right] \\ & -(\wp_r X^i)\left[\wp_m\overline{T}^r_{kj} + \wp_j\overline{T}^r_{mk} + \wp_k\overline{T}^r_{jm}\right] \\ & -\left[\wp_m\dot{\partial}_r X^i.\overline{R}^r_{kj} + \wp_j\dot{\partial}_r X^i.\overline{R}^r_{mk} + \wp_k\dot{\partial}_r X^i.\overline{R}^r_{jm}\right] \\ & -\left[\wp_m\wp_r X^i.\overline{T}^r_{kj} + \wp_j\wp_r X^i.\overline{T}^r_{mk} + \wp_k\wp_r X^i.\overline{T}^r_{jm}\right].\end{aligned}$$

Applying the Ricci commutation formula for $\wp_k X^i$, we have

$$\wp_m\wp_j\wp_k X^i - \wp_j\wp_m\wp_k X^i = \wp_k X^r.\overline{R}^i_{rjm} - \wp_r X^i.\overline{R}^r_{kjm}$$
$$-\dot{\partial}_r\wp_k X^i.\overline{R}^r_{jm} - \wp_r\wp_k X^i.\overline{T}^r_{jm}.$$

The cyclic change of indices *m, j,* and *k* in this equation gives

$$\wp_j\wp_k\wp_m X^i - \wp_k\wp_j\wp_m X^i = \wp_m X^r.\overline{R}^i_{rkj} - \wp_r X^i.\overline{R}^r_{mkj}$$
$$-\dot{\partial}_r\wp_m X^i.\overline{R}^r_{kj} - \wp_r\wp_m X^i.\overline{T}^r_{kj},$$

and

$$\wp_k\wp_m\wp_j X^i - \wp_m\wp_k\wp_j X^i = \wp_j X^r.\overline{R}^i_{rmk} - \wp_r X^i.\overline{R}^r_{jmk}$$
$$-\dot{\partial}_r\wp_j X^i.\overline{R}^r_{mk} - \wp_r\wp_j X^i.\overline{T}^r_{mk}.$$

Adding these three equations, we get

(7.6)
$$\begin{aligned}
&\wp_m\wp_j\wp_k X^i - \wp_j\wp_m\wp_k X^i + \wp_j\wp_k\wp_m X^i \\
&- \wp_k\wp_j\wp_m X^i + \wp_k\wp_m\wp_j X^i - \wp_m\wp_k\wp_j X^i \\
&= \left[\wp_k X^r.\overline{R}^i_{rjm} + \wp_m X^r.\overline{R}^i_{rkj} + \wp_j X^r.\overline{R}^i_{rmk} \right] \\
&- \wp_r X^i \left[\overline{R}^r_{kjm} + \overline{R}^r_{mkj} + \overline{R}^r_{jmk} \right] \\
&- \left[\dot{\partial}_r\wp_k X^i.\overline{R}^r_{jm} + \dot{\partial}_r\wp_m X^i.\overline{R}^r_{kj} + \dot{\partial}_r\wp_j X^i.\overline{R}^r_{mk} \right] \\
&- \left[\wp_r\wp_k X^i.\overline{T}^r_{jm} + \wp_r\wp_m X^i.\overline{T}^r_{kj} + \wp_r\wp_j X^i.\overline{T}^r_{mk} \right].
\end{aligned}$$

Since the left hand sides of (7.5) and (7.6) are the same, the right hand sides will also be the same. Hence

