On Common Fixed point Theorem in Fuzzy Metric space

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ABSTRACT
In this research article we are proving common fixed point theorem using Occasionally Weakly Compatible Mapping in fuzzy metric space.

KEYWORDS: Common Fixed point, Fuzzy Metric space, Occasionally Weakly Compatible Mapping, Continuous t-norm.

1. INTRODUCTION
It proved a turning point in the development of mathematics when the notion of fuzzy set was introduced by Zadeh [24] which laid the foundation of fuzzy mathematics. Kramosil and Michalek [11] introduced the notion of a fuzzy metric space by generalizing the concept of the probabilistic metric space to the fuzzy situation. George and Veeramani [7] modified the concept of fuzzy metric spaces introduced by Kramosil and Michalek [11]. There are many view points of the notion of the metric space in fuzzy topology for instance one can refer to Kaleva and Seikkala [10], Kramosil and Michalek [11], George and Veeramani [7].

2. PRELIMINARIES:
Definition 2.1. [24] Let X be any non empty set. A fuzzy set M in X is a function with domain X and values in [0, 1].

Definition 2.2. [19] A binary operation \( \ast : [0,1] \times [0,1] \rightarrow [0,1] \) is a continuous t-norm if it satisfy the following condition:
(i) \( \ast \) is associative and commutative.
(ii) \( \ast \) is is continuous function.
(iii) \( a \ast 1 = a \) for all \( a \in [0,1] \)
(iv) \( a \ast b \leq c \ast d \) whenever \( a \leq c \) and \( b \leq d \) and \( a, b, c, d \in [0,1] \)

Definition 2.3. [11] The 3 − tuple \( (X, M, \ast) \) is called a fuzzy metric space in the sense of Kramosil and Michalek if X is an arbitrary set, \( \ast \) is a continuous t − norm and M is a fuzzy set in \( X \times [0,\infty) \) satisfying the following conditions:
(a) \( M(x, y, t) > 0 \),
(b) \( M(x, y, t) = 1 \) for all \( t > 0 \) if and only if \( x = y \),
(c) \( M(x, y, t) = M(y, x, t) \),
(d) \( M(x, y, t) M(y, z, s) \leq M(x, z, t + s) \),
(e) \( M(x, y, .) : [0,\infty) \rightarrow [0, 1] \) is a continuous function, for all \( x, y, z \in X \) and \( t, s > 0 \).

Definition 2.4. [11] Let \( (X, M, \ast) \) be a fuzzy metric space . Then
(i) A sequence \( \{x_n\} \) in X converges to \( x \) if and only if for each \( t > 0 \) there exists \( n_0 \in \mathbb{N}, \) such that,
\[ \lim_{n \to \infty} M(x_n, x, t) = 1, \text{ for all } n \geq n_0. \]
(ii) The sequence \( \{x_n\} \in \mathbb{N} \) is called Cauchy sequence if \( \lim_{n \to \infty} M(x_n, x_{n+p}, t) = 1, \text{ for all } t > 0 \text{ and } p \in \mathbb{N}. \)
(iii) A fuzzy metric space X is called complete if every Cauchy sequence is convergent in X.

Definition 2.5. [23] Two self-mappings \( f \) and \( g \) of a fuzzy metric space \( (X, M, \ast) \) are said to be weakly commuting if \( M(fgx, gfx, t) \geq M(fx, gx, t) \), for each \( x \in X \) and for each \( t > 0 \).

Definition 2.6. [5] Two self mappings \( f \) and \( g \) of a fuzzy metric space \( (X, M, \ast) \) are called compatible if \( \lim_{n \to \infty} M(fgx_n, gfx_n, t) = 1 \) whenever \( \{x_n\} \) is a sequence in X such that \( \lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = x \) for some \( x \) in X.

Definition 2.7. [2] A pair of mappings \( f \) and \( g \) from a fuzzy metric space \( (X, M, \ast) \) into itself are weakly compatible if they commute at their coincidence points, i.e., \( fx = gx \) implies that \( fx = gx \).

Definition 2.8. Let \( X \) be a set, \( f, g \) selfmaps of \( X \). A point \( x \) in \( X \) is called a coincidence point of \( f \) and \( g \) iff \( fx = gx \). We shall call \( w = fx = gx \) a point of coincidence of \( f \) and \( g \).
Definition 2.9 [2] A pair of maps $S$ and $T$ is called weakly compatible pair if they commute at coincidence points.

Definition 2.10 [4] Two self maps $f$ and $g$ of a set $X$ are occasionally weakly compatible (owc) iff there is a point $x$ in $X$ which is a coincidence point of $f$ and $g$ at which $f$ and $g$ commute.

A. Al-Thagafi and Naseer Shahzad [4] shown that occasionally weakly compatible is weakly compatible but converse is not true.

