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A Coefficient Inequality for the Starlike Univalent Functions in the Unit Disc with Two Fixed Points of Complex Order

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Abstract
The aim of this paper is to obtained a coefficient inequality for the class of starlike function $S^*(A, B, b)$ in the unit disc U= {z: │z│< 1}, where A and B are two fixed points and b is non zero complex number.

1. Introduction

Let A_n denote the class of functions which is the following form

$$w(z)=\sum_{n=1}^{\infty} b_n z^n , \quad \{n \in N = \{1,2,3.....\}\} \tag{1.1}$$

which are analytic in the open unit disc U={ z: │z│< 1} and satisfying the condition w(0)=0 and │w(z)│< 1. Further, let S_n denote the subclass of functions in A_n which are univalent in U. Also $S_\alpha^*(n)$ represents the subclasses of A_n which is starlike functions of order α (0 ≤ α < 1). The starlike function is defined as follows

$$S_\alpha^*(n)=\left\{ f \in A_n : R\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, 0 \le \alpha < 1, z \in U \right\} \tag{1.2}$$

The present paper is devoted to a unified study of various subclasses of univalent functions. For this purpose, we introduce the new class of analytic functions S^* (A, B, b)
of the form

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n , \tag{1.3}$$

analytic in the open unit disc U and satisfying the conditions

$$\left|\frac{f'(z)-1}{b(A-B)-B\{f'(z)-1\}}\right|<1 \tag{1.4}$$

and

$$1+\frac{1}{b}\{f'(z)-1\}=\frac{1+Aw(z)}{1+Bw(z)} \tag{1.5}$$

where A and B are arbitrary fixed numbers such that -1 ≤ B < A ≤ 1 and b is non-zero complex number.

2. Preliminaries Lemmas:

Before stating and proving our main results, we need the following lemmas due to Keogh and Merkes [1] and Silverman [3].

Lemma 1: Let the function w(z) defined by

$$w(z)=\sum_{n=1}^{\infty} b_n z^n ,$$

be analytic with w(0)=0 and │w(z)│< 1 in U. If s is any complex number then

$$\left|b_2 - s b_1^2\right| \le \max(1, |s|) \tag{1.6}$$

Lemma 2: Let the function f(z) defined by

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n ,$$

then,

$$\sum_{k=n+1}^{\infty}\left(\frac{k-\alpha}{1-\alpha}\right)a_k \leq 1 \tag{1.7}$$

$$\Rightarrow f(z)\in S_n^*(\alpha) \qquad \{z\in U : n\in N : 0\leq\alpha<1\}$$

3. Main Results: (Coefficient inequalities)

Theorem 1: If the function f (z) defined by

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n,$$

belongs to the class S^* (A, B, b). If δ is any complex number, then

$$\left|a_3-\delta a_2^2\right|\leq\frac{\mu(A-B)|b|}{\alpha(\lambda,3)}\max\{1,|d|\}, \tag{1.8}$$

where $$d=\frac{B\{\alpha(\lambda,2)\}^2+\mu(A-B)b\delta\{\alpha(\lambda,3)\}}{\{\alpha(\lambda,2)\}^2}$$

The inequality (1.8) is sharp.

Proof: Since the function f (z) belongs to the class S^* (A, B, b), we have

$$1+\frac{1}{b}\left\{\frac{D^{\lambda+1}f(z)}{z}-1\right\}=(1-\mu)+\mu\left\{\frac{1+Aw(z)}{1+Bw(z)}\right\}, \tag{1.9}$$

where

$w(z)=\sum_{n=1}^{\infty}b_n z^n$, is analytic in U and satisfies the conditions

w (0) = 0, $\qquad |w(z)|<1$.

From (1.9), we have

$$w(z)=\frac{\{D^{\lambda+1}f(z)/z-1\}}{\mu(A-B)b-B\left\{\frac{D^{\lambda+1}f(z)}{z}-1\right\}}$$

$$=\frac{\sum_{n=2}^{\infty}\alpha(\lambda,n)a_n z^{n-1}}{\mu(A-B)b-B\sum_{n=2}^{\infty}\alpha(\lambda,n)a_n z^{n-1}}$$

$$=\frac{1}{\mu(A-B)b}\left[\alpha(\lambda,2)a_2 z+\alpha(\lambda,3)a_3 z^2+\frac{B\{\alpha(\lambda,2)\}^2 a_2^2 z^2}{\mu(A-B)b}+\ldots\right]$$

and then comparing the coefficients of z and z^2 on both sides, we have

$$b_1=\frac{\alpha(\lambda,2)a_2}{\mu(A-B)b}$$

and

$$b_2=\frac{\alpha(\lambda,3)a_3}{\mu(A-B)b}+\frac{B\{\alpha(\lambda,2)\}^2 a_2^2}{\mu^2(A-B)^2 b^2}$$

