

A Coefficient Inequality for the Starlike Univalent Functions in the Unit Disc with Two Fixed Points of Complex Order

Dr. R. K. Sharma (Assistant Professor)
Dept. of Mathematics, BUIT, Barkatullah University Bhopal
Email: rksharma178@rediffmail.com

Abstract

The aim of this paper is to obtain a coefficient inequality for the class of starlike function $S^*(A, B, b)$ in the unit disc $U = \{z: |z| < 1\}$, where A and B are two fixed points and b is non zero complex number.

1. Introduction

Let A_n denote the class of functions which is the following form

$$w(z) = \sum_{n=1}^{\infty} b_n z^n, \quad \{n \in N = \{1, 2, 3, \dots\}\} \quad (1.1)$$

which are analytic in the open unit disc $U = \{z: |z| < 1\}$ and satisfying the condition $w(0)=0$ and $|w(z)| < 1$. Further, let S_n denote the subclass of functions in A_n which are univalent in U . Also $S_n^*(\alpha)$ represents the subclasses of A_n which is starlike functions of order α ($0 \leq \alpha < 1$). The starlike function is defined as follows

$$S_n^*(\alpha) = \left\{ f \in A_n : R \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha, 0 \leq \alpha < 1, z \in U \right\} \quad (1.2)$$

The present paper is devoted to a unified study of various subclasses of univalent functions. For this purpose, we introduce the new class of analytic functions $S^*(A, B, b)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1.3)$$

analytic in the open unit disc U and satisfying the conditions

$$\left| \frac{f'(z)-1}{b(A-B) - B\{f'(z)-1\}} \right| < 1 \quad (1.4)$$

and

$$1 + \frac{1}{b} \{f'(z)-1\} = \frac{1 + Aw(z)}{1 + Bw(z)} \quad (1.5)$$

where A and B are arbitrary fixed numbers such that $-1 \leq B < A \leq 1$ and b is non-zero complex number.

2. Preliminaries Lemmas:

Before stating and proving our main results, we need the following lemmas due to Keogh and Merkes [1] and Silverman [3].

Lemma 1: Let the function $w(z)$ defined by

$$w(z) = \sum_{n=1}^{\infty} b_n z^n,$$

be analytic with $w(0)=0$ and $|w(z)| < 1$ in U . If s is any complex number then

$$|b_2 - sb_1^2| \leq \max(1, |s|) \quad (1.6)$$

Lemma 2: Let the function $f(z)$ defined by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

then,

$$\sum_{k=n+1}^{\infty} \left(\frac{k-\alpha}{1-\alpha} \right) a_k \leq 1 \tag{1.7}$$

$$\Rightarrow f(z) \in S_n^*(\alpha) \quad \{z \in U : n \in \mathbb{N} : 0 \leq \alpha < 1\}$$

3. Main Results: (Coefficient inequalities)

Theorem 1: If the function $f(z)$ defined by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

belongs to the class $S^*(A, B, b)$. If δ is any complex number, then

$$|a_3 - \delta a_2^2| \leq \frac{\mu(A-B)|b|}{\alpha(\lambda, 3)} \max\{1, |d|\}, \tag{1.8}$$

where
$$d = \frac{B\{\alpha(\lambda, 2)\}^2 + \mu(A-B)b\delta\{\alpha(\lambda, 3)\}}{\{\alpha(\lambda, 2)\}^2}$$

The inequality (1.8) is sharp.