Lemma 2.11 [4] Let $X$ be a set, $f, g$ owc self maps of $X$. If $f$ and $g$ have a unique point of coincidence, $w = fx = gx$, then $w$ is the unique common fixed point of $f$ and $g$.

3. IMPLICIT RELATIONS:
   
   (a) Let $\emptyset$ be the set of all real continuous functions $\emptyset : (R^+)^5 \rightarrow R^+$ satisfying the condition $\emptyset : (u, u, v, v, u) \geq 0$ imply $u \geq v$, for all $u, v \in [0,1]$.
   
   (b) Let $\emptyset$ be the set of all real continuous functions $\emptyset : (R^+)^4 \rightarrow R^+$ satisfying the condition $\emptyset : (u, v, u, u) \geq 0$ imply $u \geq v$, for all $u, v \in [0,1]$.

4. MAIN RESULTS

Theorem 4.1.: Let $(X, M, *)$ be a fuzzy metric space with $*$ continuous $t$-norm. Let $A, B, S, T$ be self mappings of $X$ satisfying

(i) The pair $(A, S)$ and $(B, T)$ be owc.

(ii) For some $\emptyset \in \Phi$ and for all $x, y \in X$ and every $t > 0$,

\[
\emptyset \{M(Ax, By, t), M(Sx, Ty, t), M(Sx, Ax, t), M(Ty, By, t), M(Sx, By, t)\} \geq 0
\]

then there exists a unique point $w \in X$ such that $Aw = Sw = w$ and a unique point $z \in X$ such that $Bz = Tz = z$.

Moreover, $z = w$, so that there is a unique common fixed point of $A, B, S$ and $T$.

Proof: Let the pairs \{A, S\} and \{B, T\} be owc, so there are points $x, y \in X$ such that $Ax = Sx$ and $By = Ty$. We claim that $Ax = By$. If not, by inequality (ii)

\[
\emptyset \{M(A, By, t), M(A, By, t), M(A, Ax, t), M(By, By, t), M(A, By, t)\} \geq 0
\]

\[
\emptyset \{M(A, By, t), M(A, By, t), 1, 1, M(A, By, t)\} \geq 0
\]

In view of $\Phi$ we get $Ax = By$ i.e. $Ax = Sx = By = Ty$

Suppose that there is a another point $z$ such that $Az = Sz$ then by (i) we have $Az = Sz = By = Ty$, so $Ax = Az$ and $w = Ax = Sx$ is the unique point of coincidence of $A$ and $S$. By Lemma 2.11 $w$ is the only common fixed point of $A$ and $S$. Similarly there is a unique point $z \in X$ such that $z = Bz = Tz$. Assume that $w \neq z$. We have

\[
\emptyset \{M(Aw, Bz, t), M(Sw, Tz, t), M(Sw, Aw, t), M(Tz, Bz, t), M(Sw, Bz, t)\} \geq 0
\]

\[
\emptyset \{M(w, z, t), M(w, z, t), M(w, w, t), M(z, z, t), M(w, z, t)\} \geq 0
\]

\[
\emptyset \{M(w, z, t), M(w, z, t), 1, 1, M(w, z, t)\} \geq 0
\]

In view of $\Phi$ we get $w = z$, by Lemma 2.11 and $z$ is a common fixed point of $A, B, S$ and $T$. The uniqueness of the fixed point holds from (ii)
Theorem 4.2.: Let \((X, M, \ast)\) be a fuzzy metric space with \(\ast\) continuous t-norm. Let \(A, B, S, T\) be self mappings of \(X\) satisfying

(i) The pair \((A, S)\) and \((B, T)\) be owc.

(ii) For some \(\emptyset \in \Phi\) and for all \(x, y \in X\) and every \(t > 0\),

\[
\emptyset(M(Sx, Ty, t), M(Sx, Ax, t), M(Ty, Ax, t)) \geq 0
\]

then there exists a unique point \(w \in X\) such that \(Aw = Sw = w\) and a unique point \(z \in X\) such that \(Bz = Tz = z\). Moreover, \(z = w\), so that there is a unique common fixed point of \(A, B, S\) and \(T\).

Proof: Let the pairs \((A, S)\) and \((B, T)\) be owc, so there are points \(x, y \in X\) such that \(Ax = Sx\) and \(By = Ty\). We claim that \(Ax = By\). If not, by inequality (ii).

\[
\emptyset(M(Ax, By, t), M(Ax, Ax, t), M(By, Ax, t)) \geq 0
\]

In view of \(\Phi\) we get \(Ax = By\) i.e. \(Ax = Sx = By = Ty\).

Suppose that there is another point \(z\) such that \(Az = Sz\) then by (i) we have \(Az = Sz = By = Ty\), so \(Ax = Az\) and \(w = Ax = Sx\) is the unique point of coincidence of \(A\) and \(S\). By Lemma 2.12 \(w\) is the only common fixed point of \(A\) and \(S\). Similarly there is a unique point \(z \in X\) such that \(Bz = Tz\).

5. REFERENCES


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