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Hyers- Ulam Rassias Stability of Exponential Primitive Mapping

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Abstract
The aim of this paper is to prove the stability of Exponential Primitive Mapping in sprit of Hyers- Ulam-Rassias.
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1. INTRODUCTION

The problem of stability of homomorphism stemmed from the question posed by S.M.Ulam in 1940 in his lecture before the mathematical club of the University of Wisconsin. He demanded an answer to the following question of stability of homomorphism for metric groups.

Let G' be a group and let G'' be a metric group with the metric d. Given $\epsilon > 0$, does there exists a $\delta > 0$ such that if a mapping $h : G' \to G''$ satisfies the following inequality $d[h(xy), h(x)h(y)] < \delta$ for all x and y in G', then there exists a homomorphism $H : G' \to G''$ with $d[h(x), H(x)] < \epsilon$ for all x in G' ?

In 1941 D.H.Hyers answered his question considering the case of Banach spaces. D.H.Hyers [18] proved the following result where E' and E'' are Banach spaces.

Result

Let $f : E' \to E''$ be a mapping between Banach spaces. If f satisfies the following inequality

$$\| f(x+y) - f(x) - f(y) \| \leq \delta$$

for all x and y in E' and some $\delta > 0$ then the limit

$$T(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$$

exists for all x in E' and $T : E' \to E''$ is a unique additive mapping such that

$$\| f(x) - T(x) \| \leq \delta \quad \text{for all } x \text{ in } E'.$$

Moreover, if f(t x) is continuous in t for each fixed x in E', then the mapping T is linear.

In 1978 Th.M.Rassias [2] generalized the result of Hyers by proving the following result.

Result

Let $f : E' \to E''$ be a mapping between Banach spaces. If f satisfies the following inequality

$$\| f(x+y) - f(x) - f(y) \| \leq \theta(\| x \|^p + \| y \|^p)$$

for all x and y in E' and for some $\theta > 0$ and some p with $0 \leq p < 1$, then there exists a unique additive mapping $T : E' \to E''$ such that

$$\| T(x) - f(x) \| \leq 2\theta \left(\frac{\| x \|^p}{2 - 2^p} \right)$$

for all x and y in E'. In addition , if f(t x) is continuous in t for each fixed x in E', then the mapping T is linear.

The method adopted by D.H.Hyers is designated as *Direct Method* and the stability of any functional inequality which had an independent bound is termed as ***Hyers - Ulam Stability***.

Th.M.Rassias and thereafter others like George Isac, P.Gauvruta, G.L.Forti, John A.Baker , F.Skof, S.M.Jung etc. obtained several useful result on the stability of functional inequalities which had bounds dependant in some way on the elements in the domain of the function under consideration. Such type of stability is termed as ***Hyers - Ulam - Rassias Stability.***

2. PRELIMINARIES

2.1 Definition of Exponential Primitive Mapping

Let G be a mapping on an open set $E \subset R^n \to R^n$. And let there be an integer $m < n$ and a real function g with domain E such that

$G(X) = \sum x_i e_i + [x_m^{g(X)}] e_m$ for all x in E and $i \neq m$.
or
$G(X) = X + [(x_m^{g(X)} - x_m)] e_m$ for all x in E.
Then G(X) is *e - primitive mapping.*

2.2 Definition of Metric Group Let G be a group. A metric d on G is said to be left invariant if for every x, y, z ∈ G, d(y, z) = d(xy, xz). Right invariant is defined similarly,and a metric is said to be bi-invariant if it is both left and right invariant. A group with a left-invariant metric such that the inversion function x → x^-1 is continuous is called a metric group.Very important **examples** of metric groups come from what are known as finitely generated groups
In particular, the Euclidean spaces are **examples** of metric groups.

3. Hyers - Ulam - Rassias Stability
3.1 Theorem
Let G be e-primitive mapping from an open set $E \subset R^n \to R^n$. Then for $\theta > 0$ and p in (0, 1] if the following inequality is satisfied
$\| G(X + Y) - G(X) - G(Y) \| \leq \theta.[\| X \|^p + \| Y \|^p]$ (3.11)
for all X and Y in E , then there exists a unique additive mapping
$T : E \subset R^n \to R^n$ such that

$$\| T(X) - G(X) \| \leq \left(\frac{\theta \| X \|^p}{1 - 2^{p-1}} \right) \qquad (3.12)$$

for all X in E and $p < 1$.

and

$$\| T(X) - G(X) \| \leq \left(\frac{\theta \| X \|^p}{1 - 2^{1-p}} \right) \qquad (3.13)$$

for all X in E and $p > 1$.
Proof
From the definition (2.1) of G(X), (3.11) is equivalent to

$\| [(X + Y) + (x_m + y_m)^{g(X+Y)} - (x_m + y_m) e_m] - [X + (x_m)^{g(X)} - (x_m) e_m)] - [Y + (y_m)^{g(Y)} - (y_m) e_m] \| \leq \theta[\| X \|^p + \| Y \|^p]$ (3.14) or

$\| [(x_m + y_m)^{g(X+Y)}] - [(x_m)^{g(X)}] - [(y_m)^{g(Y)})] \leq \theta[\| X \|^p + \| Y \|^p]$

Replace Y by X and consequently y_m by x_m in last inequality to obtain

$\| [2x_m]^{g(2X)} - 2[x_m]^{g(X)} \| \leq 2\theta.[\| X \|^p]$ for all X in E.

