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Generalization of p-Injective Rings and Projective Modules

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Abstract
Any left R-module M is said to be p-injective if for every principal left ideal I of R and any R-homomorphism g: I→M, there exists y ∈M such that $g(b) = by$ for all b in I. We find that RM is p-injective iff for each r∈R, x∈M if x∉rM then there exists c∈R with cr = 0 and cx ≠0. A ring R is said to be epp-ring if every projective R-module is p-injective. Any ring R is right epp-ring iff the trace of projective right R-module on itself is p-injective. A left epp-ring which is not right epp-ring has been constructed.
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Introduction: Any ring R is said to be QF if every injective left R-module is projective. Villamayor [3] characterises a ring R over which every simple R-module is injective Faith call it V-ring. R.R Colby defines a ring R as a left (right) IF-ring if every injective left (right) R-module is flat. R.Y.C. Ming[4] characterises the rings over which every simple left R-module is p-injective. Motivated by these ideas here we define a ring R as a left epp-ring if every left projective R-module is p-injective. Through out, R denotes an associative ring with identity and R-modules are unitary.

Definition: 1. A left R-module M is called p-injective if for any principal left ideal I of R and any left R-homomorphism g : I→ M, there exist y ∈ M such that g (b) = by for all b in I.

Definition : 2. A left R-module M is f -injective if, for any finitely generated left ideal I of R and any left R-homomorphism g:I →M , there exists y∈ M such that g (b) = b y for all b in I.

Proposition: 3(i) Direct product of p-injective modules is p-injective if and only if each factor is p-injective.
(ii) Direct sum of p-injective modules is p-injective if and only if each summand is p-injective. [4]

Proof :-(i) Let the direct product $\prod_A M_i$ be p-injective module, to show each M_i is p-injective. For this consider the homomorphism $f_i : I \to M_i$ where I = (a) be any principal left ideal of R generated by 'a'. If $j_i : M_i \to \prod_A M_i$ be the inclusion homomorphism then $j_i \circ f_i : I \to \prod_A M_i$ is a homomorphism. Since $\prod_A M_i$ is p-injective, there exists $(m_i)_{i\in A} \in \Pi M_i$ such that $j_i \circ f_i (x) = x (m_i)_{i\in A}$ for all x∈ I.
Let $q_i : \prod_A M_i \to M_i$ be the projection homomorphism.

Then $q_i \circ j_i \circ f_i = f_i$ and $q_i \circ j_i \circ f_i(a) = q_i(a(m_i)_{i\in A}) = a\, m_i = f_i (a)$. Hence M_i is p-injective.

Conversely let each M_i is p-injective and $f{:}I\to \prod_A M_i$ be any R-homomorphism. Consider the homomorphism $q_i \circ f$: I→M_i. By p-injectivity of M_i there exists m_i for each i ∈ A such that $q_i \circ f (a) = a\, m_i$. Thus f: I → $\prod_A M_i$ is given by $f (a) = a (m_i)_{i\in A}$. Hence $\prod_A M_i$ is p-injective.

(ii) Its proof is very much similar to (i).

Corollary: 4. Direct sum of injective module is p-injective.

Theorem: 5. Following conditions are equivalent for a left R-module M;
(i) M is left p-injective module.
(ii) For each r∈R, x∈M of x∉rM then there exists c∈R with cr = 0 and cx ≠ 0.

Proof: (i) ⇒ (ii) Suppose (ii) is not true that is for each r∈R, x∈M if x∉rM for some c, cr = 0 and cx = 0. Put I = (r) = Rr then there is an R-homomorphism f: I→M defined by f(r) = x. By p-injectivity of $_RM$ there exists x'∈M such that f(r) = rx' for all r∈I.

Therefore x = f(r) = rx'∈ rm which is a contradiction to the fact that x∉rM.

(ii) ⇒ (i) Let I = (r) = Rr be any principal left ideal and f : I → M be any R-homomorphism. Suppose f(r) ≠ ry for some y∈M and cr = 0. This implies that f(r)∉rm and f(cr) = cf(r) = 0 which is a contradiction to the fact that cr=0 and cx ≠ 0.

Definition: 6. A ring R is right (left) epp-ring if every right (left) projective R-module is p-injective. A ring R is epp-ring if it is both right as well as left epp-ring.

The trace of an R-module M on M is denoted by

$T_M(M) = \{Imf \mid f \in S = End_R M\}$.

Lemma: 7. [5] (i) Let $A \in |M_R$. The map $\varphi'(A) : Hom(M, A) \otimes {}_S M \to T_M(A)$ is an isomorphism if and only if M generates all kernels of homomorphism $M^n \to A$, $n \in N$.

(ii) The left S-module ${}_SM$ is flat if and only if generates all kernels of homomorphism $M^n \to M$, $n \in N$.