$$\begin{aligned}
&\left[\wp_m X^r.\overline{R}^i_{rkj} + \wp_j X^r.\overline{R}^i_{rmk} + \wp_k X^r.\overline{R}^i_{rjm} \right] + X^r \left[\wp_m \overline{R}^i_{rkj} + \wp_j \overline{R}^i_{rmk} + \wp_k \overline{R}^i_{rjm} \right] \\
&-\dot{\partial}_r X^i \left[\wp_m \overline{R}^r_{kj} + \wp_j \overline{R}^r_{mk} + \wp_k \overline{R}^r_{jm} \right] - (\wp_r X^i) \left[\wp_m \overline{T}^r_{kj} + \wp_j \overline{T}^r_{mk} + \wp_k \overline{T}^r_{jm} \right] \\
&-\left[\wp_m \dot{\partial}_r X^i.\overline{R}^r_{kj} + \wp_j \dot{\partial}_r X^i.\overline{R}^r_{mk} + \wp_k \dot{\partial}_r X^i.\overline{R}^r_{jm} \right] \\
&-\left[\wp_m\wp_r X^i.\overline{T}^r_{kj} + \wp_j\wp_r X^i.\overline{T}^r_{mk} + \wp_k\wp_r X^i.\overline{T}^r_{jm} \right] \\
&= \left[\wp_k X^r.\overline{R}^i_{rjm} + \wp_m X^r.\overline{R}^i_{rkj} + \wp_j X^r.\overline{R}^i_{rmk} \right] - \wp_r X^i \left[\overline{R}^r_{kjm} + \overline{R}^r_{mkj} + \overline{R}^r_{jmk} \right] \\
&\quad - \left[\dot{\partial}_r\wp_k X^i.\overline{R}^r_{jm} + \dot{\partial}_r\wp_m X^i.\overline{R}^r_{kj} + \dot{\partial}_r\wp_j X^i.\overline{R}^r_{mk} \right] \\
&\quad - \left[\wp_r\wp_k X^i.\overline{T}^r_{jm} + \wp_r\wp_m X^i.\overline{T}^r_{kj} + \wp_r\wp_j X^i.\overline{T}^r_{mk} \right],
\end{aligned}$$

i.e.

$$\begin{aligned}
&X^r \left[\wp_m \overline{R}^i_{rkj} + \wp_j \overline{R}^i_{rmk} + \wp_k \overline{R}^i_{rjm} \right] - \dot{\partial}_r X^i \left[\wp_m \overline{R}^r_{kj} + \wp_j \overline{R}^r_{mk} + \wp_k \overline{R}^r_{jm} \right] \\
&+ [\{\dot{\partial}_r\wp_k X^i - \wp_k\dot{\partial}_r X^i\}\overline{R}^r_{jm} + \{\dot{\partial}_r\wp_m X^i - \wp_m\dot{\partial}_r X^i\}\overline{R}^r_{kj} \\
&+ \{\dot{\partial}_r\wp_j X^i - \wp_j\dot{\partial}_r X^i\}\overline{R}^r_{mk}] + [\{\wp_r\wp_k X^i - \wp_k\wp_r X^i\}\overline{T}^r_{jm} \\
&+ \{\wp_r\wp_m X^i - \wp_m\wp_r X^i\}\overline{T}^r_{kj} + \{\wp_r\wp_j X^i - \wp_j\wp_r X^i\}\overline{T}^r_{mk}] \\
&- (\wp_r X^i)[\{\wp_m \overline{T}^r_{kj} + \wp_j \overline{T}^r_{mk} + \wp_k \overline{T}^r_{jm}\} - \{\overline{R}^r_{kjm} + \overline{R}^r_{mkj} + \overline{R}^r_{jmk}\}] = 0.
\end{aligned}$$