Replace X by $2^n X$ and consequently x_m by $2^n x_m$ in above inequality to obtain

$$\left\| [\,2^{n+1} x_m\,]^{g(2^{n+1} X)} - 2[\,2^n x_m\,]^{g(2^n X)} \right\| \leq 2\theta.[\,\|\,2^n X\,\|^p\,]$$

for all X in E.
Divide inequality just above by 2^{n+1} to obtain

$$\left\| \frac{[\,2^{n+1} x_m\,]^{g(2^{n+1} X)}}{2^{n+1}} - \frac{[\,2^n x_m\,]^{g(2^n X)}}{2^n} \right\| \leq \left(\frac{\theta.[\|\,2^n X\,\|^p]}{2^{n+1}} \right)$$

for all X in E. or

$$\left\| \frac{[\,2^{n+1} x_m\,]^{g(2^{n+1} X)}}{2^{n+1}} - \frac{[\,2^n x_m\,]^{g(2^n X)}}{2^n} \right\| \leq \left(\frac{\theta\|\,2^n X\,\|^p}{2^{n(1-p)}} \right)$$

for all X in E and n in N, which tend to zero as $n \to \infty$ for $p < 1$. Hence the sequence

$$\left\{ \frac{[\,2^n x_m\,]^{g(2^n X)}}{2^n} \right\}$$ is Cauchy and therefore convergent in R.

$$\text{Fix} \quad T(X) = \lim_{n\to\infty} \left(\frac{[\,2^n x_m\,]^{g(2^n X)}}{2^n} \right) \qquad (3.15)$$

Case 1 Let $p < 1$

From last inequality it follows that

$$\| T (X) - G(X) \| \leq \sum_{n=0}^{\infty} \frac{\left(\theta \| X \|^p \right)}{2^{n(1-p)}} \quad \text{for all } X \text{ in } E \text{ and } n \text{ in } N.$$

or

$$\| T (X) - G(X) \| \leq \left(\frac{\theta \| X \|^p}{1 - 2^{p-1}} \right)$$

for all X in E and p < 1 , this is exactly (3.12).

Additive

Replace X by 2^nX and Y by 2^nY and consequently x by 2^nx and y by 2^ny in (3.14) to obtain

$$\left\| [(2^n x_m + 2^n y_m)^{g(2^n X + 2^n Y)}] - [2^n x_m{}^{g(2^n X)}] - [2^n y_m{}^{g(2^n Y)}] \right\| \leq \theta[\| (2^nX) \|^p + \| (2^nY) \|^p]$$

Or

$$\lim_{n\to\infty} \left\| \frac{[(2^nx_m + 2^ny_m)^{g(2^n X + 2^n Y)}]}{2^n} - \frac{[2^nx_m{}^{g(2^n X)}]}{2^n} - \frac{[2^ny_m{}^{(2^n Y)}]}{2^n} \right\| = 0$$

Or

$T (X + Y) = T (X) + T (Y)$.

Uniqueness

Let $T '(X) : E \subset R^n \rightarrow R^n$ be another mapping which satisfies (3.12) and (3.14) . Obviously $T (2^nX) = 2^n T (2^nX)$ and $T '(2^nX) = 2^n T '(2^nX)$. Therefore

$$\| T (X) - T '(X) \| \leq 2^{-n} \| T (2^nX) - G (2^nX) \| + 2^{-n} \| T '(2^nX) - G (2^nX) \|$$

Or

$$\| T(X) - T'(X) \| \leq \left(\frac{2\theta \| 2^n X \|^p}{2^n(1 - 2^{p-1})} \right)$$

This tend to zero as n tends to infinity. Hence $T(X) = T'(X)$.

Case 2 Let $p > 1$.

For $p > 1$, proof runs parallel to that of case 1 . Here

$$T(X) = \lim_{n\to\infty} \left(\frac{[2^{-n} x_m]^{g(2^{-n} X)}}{2^{-n}} \right)$$

and it can be proved easily that

$$\| T(X) - G(X) \| \leq \left(\frac{\theta \| X \|^p}{1 - 2^{1-p}} \right)$$

for all X in E and $p > 1$.
That T is additive and unique follows easily.

REFRENCES

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EXPECTED OUTCOME OF THE PRESENT WORK

1. Generalization of these result to topological vector space can be further taken up .
2. In addition to these some open problems posed by TH.M.RASSIAS[2] in his Survey paper on the topic shall be taken for further investigation.

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