Thus

$$a_2=\frac{\mu(A-B)bb_1}{\alpha(\lambda,2)}$$

and

$$a_3 = \frac{\mu(A-B)b(b_2 - Bb_1^2)}{\alpha(\lambda,3)}$$

Hence

$$a_3 - \delta a_2^2 = \frac{\mu(A-B)b}{\alpha(\lambda,3)}(b_2 - db_1^2),$$

where

$$d = \frac{B\{\alpha(\lambda,2)\}^2 + \mu(A-B)b\delta\{\alpha(\lambda,3)\}}{\{\alpha(\lambda,2)\}^2}$$

therefore

$$\left|a_3 - \delta a_2^2\right| = \frac{\mu(A-B)\left|b\right|}{\alpha(\lambda,3)}\left|b_2 - db_1^2\right|$$

Using Lemma 1 in the above equation, we get

$$\left|a_3 - \delta a_2^2\right| \le \frac{\mu(A-B)\left|b\right|}{\alpha(\lambda,3)}\max\{1,\left|d\right|\},$$

Hence the result.

Theorem 2: If the function f (z) defined by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

belongs to the class S^* (A, B, b), then

$$\left|a_n\right| \le \frac{(A-B)\left|b\right|}{n} \tag{2.0}$$

The estimates are sharp.

Proof: Since the function f(z) belongs to the class S^* (A, B, b), we have

$$1 + \frac{1}{b}\{f'(z) - 1\} = \frac{1 + Aw(z)}{1 + Bw(z)} \tag{2.1}$$

where the function w(z) defined by (1.1) is regular in U and satisfies the conditions w(0)=0, $|w(z)| < 1$, for z belongs to U. Now, from (2.1) we have

$$\left[(B-A) + (B/b)\sum_{n=2}^{\infty} na_n z^{n-1}\right]w(z) = -\frac{1}{b}\sum_{n=2}^{\infty} na_n z^{n-1}$$

i.e.

$$\left[(A-B) - (B/b)\sum_{n=2}^{\infty} na_n z^{n-1}\right]\left[\sum_{n=1}^{\infty} b_n z^n\right] = \frac{1}{b}\sum_{n=2}^{\infty} na_n z^{n-1} \tag{2.2}$$

now equating the corresponding coefficients in (2.2), we observe that the coefficient a_n on the RHS of (2.2) depends only on $a_2, a_3, \ldots, a_{n-1}$ on the LHS of (2.2). Hence for $n \ge 2$, it follows from (2.2) that

$$\left[(A-B) - (B/b)\sum_{n=2}^{k-1} na_n z^{n-1}\right]w(z) = \frac{1}{b}\sum_{n=2}^{k} na_n z^{n-1} + \sum_{n=k+1}^{\infty} d_n z^{n-1}$$

This yields

$$\left[(A-B) - (B/b)\sum_{n=2}^{k-1} na_n z^{n-1}\right] = \left|\frac{1}{b}\sum_{n=2}^{k} na_n z^{n-1} + \sum_{n=k+1}^{\infty} d_n z^{n-1}\right| \tag{2.3}$$

Now squaring both sides of (2.3) and integrating round $|z| = r$, $0 < r < 1$, we obtain

$$(A-B)^2 + \frac{B^2}{b^2}\sum_{n=2}^{k-1} n^2\left|a_n\right|^2 r^{2n-2} \ge \frac{1}{\left|b\right|^2}\sum_{n=2}^{k-1} n^2\left|a_n\right|^2 r^{2n-2} + \sum_{n=k+1}^{\infty}\left|d_n\right|^2 r^{2n-2}$$

by assuming $r \to 1$, we get

$$(A-B)^2+\frac{B^2}{b^2}\sum_{n=2}^{k-1}n^2\left|a_n\right|^2\geq\frac{1}{\left|b\right|^2}\sum_{n=2}^{k}n^2\left|a_n\right|^2$$

or

$$(1-B)^2\sum_{n=2}^{k-1}n^2\left|a_n\right|^2+n^2\left|a_n\right|^2\leq(A-B)^2\left|b\right|^2 \qquad (2.4)$$

Since $-1\leq B<1,$, we obtain from (2.4)

$$n^2\left|a_n\right|^2\leq(A-B)^2\left|b\right|^2$$

This gives

$$\left|a_n\right|\leq\frac{(A-B)\left|b\right|}{n}$$

This shows the result is sharp.

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