Using the commutation formulae (4.4) and (5.7), the above equation becomes

(7.7)
$$X^r[\wp_m\overline{R}^i_{rkj}+\wp_j\overline{R}^i_{rmk}+\wp_k\overline{R}^i_{rjm}+\overline{G}^i_{hrk}\overline{R}^h_{jm}+\overline{G}^i_{hrm}\overline{R}^h_{kj}+\overline{G}^i_{hrj}\overline{R}^h_{mk}$$
$$+\overline{R}^i_{rms}\overline{T}^s_{kj}+\overline{R}^i_{rjs}\overline{T}^s_{mk}+\overline{R}^i_{rks}\overline{T}^s_{jm}]-\dot{\partial}_r X^i[\wp_m\overline{R}^r_{kj}+\wp_j\overline{R}^r_{mk}+\wp_k\overline{R}^r_{jm}$$
$$+\overline{G}^r_{hsk}y^s\overline{R}^h_{jm}+\overline{G}^r_{hsm}y^s\overline{R}^h_{kj}+\overline{G}^r_{hsj}y^s\overline{R}^h_{mk}+\overline{R}^r_{ms}\overline{T}^s_{kj}+\overline{R}^r_{js}\overline{T}^s_{mk}+\overline{R}^r_{ks}\overline{T}^s_{jm}]$$
$$-(\wp_r X^i)[\{\wp_m\overline{T}^r_{kj}+\wp_j\overline{T}^r_{mk}+\wp_k\overline{T}^r_{jm}\}-\{\overline{R}^r_{kjm}+\overline{R}^r_{mkj}+\overline{R}^r_{jmk}\}]$$
$$-(\wp_r X^i)[\overline{T}^r_{ms}\overline{T}^s_{kj}+\overline{T}^r_{js}\overline{T}^s_{mk}+\overline{T}^r_{ks}\overline{T}^s_{jm}]=0.$$

Using (7.2), we get
$$\overline{T}^r_{ms}\overline{T}^s_{kj}+\overline{T}^r_{js}\overline{T}^s_{mk}+\overline{T}^r_{ks}\overline{T}^s_{jm}=0.$$

Therefore (7.7) becomes
$$X^r[\wp_m\overline{R}^i_{rkj}+\wp_j\overline{R}^i_{rmk}+\wp_k\overline{R}^i_{rjm}+\overline{G}^i_{hrk}\overline{R}^h_{jm}+\overline{G}^i_{hrm}\overline{R}^h_{kj}+\overline{G}^i_{hrj}\overline{R}^h_{mk}$$
$$+\overline{R}^i_{rms}\overline{T}^s_{kj}+\overline{R}^i_{rjs}\overline{T}^s_{mk}+\overline{R}^i_{rks}\overline{T}^s_{jm}]-\dot{\partial}_r X^i[\wp_m\overline{R}^r_{kj}+\wp_j\overline{R}^r_{mk}+\wp_k\overline{R}^r_{jm}$$
$$+\overline{G}^r_{hsk}y^s\overline{R}^h_{jm}+\overline{G}^r_{hsm}y^s\overline{R}^h_{kj}+\overline{G}^r_{hsj}y^s\overline{R}^h_{mk}+\overline{R}^r_{ms}\overline{T}^s_{kj}+\overline{R}^r_{js}\overline{T}^s_{mk}+\overline{R}^r_{ks}\overline{T}^s_{jm}]$$
$$-(\wp_r X^i)[\{\wp_m\overline{T}^r_{kj}+\wp_j\overline{T}^r_{mk}+\wp_k\overline{T}^r_{jm}\}-\{\overline{R}^r_{kjm}+\overline{R}^r_{mkj}+\overline{R}^r_{jmk}\}]=0.$$

If the vector X^i is independent of $\dot{x}^i$ then $\dot{\partial}_r X^i=0$, and hence
$$X^r[\wp_m\overline{R}^i_{rkj}+\wp_j\overline{R}^i_{rmk}+\wp_k\overline{R}^i_{rjm}+\overline{G}^i_{hrk}\overline{R}^h_{jm}+\overline{G}^i_{hrm}\overline{R}^h_{kj}+\overline{G}^i_{hrj}\overline{R}^h_{mk}$$
$$+\overline{R}^i_{rms}\overline{T}^s_{kj}+\overline{R}^i_{rjs}\overline{T}^s_{mk}+\overline{R}^i_{rks}\overline{T}^s_{jm}]=0.$$

Since the vector X^r is arbitrary, we have

(7.8)
$$\wp_m\overline{R}^i_{rkj}+\wp_j\overline{R}^i_{rmk}+\wp_k\overline{R}^i_{rjm}+\overline{G}^i_{hrk}\overline{R}^h_{jm}+\overline{G}^i_{hrm}\overline{R}^h_{kj}+\overline{G}^i_{hrj}\overline{R}^h_{mk}$$
$$+\overline{R}^i_{rms}\overline{T}^s_{kj}+\overline{R}^i_{rjs}\overline{T}^s_{mk}+\overline{R}^i_{rks}\overline{T}^s_{jm}=0.$$

This is the second Bianchi identity.