Theorem: 8. Following conditions are equivalent for a ring R;

(i) R is right epp-ring.

(ii) The trace of projective module M_R on itself is p-injective.

(iii) The trace of right free R-module on itself is p-injective.

Proof: (i) $\Rightarrow$ (ii) Let R be any right epp-ring i.e. every right projective R-module is p-injective and M be any projective right R-module. Using the fact that every projective is flat and the tensor product of two flat modules is flat we get ${}_SM \otimes_R R \cong {}_SM$ is flat. Therefore by Lemma 7 M generates all kernels of the homomorphisms $M^n \to M$, $n \in N$,which implies that $\varphi'(M):Hom_R(M,M) \otimes M \to T_M(M)$ is an isomorphism by Lemma 7 that is $S \otimes M_R$ is an isomorphic to $T_M(M)$ or M_R is isomorphic to $T_M(M)$ and so $T_M(M)$ is p-injective as R is epp-ring.

(ii) $\Rightarrow$ (iii) obvious

(iii) $\Rightarrow$ (i) Assume that for every free right R-module M, $T_M(M)$ is p-injective. M would be flat too (free $\Rightarrow$ projective $\Rightarrow$ flat). Using the same arguments as in (i) $\Rightarrow$ (ii) we get $T_M(M)$ is isomorphic to M therefore M is p-injective. Let K be any projective R-module than K would be direct summand of some free R-module say M but M is p-injective. Since direct summand of p-injective is p-injective [proposition 3(ii)] hence K is p-injective that is R is a epp-ring.

This theorem is true for a left epp-ring and left projective R-module M too.

Example: 9 [2, R.R. Colby Example 1].

A commutative epp-ring which is not a epp-ring modulo its radical. Let $R = Z \oplus \frac{Q}{Z}$ with multiplication defined by $(n_1, q_1)(n_2, q_2) = (n_1 n_2, n_1 q_2 + n_2 q_1)$, $n_i \in Z$, $q_i \in Q$. Then R is a commutative coherent ring with Jacobson radical

$$J(R) = \left\{ \frac{(n,\ q)}{n} = 0 \right\} = \left(0, \frac{Q}{Z} \right)$$

It is obvious that each finitely generated ideal is principal. Thus R is epp-ring but $\frac{R}{J(R)} \cong Z$ which is not a epp-ring as the homomorphism $f: Z \to Z$ defined by $f(nx) = x$ can not be extended to a homomorphism from $Z \to Z$.

Example 10: A left epp-ring which is not a right epp-ring [2, Example 2].

Let R be an algebra over a field F with basis $\{1, e_0, e_1, \ldots. x_1, x_2, \ldots..\}$ for all i, j

$e_i e_j = \delta_{i,j} e_j$

$x_i e_j = \delta_{i,j+1} x_i$

$e_i x_j = \delta_{ij} x_j$

$x_i x_j = 0$

It can be easily verified that R is left coherent and every R-homomorphism $f: {}_RI \to {}_RR$ extends from ${}_RR \to {}_RR$. Thus ${}_RR$ is p-injective that is R is left epp-ring. However R is not right epp-ring, since the homomorphism $x_1R \to e_0R$ via $x_1 r \to e_0 r$ can not be extended over R.

Preposition: 11. Let R and S are Marita equivalent ring then R is epp-ring if and only if S is epp-ring.

Proof: Let $F: {}_R|M \to {}_S|M$ and $G: {}_S|M \to {}_R|M$ are category equivalences where ${}_R|M$ and ${}_S|M$ denote the categories of left R-module and left S-module respectively. Since projectivity is a categorical property we have to show only that M is p-injective in ${}_R|M$ if and only if F (M) is p-injective in ${}_S|M$. The sequence with principal ideal I,

$$0 \to I \to R \to \frac{R}{I} \to 0$$

is exact in ${}_S|M$ if and only if

$$0 \to G(I) \to G(R) \to G\left(\frac{R}{I}\right) \to 0$$

is exact in ${}_R|M$ [1, page 224]. And the sequence

$$0 \rightarrow \mathrm{Hom}_R\left(G\left(\frac{R}{I}\right), M\right) \rightarrow \mathrm{Hom}_R(G(R), M) \rightarrow \mathrm{Hom}_R(G(I), M) \rightarrow 0$$

is exact in ${}_R|M$ if and only if

$$0 \rightarrow \mathrm{Hom}_S\left(\frac{R}{I}, F(M)\right) \rightarrow \mathrm{Hom}_S(R, F(M)) \rightarrow \mathrm{Hom}_S(I, F(M)) \rightarrow 0$$

is an exact sequence in ${}_S|M$.

That is if M is p-injective in ${}_R|M$ if and only if F(M) is p-injective in ${}_S|M$.

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