Proof: Since the function $f(z)$ belongs to the class $S^*(A, B, b)$, we have

$$1 + \frac{1}{b} \left\{ \frac{D^{\lambda+1} f(z)}{z} - 1 \right\} = (1 - \mu) + \mu \left\{ \frac{1 + Aw(z)}{1 + Bw(z)} \right\}, \tag{1.9}$$

where

$$w(z) = \sum_{n=1}^{\infty} b_n z^n, \text{ is analytic in } U \text{ and satisfies the conditions}$$

$$w(0) = 0, \quad |w(z)| < 1.$$

From (1.9), we have

$$\begin{aligned} w(z) &= \frac{\{D^{\lambda+1} f(z) / z - 1\}}{\mu(A-B)b - B \left\{ \frac{D^{\lambda+1} f(z)}{z} - 1 \right\}} \\ &= \frac{\sum_{n=2}^{\infty} \alpha(\lambda, n) a_n z^{n-1}}{\mu(A-B)b - B \sum_{n=2}^{\infty} \alpha(\lambda, n) a_n z^{n-1}} \\ &= \frac{1}{\mu(A-B)b} \left[\alpha(\lambda, 2) a_2 z + \alpha(\lambda, 3) a_3 z^2 + \frac{B\{\alpha(\lambda, 2)\}^2 a_2^2 z^2}{\mu(A-B)b} + \dots \right] \end{aligned}$$

and then comparing the coefficients of z and z^2 on both sides, we have

$$b_1 = \frac{\alpha(\lambda, 2) a_2}{\mu(A-B)b}$$

and

$$b_2 = \frac{\alpha(\lambda, 3) a_3}{\mu(A-B)b} + \frac{B\{\alpha(\lambda, 2)\}^2 a_2^2}{\mu^2(A-B)^2 b^2}$$

Thus

$$a_2 = \frac{\mu(A-B)bb_1}{\alpha(\lambda, 2)}$$

and

$$a_3 = \frac{\mu(A-B)b(b_2 - Bb_1^2)}{\alpha(\lambda, 3)}$$

Hence

$$a_3 - \delta a_2^2 = \frac{\mu(A-B)b}{\alpha(\lambda, 3)}(b_2 - db_1^2),$$

where

$$d = \frac{B\{\alpha(\lambda, 2)\}^2 + \mu(A-B)b\delta\{\alpha(\lambda, 3)\}}{\{\alpha(\lambda, 2)\}^2}$$

therefore

$$|a_3 - \delta a_2^2| = \frac{\mu(A-B)|b|}{\alpha(\lambda, 3)}|b_2 - db_1^2|$$

Using Lemma 1 in the above equation, we get

$$|a_3 - \delta a_2^2| \leq \frac{\mu(A-B)|b|}{\alpha(\lambda, 3)} \max\{1, |d|\},$$

Hence the result.

Theorem 2: If the function $f(z)$ defined by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

belongs to the class $S^*(A, B, b)$, then

$$|a_n| \leq \frac{(A-B)|b|}{n} \tag{2.0}$$

The estimates are sharp.

Proof: Since the function $f(z)$ belongs to the class $S^*(A, B, b)$, we have

$$1 + \frac{1}{b}\{f'(z) - 1\} = \frac{1 + Aw(z)}{1 + Bw(z)} \tag{2.1}$$

where the function $w(z)$ defined by (1.1) is regular in U and satisfies the conditions $w(0)=0, |w(z)| < 1$, for z belongs to U . Now, from (2.1) we have

$$\begin{aligned} & \left[(B-A) + (B/b) \sum_{n=2}^{\infty} na_n z^{n-1} \right] w(z) = -\frac{1}{b} \sum_{n=2}^{\infty} na_n z^{n-1} \\ \text{i.e.} \quad & \left[(A-B) - (B/b) \sum_{n=2}^{\infty} na_n z^{n-1} \right] \left[\sum_{n=1}^{\infty} b_n z^n \right] = \frac{1}{b} \sum_{n=2}^{\infty} na_n z^{n-1} \end{aligned} \tag{2.2}$$

now equating the corresponding coefficients in (2.2), we observe that the coefficient a_n on the RHS of (2.2) depends only on a_2, a_3, \dots, a_{n-1} on the LHS of (2.2). Hence for $n \geq 2$, it follows from (2.2) that

$$\left[(A-B) - (B/b) \sum_{n=2}^{k-1} na_n z^{n-1} \right] w(z) = \frac{1}{b} \sum_{n=2}^k na_n z^{n-1} + \sum_{n=k+1}^{\infty} d_n z^{n-1}$$