Transvecting (7.8) by $\dot{x}^r$ and using (2.6) and (5.11), we have
$$\wp_m\overline{R}^i_{kj}+\wp_j\overline{R}^i_{mk}+\wp_k\overline{R}^i_{jm}+\overline{G}^i_{hrk}\overline{R}^h_{jm}\dot{x}^r+\overline{G}^i_{hrm}\overline{R}^h_{kj}\dot{x}^r+\overline{G}^i_{hrj}\overline{R}^h_{mk}\dot{x}^r$$
$$+\overline{R}^i_{ms}\overline{T}^s_{kj}+\overline{R}^i_{js}\overline{T}^s_{mk}+\overline{R}^i_{ks}\overline{T}^s_{jm}=0.$$

This is the third Bianchi identity.

8. Commutation formula for Berwald Covariant Differential Operator and Semi-Symmetric Projective Covariant Differential Operator

The semi-symmetric projective covariant derivative of $B_k X^i$ is given by

(8.1) $$\wp_h(B_k X^i)=\partial_h(B_k X^i)-\dot{\partial}_r(B_k X^i)\overline{G}^r_h+(B_k X^r)\overline{G}^i_{rh}-(B_r X^i)\overline{G}^r_{kh},$$

which, in view of (1.9.5), may be written as

(8.2)
$$\wp_h(B_k X^i)=\partial_h\{\partial_k X^i-(\dot{\partial}_r X^i)G^r_k+X^r G^i_{rk}\}-\dot{\partial}_r\{\partial_k X^i-(\dot{\partial}_s X^i)G^s_k+X^s G^i_{sk}\}\overline{G}^r_h$$
$$+\{\partial_k X^r-(\dot{\partial}_s X^r)G^s_k+X^s G^r_{sk}\}\overline{G}^i_{rh}-\{\partial_r X^i-(\dot{\partial}_s X^i)G^s_r+X^s G^i_{sr}\}\overline{G}^r_{kh}.$$

On simplifying we get

$$\begin{aligned}\wp_h(B_k X^i) = {} & \partial_h\partial_k X^i - \partial_h(\dot{\partial}_r X^i)G^r_k - (\dot{\partial}_r X^i)(\partial_h G^r_k) + (\partial_h X^r)G^i_{rk} \\ & + X^r(\partial_h G^i_{rk}) - \dot{\partial}_r(\partial_k X^i)\bar{G}^r_h + (\dot{\partial}_r\dot{\partial}_s X^i)G^s_k\bar{G}^r_h + (\dot{\partial}_s X^i)G^s_{rk}\bar{G}^r_h \\ & - (\dot{\partial}_r X^s)G^i_{sk}\bar{G}^r_h - X^s G^i_{rsk}\bar{G}^r_h + (\partial_k X^r)\bar{G}^i_{rh} - (\dot{\partial}_s X^r)G^s_k\bar{G}^i_{rh} \\ & + X^s G^r_{sk}\bar{G}^i_{rh} - (\partial_r X^i)\bar{G}^r_{kh} + (\dot{\partial}_s X^i)G^s_r\bar{G}^r_{kh} - X^s G^i_{sr}\bar{G}^r_{kh}.\end{aligned} \tag{8.3}$$

Berwald covariant derivative of $\wp_h X^i$ is given by

$$B_k(\wp_h X^i) = \partial_k(\wp_h X^i) - \dot{\partial}_r(\wp_h X^i)G^r_k + (\wp_h X^r)G^i_{rk} - (\wp_r X^i)G^r_{hk}. \tag{8.4}$$

which, in view of (3.1), may be written as

$$\begin{aligned}B_k(\wp_h X^i) = {} & \partial_k\{\partial_h X^i - (\dot{\partial}_r X^i)\bar{G}^r_h + X^r\bar{G}^i_{rh}\} - \dot{\partial}_r\{\partial_h X^i - (\dot{\partial}_s X^i)\bar{G}^s_h + X^s\bar{G}^i_{sh}\}G^r_k \\ & + \{\partial_h X^r - (\dot{\partial}_s X^r)\bar{G}^s_h + X^s\bar{G}^r_{sh}\}G^i_{rk} - \{\partial_r X^i - (\dot{\partial}_s X^i)\bar{G}^s_r + X^s\bar{G}^i_{sr}\}G^r_{hk}.\end{aligned} \tag{8.5}$$

