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Common Fixed Point Theorem for Occasionally Weakly Compatible Mapping in Q-Fuzzy Metric Space

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Abstract
The Fixed Point theory is an important and major topic of the nonlinear functional analysis that deals with the investigation leading to the existence and approximation of a "Fixed Point". To enhance its literature, the Common Fixed Point Theorem in Q Fuzzy Metric Space, is established as a prime objective of this paper .The goal is achieved by taking four self mappings on a Q Fuzzy Metric Space, satisfying the general contractive condition along with the definition of occasionally weakly compatible. The theorem is also illustrated with example and a unique fixed point is derived. Our result is independent of the continuity requirement of the maps and completeness of the space.
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1. Introduction

The concept of fuzzy sets introduced by Zadeh [12] in 1965, plays an important role in topology and analysis. Since then, there are many author to study the fuzzy set with application. Espectially, Kromosil and Michalek[10] put forward a new concept of fuzzy metric spaces. George and Vermani [6] revised the notion of fuzzy metric spaces with the help of continuous tnorm. As a result of many fixed point theorem for various forms of mapping are obtained in fuzzy metric spaces. Dhage [5] introduced the defination of D metric space and proved many new fixed point theorem in D-metric spaces. Recently, Mustafa and Sims[13] presented a new definition of G-metric space and made great contribution to the development of Dhage theory. On the other hand, Lopez-Rodrigues and Romaguera [11] introduced the concept of Hausdorff fuzzy metric in a more general space .The Q-fuzzy metrics spaces is introduced by Guangpeng Sun and kai Yang[7] which can be consider as a Generalization of fuzzy metric spaces. The concept of compatible maps by [10] and weakly compatible maps by [8] in fuzzy metric space is generalized by A.Al Thagafi and Naseer Shahzad [1] by introducing the concept of occasionally weakly compatible mappings. Recent results on fixed point in Q-fuzzy metric space can be viewed in[7].

The main purpose of our paper is to prove common fixed point theorem in Q fuzzy metric space under general contractive conditions satisfying the definition of occasionally weakly compatible (owc) map. It extends the scope of study common fixed point theorems from the class of weakly compatible mapping to wider class of mappings. This result generalizes and extends several known fixed point theorems for owc maps on G metric space.

Our improvements in this paper are five-fold as;

(i) Relaxed the continuity of maps completely
(ii) Completeness of the space removed
(iii) Minimal type contractive condition used
(iv) The condition $\lim_{t\to\infty} M(x,y,t)$ =1 not used
(v) Weakened the concept of compatibility by a more general concept of occasionally weakly Compatible (owc) maps.

2. Preliminary Notes

Definition 2.1[2]

A binary operation $*:[0,1]\times[0,1]\to[0,1]$ is a continuous t norm if it satisfy the following condition:

(i) * is associative and commutative .
(ii) * is continous function.
(iii) a *1=a for all a Ɛ[0,1]
(iv) a *b ≤ c*d whenever a ≤ c and b ≤ d and a, b, c, d Ɛ[0,1]

Definition 2.2

A 3-tuple (X, M, *) is said to be a fuzzy metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set on $X^2 \times (0,\infty)$ satisfying the following conditions:
for all x, y, z ϵ X, s, t > 0,

(fm1) $M(x, y, t) > 0$;
(fm2) $M(x, y, t) = 1$ if and only if $x = y$;
(fm3) $M(x, y, t) = M(y, x, t)$;
(fm4) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$;
(fm5) $M(x, y, *) : (0,\infty) \to (0, 1]$ is continuous.
Then M is called a fuzzy metric on X.
The function M(x, y, t) denote the degree of nearness between x and y with respect to t.

Definition 2.3[7]:
A 3-tuple (X,Q, *) is called a Q-fuzzy metric space if X is an arbitrary (non-empty) set * is a continuous t -norm, and Q is a fuzzy set on $X^3\times(0,\infty)$,satisfying the following conditions for each x, y, z, a Ɛ X and t ,s> 0 :

(Qm1) $Q(x,x,y,t)>0$ and $Q(x,x,y,t)\le Q(x,y,z,t)$ for all x,y,z Ɛ X with z=y
(Qm2) $Q(x,y,z,t)=1$ if and only if $x = y = z$
(Qm3) $Q(x,y,z,t) = Q(p(x,y,z),t)$,(symmetry) where p is a permutation function ,
(Qm4) $Q(x,a,a,t) * Q(a,y,z,s)\le Q(x,y,z,t+s)$,
(Qm5) $Q(x,y,z,.):(0,\infty)\to[0,1]$ is continuous

A Q-fuzzy metric space is said to be symmetric if $Q(x,y,y,t)=Q(x,x,y,t)$ for all x,y Ɛ X.