This yields

$$\left[(A-B) - (B/b) \sum_{n=2}^{k-1} na_n z^{n-1} \right] = \left| \frac{1}{b} \sum_{n=2}^k na_n z^{n-1} + \sum_{n=k+1}^{\infty} d_n z^{n-1} \right| \tag{2.3}$$

Now squaring both sides of (2.3) and integrating round $|z|=r, 0 < r < 1$, we obtain

$$(A-B)^2 + \frac{B^2}{b^2} \sum_{n=2}^{k-1} n^2 |a_n|^2 r^{2n-2} \geq \frac{1}{|b|^2} \sum_{n=2}^{k-1} n^2 |a_n|^2 r^{2n-2} + \sum_{n=k+1}^{\infty} |d_n|^2 r^{2n-2}$$

by assuming $r \rightarrow 1$, we get

$$(A - B)^2 + \frac{B^2}{b^2} \sum_{n=2}^{k-1} n^2 |a_n|^2 \geq \frac{1}{|b|^2} \sum_{n=2}^k n^2 |a_n|^2$$

or

$$(1 - B)^2 \sum_{n=2}^{k-1} n^2 |a_n|^2 + n^2 |a_n|^2 \leq (A - B)^2 |b|^2 \quad (2.4)$$

Since $-1 \leq B < 1$, we obtain from (2.4)

$$n^2 |a_n|^2 \leq (A - B)^2 |b|^2$$

This gives

$$|a_n| \leq \frac{(A - B)|b|}{n}$$

This shows the result is sharp.

References:

1. F. R. Keogh and E. P. Merkes: A coefficient inequality for certain classes of analytic functions Proc. Amer. Math. Soc. 20, 8-14 (1969).
2. I. S. Jack: Functions Starlike and convex of order α , J. London Math. Soc. 3, 469-474 (1971).
3. H. Silverman: Univalent functions with negative coefficients, Proc. Amer. Math. Soc. 51, 109-116 (1975).
4. P. Singh and M. Tygel: On some univalent functions in the unit disc, Indian Jour. pure app. Math. 12(4) (1981), 513-520 (1981).
5. V. Singh: On some criteria for univalence and starlikeness Ind. Jour. of Pure and App. Math. 34(4), 569-577 (2003).
6. Y. Polatoglu and M. Bolcal: A coefficient inequality for the class of analytic functions in the unit disc, I.J.M.M.S. Vol. 59, 3753-3759 (2003).
7. R.K. Sharma and D. Singh: "An Inequality of Subclasses of Univalent Functions Related to Complex Order by Convolution Method", General Mathematics Notes Vol. 3, No. 2, (2011).

This academic article was published by The International Institute for Science, Technology and Education (IISTE). The IISTE is a pioneer in the Open Access Publishing service based in the U.S. and Europe. The aim of the institute is Accelerating Global Knowledge Sharing.

More information about the publisher can be found in the IISTE's homepage:

<http://www.iiste.org>

CALL FOR PAPERS

The IISTE is currently hosting more than 30 peer-reviewed academic journals and collaborating with academic institutions around the world. There's no deadline for submission. **Prospective authors of IISTE journals can find the submission instruction on the following page:** <http://www.iiste.org/Journals/>

The IISTE editorial team promises to review and publish all the qualified submissions in a **fast** manner. All the journals articles are available online to the readers all over the world without financial, legal, or technical barriers other than those inseparable from gaining access to the internet itself. Printed version of the journals is also available upon request of readers and authors.

IISTE Knowledge Sharing Partners

EBSCO, Index Copernicus, Ulrich's Periodicals Directory, JournalTOCS, PKP Open Archives Harvester, Bielefeld Academic Search Engine, Elektronische Zeitschriftenbibliothek EZB, Open J-Gate, OCLC WorldCat, Universe Digital Library, NewJour, Google Scholar