After simplifying (8.5), we obtain

$$\begin{aligned}B_k(\wp_h X^i) = {} & \partial_k\partial_h X^i - \partial_k(\dot{\partial}_r X^i)\bar{G}^r_h - (\dot{\partial}_r X^i)(\partial_k\bar{G}^r_h) + (\partial_k X^r)\bar{G}^i_{rh} \\ & + X^r(\partial_k\bar{G}^i_{rh}) - \dot{\partial}_r(\partial_h X^i)G^r_k + (\dot{\partial}_r\dot{\partial}_s X^i)\bar{G}^s_h G^r_k + (\dot{\partial}_s X^i)(\dot{\partial}_r\bar{G}^s_h)G^r_k \\ & - (\dot{\partial}_r X^s)\bar{G}^i_{sh}G^r_k - X^s\bar{G}^i_{rsh}G^r_k + (\partial_h X^r)G^i_{rk} - (\dot{\partial}_s X^r)\bar{G}^s_h G^i_{rk} \\ & + X^s\bar{G}^r_{sh}G^i_{rk} - (\partial_r X^i)G^r_{hk} + (\dot{\partial}_s X^i)\bar{G}^s_r G^r_{hk} - X^s\bar{G}^i_{sr}G^r_{hk}.\end{aligned} \tag{8.6}$$

From (8.3) and (8.6), we have

$$\begin{aligned}B_k(\wp_h X^i) - \wp_h(B_k X^i) = {} & X^r\{\partial_k\bar{G}^i_{rh} - \partial_h G^i_{rk} - \bar{G}^i_{srh}G^s_k + G^i_{srk}\bar{G}^s_h \\ & + \bar{G}^s_{rh}G^i_{sk} - G^s_{rk}\bar{G}^i_{sh} - \bar{G}^i_{rs}G^s_{hk} + G^i_{rs}\bar{G}^s_{kh}\} \\ & - \dot{\partial}_r X^i\{\partial_k\bar{G}^r_h - \partial_h G^r_k + G^r_{sk}\bar{G}^s_h \\ & - (\dot{\partial}_s\bar{G}^r_h)G^s_k + G^r_s\bar{G}^s_{kh} - \bar{G}^r_s G^s_{hk}\} \\ & - \partial_r X^i\{G^r_{hk} - \bar{G}^r_{kh}\}.\end{aligned} \tag{8.7}$$

In view of (3.1) and (2.7), (8.7) may be written as

$$B_k(\wp_h X^i) - \wp_h(B_k X^i) = X^r\mathfrak{R}^i_{rhk} - (\dot{\partial}_r X^i)\mathfrak{R}^r_{hk} - (\wp_r X^i)T^r_{hk}, \tag{8.8}$$

where

$$\mathfrak{R}^i_{rhk} = \partial_k\bar{G}^i_{rh} - \partial_h G^i_{rk} + G^i_{srk}\bar{G}^s_h - \bar{G}^i_{srh}G^s_k + \bar{G}^s_{rh}G^i_{sk} - G^s_{rk}\bar{G}^i_{sh}, \tag{8.9}$$

$$\mathfrak{R}^r_{hk} = \partial_k\bar{G}^r_h - \partial_h G^r_k + G^r_{sk}\bar{G}^s_h - (\bar{G}^r_{sh} + \bar{G}^r_{sjh}\dot{x}^j)G^s_k \tag{8.10}$$

and

$$T^r_{hk} = G^r_{hk} - \bar{G}^r_{kh}. \tag{8.11}$$

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