Example 2.1: Let X is a nonempty set and G is the G-metric on X. Denote a*b = a.b for all a,bƐ[0,1]. For each t>0: $Q(x,y,z,t) = \frac{t}{t+G(x,y,z,t)}$ Then (X,Q,*) is a Q-fuzzy metric

2.4 Comparative study of Fuzzy Metric Space and Q Fuzzy Metric Space :

1) In fuzzy metric space the fuzzy set M is defined on X^2 x $(0,\infty)$ where as in Q fuzzy metric space the fuzzy set Q is defined on X^3 x $(0,\infty)$.Thus it can be said that a Q Fuzzy Metric Space is the extended version of the Fuzzy Metric Space in which Triangle Inequality is replaced by Rectangle Inequality.
2) The concept of Q Fuzzy Metric Space is on the G metric space which is a generalization of ordinary metric space .Therefore the Q Fuzzy Metric Space is also called as the Generalized Fuzzy Metric Space.

Example 2.2 :
Let (X,M,*) be a Fuzzy Metric Space. If we define Q: $X^3x(0,\infty) \to [0,1]$ by

$$Q(x,y,z,t)= \min \{M(x,y,t),M(y,z,t),M(z,x,t)\} \text{ for every } x,y,z \text{ in } X ,$$

then (X,Q,*) is a Q Fuzzy Metric Space.

Solution
We will only verify (Qm5)
$Q(x,y,z,t)= \min \{M(x,y,t),M(y,z,t),M(z,x,t)\}$ $Q(x,a,a,t) = M(x,a,t)$
$Q(a,y,z,t)= \min \{M(a,y,t),M(y,z,t),M(z,a,t)\}$

$$\begin{aligned}
Q(x,a,a,t) * Q(a,y,z,s) &= M(x,a,t) * \min \{M(a,y,s),M(y,z,s),M(z,a,s)\} \\
&\le \min \{ M(x,a,t) * M(a,y,s), M(x,a,t) * M(y,z,s), M(x,a,t) * M(z,a,s) \} \\
&\le \min \{ M(x,y,t+s), M(y,z,s), M(x,z,t+s) \} \\
&\le \min \{ M(x,y,t+s), M(y,z,t+s), M(x,z,t+s) \} \\
&\le Q(x,y,z,t+s)
\end{aligned}$$

Thus (Qm5) holds
Hence (X,Q,*) is a Q Fuzzy Metric Space.

Definition 2.5[6]
Let (X,Q,*) be a Q-fuzzy metric space. A sequence $\{x_n\}$ in X converges to x if and only if Q $(x_m,x_n,x,t) \to 1$ as $n\to\infty$,for each t>0. It is called a Cauchy sequence if for each
$0 <\varepsilon<1$ and t>0, there exist n_0ƐN such that $Q(x_m,x_n,x1)>1- \varepsilon$ for each ,n,m $\ge n_0$.

Definition 2.6[3]:
Let *X* be a set, f and g self maps of X. A point x in X is called a coincidence point of f and g iff fx= gx .We shall call w=fx=gx a point of coincidence of f and g.

Definition 2.7[7]:
Let f and g be self maps on a Q-fuzzy metric space (X, Q,*) then the mappings are said to be weakly compatible if they commute at their coincidence point, that is, f x = gx implies that
fgx = gfx .

Definition 2.8[1]:
Let f and g be self maps on a Q-fuzzy metric space (X, Q,*) then the mapping are occasionally weakly compatible (owc) iff there is a point x in X which is coincidence point of f and g at which f and g commute.
Al-ThagaW and Naseer [1](2008) shown that occasionally weakly is weakly compatible but converse is not true.

Example 2.3: Let R be the usual metric space. Define S T: R→R by Sx = x and
$Tx = x^3$ for all x Ɛ R. Then Sx = Tx for x = 0, 2 but ST0 = TS0 and

ST2 ≠TS2. S and T are occasionally weakly compatible self maps but not weakly compatible

Lemma 2.7[7]:
If (X,Q, *) be a Q-fuzzy metric space,then Q(x,y,z,t) is non decreasing with respect to t for all x,y,z in X .

Proof : Proof is this is implicated in [7]

Lemma 2.8:
Let (X,Q,*) be a Q-fuzzy metric space.,if there exists k Ɛ(0,1) such that Q(x, y, k t) ≥ Q(x, y, t) for all x, y Ɛ X and t>0, then x = y.

Proof: By the assume $\lim_{t\to\infty}$ Q(x,y,z,t) = 1 and the property of non-decreasing, it is easy to get the results.

Lemma 2.9 [1]: Let X be a set, A , B owc self maps of X. If A and B have unique point of coincidence, w = A x = B x, then w is the unique common fixed point of A and B.

Proof:
Since A and B are owc, there exists a point x in X such that Ax = Bx = w and ABx = BAx. Thus, AAx = ABx = BAx, which says that AAx is also a point of coincidence of A and B. Since the point of coincidence w = Ax is unique by hypothesis, BAx = AAx = Ax, and w = Ax is a common fixed point of A and B.
Moreover, if z is any common fixed point of A and B, then z = Az =Bz = w by the uniqueness of the point of coincidence point of A and B.

Main Result :

Theorem 3.1
Let A, B,S ,T be a self mappings of the Symmetric Q Fuzzy metric space with continuous t norm satisfying the following condition :

1) The pair {A,S} and {B,T} is Occasionally Weakly Compatible
2) There exist k Ɛ (0,1) such that
   Q(Ax ,By ,By ,k t) ≥ min {Q(Sx, Ty ,Ty,t) , Q(Sx, Ax, Ax,t) , Q(Ty, By,By ,t) ,
   Q(Sx, By, By, t) , Q(Ty, Ax, Ax, t) }……………..(3.1.1)
   for all x, y in X and t>0. Then A, B, S, T have Unique Common Fixed Point in X.

Proof:
Since by hypothesis, the pair {A,S} and {B,T} is Occasionally weakly Compatible then there exists points x, y in X such that Ax = Sx and By = Ty. We claim that Ax =By.
From Equation (3.1) we have
Q(Ax ,By ,By ,kt) ≥ min {Q(Ax, By ,By,t) , Q(Ax, Ax, Ax,t) , Q(By, By ,By ,t) ,
Q(Ax, By, By, t) , Q(By, Ax, Ax, t) }
Q(Ax ,By ,By ,kt) ≥ min {Q(Ax, By ,By,t) , 1,1, Q(Ax, By, By, t) , Q(Ax, By,By, t) }
Q(Ax ,By ,By ,kt) ≥ Q(Ax, By ,By,t) for all x,y in X and t>0.
By lemma 2.8 we have Ax =By.
So Ax= By= Sx = Ty
Moreover, if there is another point z such that Az = Sz, then, using (3.1.1) it follows that Az = Sz = By = Ty or Ax = Az and w = Ax = Sx is the unique point of coincidence of A and S. Then by lemma 2.9, it follows that w is the unique common fixed point of A and S. By symmetry, there is a unique common fixed point z in X such that z = Bz = Tz.
Now, we claim that w = z. Suppose that w ≠ z.
Using equation (3.1) we have
Q(w, z, z, kt) = Q(Aw, Bz, Bz, kt)
≥ min {Q(Sw, Tz ,Tz,t) , Q(Sw, Aw, Aw ,t) , Q(Tz, Bz ,Bz ,t) ,
Q(Sw, Bz, Bz, t) , Q(Tz Aw, Aw, t) }
≥Q(Sw, Tz ,Tz,t)
≥Q(w, z ,z,t)
By lemma 2.8 we have w=z
Therefore w is a unique point of coincidence of A,B, S, T then by lemma 2.9
w is the unique common fixed point of A, B ,S, T.

Example 3.2:
Let X=[0,1] and G is the G Symmetric metric space on X such that
G(x, y, z)=max{|x-y|+|y-z|+|z-x|.
Denote a*b=ab for all a,b in [0,1] and for each t>0 define a fuzzy set Q as
$$Q(x,y,z,t)=\frac{t}{t+G(x,y,z)}$$
Then (X,Q,*) is a Q Fuzzy Metric Space Define a mappings A,B,S,T as
Ax=x , Bx=3x, Tx= $\frac{x^3}{4}$ and Sx= x^2
We claim that the pair {A,S} and {B,T} is Occasionally weakly Compatible.

At x=0 we have A(0) = 0and S(0)=0 also AS(0)=A[S(0)]=A(0)=0 and SA(0)=S[A(0)]=S(0)=0
Thus the pair {A,S} is owc map. Similarly we can show the pair {B,T} is owc.
For k Ɛ (0,1) and for all t>0 and x= 0 Ɛ X, the mappings satisfy equation (3.1.1)
Thus all the condition of theorem are verified
Hence 0 is the Common Fixed Point of A, B ,S ,T.

Corollary 3.1: Let (*X, G*) be a Symmetric *G*-metric space. Suppose that *A, B, S, T* are self maps on *X* and that the pairs {*A , S*} and {*B , T*} are each *owc.*

$$Q(Ax ,By ,By ,kt) \geq \min \{Q(Sx, Ty ,Ty,t) , Q(Sx, Ax, Ax,t) , Q(Ty, By ,By ,t) , \frac{1}{2} [Q(Sx, By, By, t) + Q(Ty, Ax, Ax, t)]\}$$

for all x, y in X and 0<k < 1, then A,B, S and T have a unique common fixed point in X.

Proof: Since (3) is a special case of (2), the result follows immediately from Theorem 2.2.

4.1 CONCLUSIONS

In this paper, as an application of occasionally weakly compatible mappings, we prove common fixed point theorems under contractive conditions that extend the scope of the study of common fixed point theorems from the class of weakly compatible mappings to a wider class of mappings. Our result substantially generalizes and improves a multitude of relevant common fixed point theorems of the existing literature in Fuzzy Metric Space.

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